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Ebihara, Yuki Yoshi

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On the Semilinear Heat Equations and Wave Equations with Unbounded Coefficient

By

Yukiyoshi EBIHARA

(Fukuoka University, Japan)

We consider first initial-boundary value problems to semilinear heat equation

$$(1) \quad u_t - u_{xx} + \frac{1}{\rho(x)} u^q = f(x, t),$$

and semilinear wave equation

$$(2) \quad u_{tt} - u_{xx} + \frac{1}{\rho(x)} u^q = g(x, t)$$

where $-\infty < a < x < b < \infty$, $t > 0$, q is a positive integer, f and g are given smooth functions in (x, t) and $\rho(x)$ is a smooth function in $x \in (a, b)$ and it has zero points in $[a, b]$, i.e., $1/\rho(x)$ is unbounded in $[a, b]$.

Under suitable conditions on $\rho(x)$, we first show the local existence of classical solutions to (1), (2), and in the second, we find sufficient conditions for obtaining global existence.

§ 0. Introduction

Recently, various kinds of nonlinear partial differential equations have been investigated by many authors in pure and applied mathematical point of view. Especially, as is known, in the course of study of nonlinear parabolic and hyperbolic equations, there are so many results (existence, uniqueness, regularity, blow-up, asymptotic behavior). However it seems few which treats the cases that equations have nonlinearities with variable coefficients having singularities (cf. [1, 2, 4, 5]). In this paper, having interests in these equations, we extend the existence theories on this field. For our purpose, we apply an elementary inequality (Lemmas 1~2) and standard energy method with Galerkin approximate scheme. From our results, one can see that equations of typical form

[Example 1]

$$u_t - u_{xx} + x^{-\alpha}(1-x)^{-\alpha}u^5 = f(x, t), \quad 0 < x < 1$$

[Example 2]

$$u_{tt} - u_{xx} + x^{-\beta}(1-x)^{-\beta}u^{2p+1} = g(x, t), \quad 0 < x < 1$$

have global smooth solutions if $\alpha < 5/3$, $\beta < 1$ and initial data are smooth. Our plan of this paper is as follows:

- §1. Preliminaries
- §2. Local existence
- §3. Global existence for the heat equation
- §4. Global existence for the wave equation
- §5. Remarks

§1. Preliminaries

Throughout this paper, all functions are considered on real field. Let V be a totality of all finite linear combinations of $\{\phi_j\}$ which are eigen functions of the operator $-d^2/dx^2 \equiv -\partial^2$ considered on $\dot{H}^1(a, b) \cap H^2(a, b)$. We put V_k ($k=0, 1, 2, \dots$) as a closure of V by the norm $|\cdot|_k^2 \equiv (\cdot, \cdot)_k$, where $(\cdot, \cdot)_k \equiv ((-\partial^2)^k \cdot, \cdot)$ ($(\cdot, \cdot) \equiv (\cdot, \cdot)_0$ is a usual $L^2(a, b)$ inner-product). We put N as a set of all positive integers. We use $C^k(0, T; X)$, $L^p(0, T; X)$ as known notations of function spaces, where X denotes a norm space.

We first note:

Lemma 1. Let $v(x) \in V_2$, then we have

$$\sup_{a < x < b} \frac{|v(x)|}{(b-x)(x-a)} \leq \text{Const. } |v|_2.$$

Proof. Since $v(x) \in V_2$, we know that $v(x) \in C^1[a, b]$, $v(a) = v(b) = 0$, then for $a < x \leq (a+b)/2$, we get by mean value theorem $v(x) = v(a) + v'(\xi)(x-a)$ ($a < \xi < x$), this implies

$$|v(x)| \leq (x-a)|v'|_{\infty} \leq \text{Const. } (x-a)|v|_2,$$

since $a < x \leq (a+b)/2$, we get $2(b-x)/(b-a) \geq 1$, thus we obtain

$$|v(x)| \leq \frac{2(b-x)(x-a)}{(b-a)} \text{Const. } |v|_2, \quad a < x \leq \frac{a+b}{2}.$$

Analogously, we also have

$$|v(x)| \leq \frac{2(x-a)(b-x)}{b-a} \text{Const. } |v|_2, \quad \frac{a+b}{2} \leq x < b.$$

This shows our assertion.

Q. E. D.

Throughout this paper we put $S(x)$ as a function which satisfies:

$$\begin{aligned} S(x) &\in C^\infty(a, b), \quad S(a) = S(b) = 0, \\ S(x) &\geq c_0(b-x)(x-a), \quad a \leq x \leq b \end{aligned}$$

for some $c_0 > 0$.

Then we can easily verify:

Lemma 2. Let $v(x) \in V_2$, then we have

$$\sup_{a < x < b} \frac{|v(x)|}{S(x)} \leq c_1 |v|_2$$

for some $c_1 > 0$.

We frequently apply Lemma 2 and Sobolev inequalities, i.e., $|u|_k \leq \gamma |u|_{k+1}$, $|u|_\infty \leq \gamma |u|_1$.

We next have:

Lemma 3. Let $v(x) \in V_2$, $\rho \in C^1(a, b)$ and $q \in N$, then

$$\left| \frac{v^q}{\rho} \right|_1 \leq \left\{ \beta_1 \left| \frac{\rho_x S^q}{\rho^2} \right|_0^2 + \beta_2 \left| \frac{S^{q-1}}{\rho} \right|_0^2 \right\}^{1/2} |v|_2^q,$$

for some constants $\beta_1 > 0$, $\beta_2 > 0$.

Proof. We obtain

$$\begin{aligned} \left| \frac{v^q}{\rho} \right|_1^2 &= \int_a^b \left\{ \frac{-\rho_x v^q}{\rho^2} + \frac{q v^{q-1} v_x}{\rho} \right\}^2 dx \\ &\leq 2 \int_a^b \left| \frac{\rho_x v^q}{\rho^2} \right|^2 dx + 2q^2 \int_a^b \left| \frac{v^{q-1} v_x}{\rho} \right|^2 dx \\ &= 2 \int_a^b \left| \frac{v}{S} \right|^{2q} \left| \frac{\rho_x S^q}{\rho^2} \right|^2 dx + 2q^2 \int_a^b \left| \frac{v}{S} \right|^{2(q-1)} |v_x|^2 \left| \frac{S^{q-1}}{\rho} \right|^2 dx \\ &\leq 2c_1^{2q} |v|_2^{2q} \cdot \left| \frac{\rho_x S^q}{\rho^2} \right|_0^2 + 2q^2 c_1^{2(q-1)} \gamma^2 |v|_2^{2q} \left| \frac{S^{q-1}}{\rho} \right|_0^2 \end{aligned}$$

from Lemma 2. Then, putting $\beta_1 \equiv 2c_1^{2q}$, $\beta_2 \equiv 2q^2 c_1^{2(q-1)} \gamma^2$ we have the assertion. Q. E. D.

Lemma 4. Let $v, w \in V_2$, $\rho(x) \in C^1(a, b)$, $q \in N$. Then

$$\left| \frac{v^{q-1} w}{\rho} \right|_1 \leq \left\{ c_1^q \left| \frac{\rho_x S^q}{\rho^2} \right|_0 + q c_1^{q-1} \gamma \left| \frac{S^{q-1}}{\rho} \right|_0 \right\} |v|_2^{q-1} |w|_2.$$

Proof. We have

$$\begin{aligned}
\left| \frac{v^{q-1}w}{\rho} \right|_1 &\leq \left| \frac{\rho_x v^{q-1}w}{\rho^2} \right|_0 + (q-1) \left| \frac{v^{q-2}v_x w}{\rho} \right|_0 + \left| \frac{v^{q-1}w_x}{\rho} \right|_0 \\
&= \left| \left(\frac{v}{S} \right)^{q-1} \left(\frac{w}{S} \right) \frac{\rho_x S^q}{\rho^2} \right|_0 + (q-1) \left| \left(\frac{v}{S} \right)^{q-2} \left(\frac{w}{S} \right) \frac{v_x S^{q-1}}{\rho} \right|_0 \\
&\quad + \left| \left(\frac{v}{S} \right)^{q-1} \frac{w_x S^{q-1}}{\rho} \right|_0 \\
&\leq c_1^q |v|_2^{q-1} |w|_2 \left| \frac{\rho_x S^q}{\rho^2} \right|_0 + (q-1) c_1^{q-1} \gamma |v|_2^{q-1} |w|_2 \left| \frac{S^{q-1}}{\rho} \right|_0 \\
&\quad + c_1^{q-1} \gamma |v|_2^{q-1} |w|_2 \left| \frac{S^{q-1}}{\rho} \right|_0
\end{aligned}$$

This shows the assertion.

Q. E. D.

Lemma 5. Let $v \in V_3$, $\rho(x) \in C^2(a, b)$, $q \in \mathbb{N}$. Then

$$\begin{aligned}
\left| \frac{v^q}{\rho} \right|_2 &\leq c \left\{ \left| \frac{\rho_x^2}{\rho^3} S^q \right|_0 + \left| \frac{\rho_{xx}}{\rho^2} S^q \right|_0 + \left| \frac{\rho_x}{\rho^2} S^{q-1} \right|_0 \right. \\
&\quad \left. + \left| \frac{S^{q-2}}{\rho} \right|_0 \right\} |v|_2^q + c \left| \frac{S^{q-1}}{\rho} \right|_0 |v|_2^{q-1} |v|_3
\end{aligned}$$

for some $c > 0$.

Proof.

$$\begin{aligned}
\left| \frac{v^q}{\rho} \right|_2 &\leq 2 \left| \frac{\rho_x^2 v^q}{\rho^3} \right|_0 + \left| \frac{\rho_{xx} v^q}{\rho^2} \right|_0 + 2q \left| \frac{\rho_x v^{q-1} v_x}{\rho^2} \right|_0 \\
&\quad + q(q-1) \left| \frac{v^{q-2} (v_x)^2}{\rho} \right|_0 + q \left| \frac{v^{q-1} v_{xx}}{\rho} \right|_0 \\
&\leq 2c_1^q \left| \frac{\rho_x^2 S^q}{\rho^3} \right|_0 |v|_2^q + c_1^q \left| \frac{\rho_{xx} S^q}{\rho^2} \right|_0 |v|_2^q \\
&\quad + 2qc_1^{q-1} \gamma \left| \frac{\rho_x S^{q-1}}{\rho^2} \right|_0 |v|_2^q + q(q-1)c_1^{q-2} \gamma^2 \left| \frac{S^{q-2}}{\rho} \right|_0 |v|_2^q \\
&\quad + qc_1^{q-1} \gamma \left| \frac{S^{q-1}}{\rho} \right|_0 |v|_2^{q-1} |v|_3.
\end{aligned}$$

This shows the statement.

Q. E. D.

§2. Local existence

In this section we prove the local existence of classical solutions to the problems:

$$\begin{aligned}
 \text{(I)} \quad & \begin{cases} u' - u_{xx} + \frac{1}{\rho(x)} u^q = f(x, t) & a < x < b, \quad t > 0 \\ u(x, 0) = u_0(x), & a < x < b \\ u(a, t) = u(b, t) = 0, & t \geq 0 \end{cases} \\
 \text{(II)} \quad & \begin{cases} u'' - u_{xx} + \frac{1}{\rho(x)} u^q = g(x, t), & a < x < b, \quad t > 0 \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), & a < x < b, \\ u(a, t) = u(b, t) = 0, & t \geq 0. \end{cases}
 \end{aligned}$$

We assume the following conditions:

$$(A.1) \quad q \in \mathbb{N}, \quad 0 < T < \infty,$$

$$(A.2.1) \quad f \in C^1(0, T; V_1), \quad f(x, 0) \in V_2,$$

$$(A.2.2) \quad g \in C^2(0, T; V_1), \quad g(x, 0) \in V_2,$$

$$(A.3.1) \quad u_0 \in V_4,$$

$$(A.3.2) \quad u_1 \in V_3,$$

$$(A.4.1) \quad \rho(x) \in C^2(a, b),$$

$$(A.4.2.1) \quad \lim_{x \downarrow a} \frac{S^q(x)}{|\rho(x)|} = \lim_{x \uparrow b} \frac{S^q(x)}{|\rho(x)|} = 0,$$

$$(A.4.2.2) \quad \lim_{x \downarrow a} \frac{|\rho_x(x)| S^q(x)}{\rho^2(x)} = \lim_{x \uparrow b} \frac{|\rho_x(x)| S^q(x)}{\rho^2(x)} = 0,$$

$$(A.4.3) \quad \left| \frac{S^{q-2}(x)}{\rho(x)} \right|_0 \equiv \alpha_1 < \infty,$$

$$(A.4.4) \quad \left| \frac{\rho_x(x) S^{q-1}(x)}{\rho^2(x)} \right|_0 \equiv \alpha_2 < \infty,$$

$$(A.4.5) \quad \left| \frac{\rho_x^2(x) S^q(x)}{\rho^3(x)} \right|_0 \equiv \alpha_3 < \infty,$$

$$(A.4.6) \quad \left| \frac{\rho_{xx}(x) S^q(x)}{\rho^2(x)} \right|_0 \equiv \alpha_4 < \infty.$$

Then we have:

Theorem 2.1. *For the problem (I), we obtain a unique solution in the class $C^0(0, \delta_1; V_2 \cap H^3(a, b)) \cap C^1(0, \delta_1; V_1)$ for some δ_1 ($0 < \delta_1 \leq T$).*

Theorem 2.2. For the problem (II), we have a unique solution in the class $C^0(0, \delta_2; V_2 \cap H^3(a, b)) \cap C^2(0, \delta_2; V_1)$ for some δ_2 ($0 < \delta_2 \leq T$).

For the purpose of the proofs of Theorems 2.1~2.2 we employ Galerkin procedure.

Here we may assume that there exist sequences $\{a_k\}$, $\{b_k\}$ such that

$$(H.1) \quad u_{0j} \equiv \sum_{k=1}^j a_k \phi_k \longrightarrow u_0 \quad \text{strongly in } V_4 \quad (j \rightarrow \infty)$$

$$(H.2) \quad u_{1j} \equiv \sum_{k=1}^j b_k \phi_k \longrightarrow u_1 \quad \text{strongly in } V_3 \quad (j \rightarrow \infty).$$

Proof of Theorem 2.1.

We first put $u_j(x, t) \equiv \sum_{k=1}^j \chi_{j,k}(t) \phi_k(x)$ as a solution of the problem:

$$(I_j) \quad \begin{cases} \left(u_j' - u_{jxx} + \frac{1}{\rho} u_j^q - f, \phi_k \right) = 0 & (k=1, 2, \dots, j) \\ u_j(0) = u_{0j}. \end{cases}$$

Then we may have $u_j \in C^2(0, \tau_j; \langle \phi_1, \dots, \phi_j \rangle)$ for some $\tau_j > 0$, where $\langle \phi_1, \dots, \phi_j \rangle$ is a linear hull by $\{\phi_1, \phi_2, \dots, \phi_j\}$.

Next, substituting $(-\partial^2 u_j')$ for ϕ_k , we get

$$|u_j'|_1^2 + \frac{1}{2} (|u_j|_2^2)' + \int_a^b \frac{1}{\rho} u_j^q (-\partial^2 u_j') dx = (f, u_j')_1.$$

Here we know from Lemma 2 that

$$\left| \frac{v^q(x)}{\rho(x)} \right| = \left| \frac{v^q(x)}{S^q(x)} \right| \left| \frac{S^q(x)}{\rho(x)} \right| \leq c_1^q |v|_2^q \left| \frac{S^q(x)}{\rho(x)} \right|$$

for $v \in V$. Then from (A.4.2.1) we get

$$\lim_{x \downarrow a} \left| \frac{(u_j(x))^q}{\rho(x)} \right| = \lim_{x \uparrow b} \left| \frac{(u_j(x))^q}{\rho(x)} \right| = 0.$$

Thus, we have

$$\begin{aligned} |u_j'|_1^2 + \frac{1}{2} (|u_j|_2^2)' &\leq \int_a^b \left| \partial \left(\frac{u_j^q}{\rho} \right) \right| |\partial u_j'| dx + |f|_1 |u_j'|_1 \\ &\leq \frac{1}{2} |f|_1^2 + |u_j'|_1^2 + \frac{1}{2} \left| \frac{u_j^q}{\rho} \right|_1^2, \end{aligned}$$

then from Lemma 3 and (A.4.3), (A.4.4),

$$(|u_j(t)|_2^2)' \leq |f(t)|_1^2 + \{\beta_1 |S(x)|_\infty^2 \alpha_2^2 + \beta_2 |S(x)|_\infty^2 \alpha_1^2\} |u_j|_2^{2q}.$$

Putting $\beta_3 \equiv \{\beta_1 |S|_\infty^2 \alpha_2^2 + \beta_2 |S|_\infty^2 \alpha_1^2\}$, $K \equiv \max \{\beta_3, \max_{t \in [0, T]} |f(t)|_1^2\}$, we obtain

$$(|u_j(t)|_2^2)' \leq K(1 + |u_j(t)|_2^2)^q,$$

hence we get

$$|u_j(t)|_2^2 \leq (1 + |u_j(0)|_2^2)/(1 - K(q-1)(1 + |u_j(0)|_2^2)^{q-1}t)^{1/(q-1)}.$$

Therefore, applying (H.1), we find $j_0 \in N$ such that for $j \geq j_0$

$$|u_j(t)|_2^2 \leq \left(\frac{3}{2} + |u_0|_2^2\right) / \left(1 - K(q-1)\left(\frac{3}{2} + |u_0|_2^2\right)^{q-1}t\right)^{1/(q-1)}.$$

Thus, we get

$$(2.1) \quad \sup_{j \geq j_0} \sup_{t \in [0, \delta_1]} |u_j(t)|_2 < \infty,$$

where $0 < \delta_1 < \min \{T, [K(q-1)(3/2 + |u_0|_2^2)^{q-1}]^{-1}\}$. We next have by differentiating both sides of (I_j),

$$\left(u_j'' - \partial^2 u_j' + \frac{qu_j^{q-1}u_j'}{\rho} - f', \phi_k\right) = 0 \quad (k=1, 2, \dots, j),$$

then

$$|u_j''|_1^2 + \frac{1}{2}(|u_j'|_2^2)' + \left(\frac{qu_j^{q-1}u_j'}{\rho}, -\partial^2 u_j''\right) = (f', u_j'')_1.$$

Applying (A.4.2.1), Lemma 4, and (A.4.3), (A.4.4), (2.1) we have

$$\begin{aligned} \left|\left(\frac{u_j^{q-1}u_j'}{\rho}, -\partial^2 u_j''\right)\right| &\leq |u_j''|_1 \left|\frac{u_j^{q-1}u_j'}{\rho}\right|_1 \\ &\leq |u_j''|_1 |u_j|_2^{q-1} |u_j'|_2 \left\{c_1^q \left|\frac{\rho_x}{\rho^2} S^q\right|_0 + qc_1^{q-1} \gamma \left|\frac{S^{q-1}}{\rho}\right|_0\right\} \\ &\leq \text{Const. } |u_j''|_1 |u_j'|_2, \quad t \in [0, \delta_1]. \end{aligned}$$

Therefore for $t \in [0, \delta_1]$, we get

$$|u_j''|_1^2 + \frac{1}{2}(|u_j'|_2^2)' \leq |f'|_1 |u_j''|_1 + \text{Const. } |u_j''|_1 |u_j'|_2$$

and then

$$(|u_j'|_2^2)' \leq |f'|_1^2 + c|u_j'|_2^2, \quad t \in [0, \delta_1].$$

So, if we show the boundedness of $\{u_j'(0)|_2\}$, we get

$$(2.2) \quad \sup_{j \geq j_0} \sup_{t \in [0, \delta_1]} |u_j'(t)|_2 < \infty.$$

Now, letting $t \rightarrow 0$ in (I_j) and substituting $\partial^4 u'_j(0)$ for ϕ_k , we obtain

$$|u'_j(0)|_2^2 + (\partial^2 u_{0j}, \partial^4 u'_j(0)) + \left(\frac{1}{\rho} u_{0j}^q, \partial^4 u'_j(0) \right) = (f, \partial^4 u'_j(0)).$$

Here applying (A.4.2.1), (A.4.2.2), (A.4.3)~(A.4.6) and Lemma 5 we know

$$\begin{aligned} \left| \left(\frac{u_{0j}^q}{\rho}, \partial^4 u'_j(0) \right) \right| &\leq |u'_j(0)|_2 \left| \frac{u_{0j}^q}{\rho} \right|_2 \\ &\leq \text{Const. } |u'_j(0)|_2 |u_{0j}|_2^{q-1} |u_{0j}|_3. \end{aligned}$$

Therefore

$$|u'_j(0)|_2 \leq |u_{0j}|_4 + |f(0)|_2 + \text{Const. } |u_{0j}|_2^{q-1} |u_{0j}|_3.$$

This shows

$$\sup_{j \geq j_0} |u'_j(0)|_2 < \infty.$$

Next, we get for $0 \leq t < s \leq \delta_1$,

$$\begin{aligned} |u'_j(s) - u'_j(t)|_1^2 &= \int_t^s (u''_j(\tau), u'_j(s) - u'_j(t))_1 d\tau \\ &= \int_t^s \{ (u''_j(\tau), u'_j(s))_1 - (u''_j(\tau), u'_j(t))_1 \} d\tau, \end{aligned}$$

where, for $\xi = s$ or t ,

$$\begin{aligned} |(u''_j(\tau), u'_j(\xi))_1| &= \left| \left(\partial^2 u'_j(\tau) - \frac{q u_j^{q-1}(\tau) u'_j(\tau)}{\rho} + f'(\tau), u'_j(\xi) \right)_1 \right| \\ &\leq |u'_j(\xi)|_2 \left\{ |u'_j(\tau)|_2 + \left| \frac{q u_j^{q-1}(\tau) u'_j(\tau)}{\rho} \right|_0 + |f'(\tau)|_0 \right\} \\ &\leq \text{Const.} \end{aligned}$$

from our results.

Therefore, we can conclude

$$(2.3) \quad |u'_j(s) - u'_j(t)|_1^2 \leq \text{Const. } |t - s|$$

for $0 \leq t, s \leq \delta_1$.

From the standard compactness argument we have a subsequence $\{u_{j_k}(t)\}$ of $\{u_j(t)\}$ and $u(t)$ such that

$$\begin{aligned} u_{j_k}(t) &\longrightarrow u(t) \text{ weakly in } C(0, \delta_1, V_2) \quad (j_k \rightarrow \infty), \\ u_{j_k}(t) &\longrightarrow u(t) \text{ strongly in } C(0, \delta_1, V_1) \quad (j \rightarrow \infty), \\ u'_{j_k}(t) &\longrightarrow u'(t) \text{ weakly } * \text{ in } L^\infty(0, \delta_1, V_2) \quad (j_k \rightarrow \infty), \end{aligned}$$

$$u'_{jk}(t) \longrightarrow u'(t) \text{ weakly in } C(0, \delta_1, V_1) \quad (j_k \rightarrow \infty),$$

and then $u(t)$ satisfies

$$\begin{cases} \left(u'(t) - u_{xx}(t) + \frac{1}{\rho} u^q(t) - f(t), \phi_k \right) = 0 & (k=1, 2, \dots), \quad 0 \leq t \leq \delta_1 \\ u(0) = u_0 \end{cases}$$

i.e.,

$$\begin{cases} u'(t) - u_{xx}(t) + \frac{1}{\rho} u^q(t) = f(t), & \text{a.e. } x \in [0, \delta_1], \quad 0 \leq t \leq \delta_1, \\ u(0) = u_0. \end{cases}$$

Here, since

$$u'(t) + \frac{1}{\rho} u^q - f \in C(0, \delta_1, V_1),$$

then $u(t)$ belongs to $C(0, \delta_1; H^3(a, b))$ and satisfies (I). Uniqueness of the solution for the Problem (I) is apparent in the class to which $u(t)$ belongs.

This completes the proof.

Q. E. D.

Proof of Theorem 2.2.

As in the previous manner we put $u_j(x, t) \equiv \sum_{k=1}^j \mu_{j,k}(t) \phi_k(x)$ as a solution of the problem:

$$(II_j) \quad \begin{cases} \left(u_j'' - \partial^2 u_j + \frac{1}{\rho} u_j^q - g, \phi_k \right) = 0 & (k=1, 2, \dots, j) \\ u_j(0) = u_{0j}, \quad u_j'(0) = u_{1j}. \end{cases}$$

We may assume $u_j \in C^4(0, \tau'_j; \langle \phi_1, \dots, \phi_j \rangle)$ for some $\tau'_j > 0$. Then substituting $(-\partial^2 u'_j)$ for ϕ_k we get

$$\frac{1}{2} \{ |u'_j|_1^2 + |u_j|_2^2 \}' + \left(\frac{u_j^q}{\rho}, (-\partial^2 u'_j) \right) = (g, u'_j)_1,$$

Applying (A.4.2.1), Lemma 3, (A.4.3), (A.4.4) again, we obtain

$$(|u'_j(t)|_1^2 + |u_j(t)|_2^2)' \leq |g(t)|_1^2 + 2|u'_j(t)|_1^2 + \tilde{\beta}|u_j(t)|_2^q$$

for some $\tilde{\beta} > 0$.

Thus, putting $\tilde{K} = \max \{ \max_{t \in [0, T]} |g(t)|_1^2, 2, \tilde{\beta} \}$, we get for $t \in [0, \tau_j)$,

$$(|u'_j(t)|_1^2 + |u_j(t)|_2^2)' \leq \tilde{K}(1 + |u'_j(t)|_1^2 + |u_j(t)|_2^q).$$

From the analogue of previous arguments in the proof of Theorem 2.1, we can say

$$(2.4) \quad \sup_{j \geq j_0} \sup_{t \in [0, \delta_2]} (|u'_j(t)|_1 + |u_j(t)|_2) < \infty,$$

for some j_0 and δ_2 ($0 < \delta_2 \leq T$). We next have by differentiating both sides of (II_j) and substituting $(-\partial^2 u''_j)$ for ϕ_k ,

$$\frac{1}{2} \{ |u''_j|_1^2 + |u'_j|_2^2 \}' + q \left(\frac{u_j^{q-1} u'_j}{\rho}, (-\partial^2 u''_j) \right) \leq |g'|_1 |u''_j|_1$$

in $t \in (0, \delta_2]$. Then, we know from Lemma 4 and our assumptions,

$$\begin{aligned} \left| q \left(\frac{u_j^{q-1} u'_j}{\rho}, (-\partial^2 u''_j) \right) \right| &= \left| q \left(\partial \left(\frac{u_j^{q-1} u'_j}{\rho} \right), \partial u''_j \right) \right| \\ &\leq q |u''_j|_1 \left\{ c_1^q |u_j|_2^{q-1} |u'_j| \left| \frac{\rho_x}{\rho^2} S^q \right|_0 + (q-1) c_1^{q-1} \gamma |u_j|_2^{q-1} |u'_j|_2 \left| \frac{1}{\rho} S^{q-1} \right|_0 \right. \\ &\quad \left. + c_1^{q-1} \gamma |u_j|_2^{q-1} |u'_j|_2 \left| \frac{1}{\rho} S^{q-1} \right|_0 \right\} \\ &\leq \text{Const.} |u''_j|_1 |u'_j|_2, \end{aligned}$$

which implies

$$(|u''_j(t)|_1^2 + |u'_j(t)|_2^2)' \leq \text{Const.} (|u''_j(t)|_1^2 + |u'_j(t)|_2^2) + |g'(t)|_1^2, \quad t \in (0, \delta_2].$$

Thus,

$$\begin{aligned} \sup_{j \geq j_0} \sup_{t \in [0, \delta_2]} (|u''_j(t)|_1^2 + |u'_j(t)|_2^2) \\ \leq \text{Const.} \{ 1 + \sup_{j \geq j_0} (|u''_j(0)|_1^2 + |u'_j(0)|_2^2) \}. \end{aligned}$$

We now prove the boundedness of $\{ |u''_j(0)|_1^2 \}$. Let $t \rightarrow 0$ in (II_j) and substitute $(-\partial^2 u''_j(0))$ for ϕ_k , then we get

$$\begin{aligned} |u''_j(0)|_1^2 + (\partial^2 u_{0j}, \partial^2 u''_j(0)) + \left(\frac{(u_{0j})^q}{\rho}, (-\partial^2 u''_j(0)) \right) \\ = (g(0), (-\partial^2 u''_j(0))), \end{aligned}$$

then

$$\sup_{j \geq j_0} |u''_j(0)|_1 \leq \sup_{j \geq j_0} \left\{ |u_{0j}|_3 + \left| \frac{(u_{0j})^q}{\rho} \right|_1 + |g(0)|_1 \right\} < \infty.$$

Therefore we have

$$(2.5) \quad \sup_{j \geq j_0} \sup_{t \in [0, \delta_2]} (|u''_j(t)|_1 + |u'_j(t)|_2) < \infty.$$

Next, differentiating both sides of (II_j) once more and substituting $(-\partial^2 u^{(3)}_j)$ for ϕ_k , we obtain

$$\begin{aligned} & \frac{1}{2} (|u_j^{(3)}|_1^2 + |u_j''|_2^2)' + \left(\frac{qu_j^{q-1}u_j'' + q(q-1)u_j^{q-2}(u_j')^2}{\rho}, (-\partial^2 u_j^{(3)}) \right) \\ & = (g'', u_j^{(3)})_1, \end{aligned}$$

then applying (A.4.2.1) and Lemma 4, (2.4), (2.5), we get

$$\left| \frac{qu_j^{q-1}u_j'' + q(q-1)u_j^{q-2}(u_j')^2}{\rho} \right|_1 \leq \text{Const.} (1 + |u_j''|_2)$$

for $t \in [0, \delta_2]$. Thus, we have for $t \in [0, \delta_2]$

$$(|u_j^{(3)}|_1^2 + |u_j''|_2^2)' \leq |g''|_1^2 + \text{Const.} (1 + |u_j^{(3)}|_1^2 + |u_j''|_2^2).$$

Here we show the boundedness of $\{|u_j''(0)|_2\}\{|u_j^{(3)}(0)|_1\}$. In (II)_j, let $t \rightarrow 0$ and substitute $\partial^4 u_j''(0)$, then we obtain

$$\begin{aligned} & |u_j''(0)|_2^2 + (\partial^2 u_{0j}, \partial^4 u_j''(0)) + \left(\frac{(u_{0j})^q}{\rho}, \partial^4 u_j''(0) \right) \\ & = (g(0), \partial^4 u_j''(0)), \end{aligned}$$

then from (A.4.2.1)~(A.4.2.2), we have

$$|u_j''(0)|_2 \leq |u_{0j}|_4 + \left| \frac{(u_{0j})^q}{\rho} \right|_2 + |g(0)|_2.$$

The right-hand side is bounded by (H.1) and Lemma 5.

Differentiate (II)_j, let $t \rightarrow 0$ and substitute $(-\partial^2 u_j^{(3)}(0))$. We have

$$\begin{aligned} & |u_j^{(3)}(0)|_1^2 + (\partial^2 u_{1j}, \partial^2 u_j^{(3)}(0)) + \left(q \frac{(u_{0j})^{q-1}(u_{1j})}{\rho}, (-\partial^2 u_j^{(3)}(0)) \right) \\ & = (g'(0), (-\partial^2 u_j^{(3)}(0))), \end{aligned}$$

then from (A.4.2.1),

$$|u_j^{(3)}(0)|_1 \leq |u_{1j}|_3 + \left| q \frac{(u_{0j})^{q-1}(u_{1j})}{\rho} \right|_1 + |g'(0)|_1.$$

The right-hand side is bounded by (H.1), (H.2) and Lemma 4. Consequently, we have

$$(2.6) \quad \sup_{j \geq j_0} \sup_{t \in [0, \delta_2]} (|u_j^{(3)}(t)|_1 + |u_j''(t)|_2) < \infty.$$

Therefore, we can assert that there exist a subsequence $\{u_{j_k}(t)\}$ of $\{u_j(t)\}$ and $u(t)$ such that

$$u_{j_k}(t) \longrightarrow u(t) \text{ weakly in } C(0, \delta_2; V_2) \quad (j_k \rightarrow \infty),$$

$$u'_{j_k}(t) \longrightarrow u'(t) \text{ weakly in } C(0, \delta_2; V_2) \quad (j_k \rightarrow \infty),$$

$$u''_{j_k}(t) \longrightarrow u''(t) \text{ weakly in } C(0, \delta_2; V_1) \quad (j_k \rightarrow \infty),$$

and $u(t)$ satisfies

$$\begin{cases} u''(t) - \partial^2 u(t) + \frac{u^q(t)}{\rho} = g(t) & \text{a.e. } x \in (a, b), \quad t \geq 0 \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases}$$

Again, since $u''(t) + u^q(t)/\rho - g(t) \in C(0, \delta_2; V_1)$, we get $u(t) \in C(0, \delta_2; H^3(a, b))$. Uniqueness is also trivial in the class to which $u(t)$ belongs.

This completes the proof.

Q. E. D.

§ 3. Global existence for the heat equation

In this section we further assume that

$$(A.1)' \quad q \text{ is an odd integer with } q \geq 3,$$

$$(A.5) \quad \rho(x) \geq 0, \quad x \in [a, b],$$

$$(A.6) \quad \text{there exists } \varepsilon > 0 \quad (0 < \varepsilon \leq 1) \text{ such that}$$

$$\int_a^b \rho^{-3}(x) S^{4\varepsilon}(x) dx \equiv \alpha_\varepsilon < \infty.$$

Note 1. We can easily check that a function $\rho(x) = (x-a)^\alpha(b-x)^\alpha$ ($a \leq x \leq b$) with $0 < \alpha < 5/3$ satisfies all conditions when $q \geq 5$.

Then assuming (A.1)', (A.2.1), (A.3.1), (A.4), (A.5) and (A.6), we have the following assertion.

Theorem 3.1.

(3.1) If $0 < \varepsilon < 1$, we have a unique solution for (I) in $[0, T]$.

(3.2) If $\varepsilon = 1$, with the smallness conditions to

$$E_0 \equiv |u_0|_1^2 + \frac{2}{q+1} \int_a^b \frac{u_0^{q+1}}{\rho} dx + \frac{1}{2} \int_0^T |f(t)|_0^2 dt$$

we obtain the same conclusion as in (3.1).

Proof. One can see that if the formal solution $u(t)$ of the problem (I) satisfies

$$(3.5) \quad \sup_{t \in [0, T]} |u(t)|_2 < \infty,$$

then the assertion will follow from the arguments in the proof of Theorem 2.1. Multiplying both sides of the equation in (I) by $u'(t)$, we get

$$\begin{aligned} |u'(t)|_0^2 + \left(\frac{1}{2} |u(t)|_1^2\right)' + \left(\frac{1}{q+1} \int_a^b \frac{(u(t))^{q+1}}{\rho} dx\right)' &= (f(t), u'(t)) \\ &\leq |f(t)|_0 |u'(t)|_0 \leq \frac{1}{4} |f(t)|_0^2 + |u'(t)|_0^2, \end{aligned}$$

then $|u(t)|_1^2 + \frac{2}{q+1} \int_a^b \frac{(u(t))^{q+1}}{\rho} dx \leq E_0$, $t \in [0, T]$. In the next, differentiating the equation by t and multiplying by $u'(t)$, we get

$$\begin{aligned} \left(\frac{1}{2} |u'(t)|_0^2\right)' + |u'(t)|_1^2 + q \int_a^b \frac{(u(t))^{q-1} (u'(t))^2}{\rho} dx &= (f'(t), u'(t)) \\ &\leq |f'(t)|_0 |u'(t)|_0 \leq |u'(t)|_1^2 + c |f'(t)|_0^2, \end{aligned}$$

then

$$\begin{aligned} |u'(t)|_0^2 &\leq |u'(0)|_0^2 + c \int_0^t |f'(s)|_0^2 ds \\ &\leq \left(|u_0|_2 + \left|\frac{1}{\rho} u_0^q\right|_0 + |f(0)|_0\right)^2 + c \int_0^t |f'(s)|_0^2 ds \\ &\leq \text{Const.}, \quad t \in [0, T]. \end{aligned}$$

Next, we have

$$|u(t)|_2 \leq |u'(t)|_0 + |f(t)|_0 + \left|\frac{1}{\rho} u^q(t)\right|_0.$$

Here,

$$\begin{aligned} \left|\frac{1}{\rho} u^q(t)\right|_0 &= \left[\int_a^b \frac{u^{2q}(t)}{\rho^2} dx\right]^{1/2} = \left[\int_a^b \frac{(u(t))^{2\varepsilon}}{S^{2\varepsilon}} \cdot \frac{(u(t))^{2q-2\varepsilon} S^{2\varepsilon}}{\rho^2} dx\right]^{1/2} \\ &\leq c_1^\varepsilon |u(t)|_2^\varepsilon \cdot \left[\int_a^b \frac{(u(t))^{(q+1)/2}}{\rho^{1/2}} \cdot \frac{(u(t))^{2q-2\varepsilon-(q+1)/2} S^{2\varepsilon}}{\rho^{3/2}} dx\right]^{1/2} \\ &\leq c_1^\varepsilon |u(t)|_2^\varepsilon \left[\int_a^b \frac{(u(t))^{q+1}}{\rho} dx\right]^{1/4} \left[\int_a^b \frac{(u(t))^{3q-4\varepsilon-1} S^{4\varepsilon}}{\rho^3} dx\right]^{1/4} \\ &\leq c_1^\varepsilon |u(t)|_2^\varepsilon \gamma^{(3q-4\varepsilon-1)/4} |u(t)|_1^{(3q-4\varepsilon-1)/4} \left[\int_a^b \frac{(u(t))^{q+1}}{\rho} dx\right]^{1/4} \alpha_\varepsilon^{1/4} \\ &\leq c_1^\varepsilon \gamma^{(3q-4\varepsilon-1)/4} E_0^{(3q-4\varepsilon-1)/8} \left(\frac{q+1}{2} E_0\right)^{1/4} \alpha_\varepsilon^{1/4} |u(t)|_2^\varepsilon \\ &= c_1^\varepsilon \gamma^{(3q-4\varepsilon-1)/4} \left(\frac{q+1}{2}\right)^{1/8} \alpha_\varepsilon^{1/4} E_0^{(3q-4\varepsilon+1)/8} |u(t)|_2^\varepsilon \\ &\equiv K_\varepsilon E_0^{(3q-4\varepsilon+1)/8} |u(t)|_2^\varepsilon \end{aligned}$$

Therefore, if $0 < \varepsilon < 1$, we get

$$|u(t)|_2 \leq |u'(t)|_0 + |f(t)|_0 + \varepsilon |u(t)|_2 + (1 - \varepsilon)(K_\varepsilon E_0^{(3q-4\varepsilon+1)/8})^{1/(1-\varepsilon)}$$

i.e.,

$$\begin{aligned} |u(t)|_2 &\leq \frac{1}{1-\varepsilon} \{|u'(t)|_0 + |f(t)|_0\} + (K_\varepsilon E_0^{(3q-4\varepsilon+1)/8})^{1/(1-\varepsilon)} \\ &\leq \text{Const.}, \quad t \in [0, T]. \end{aligned}$$

This shows (3.5). And if $\varepsilon = 1$, we choose E_0 as

$$K_1 E_0^{(3q-3)/8} < 1,$$

then

$$\begin{aligned} |u(t)|_2 &\leq (1 - K_1 E_0^{(3q-3)/8})^{-1} \{|u'(t)|_0 + |f(t)|_0\} \\ &\leq \text{Const.}, \quad t \in [0, T]. \end{aligned}$$

This shows (3.5).

Q. E. D.

§4. Global existence for the wave equation

In this section we further put the following assumptions. There exist $\mu > 0$ and $\nu > 0$ such that

$$(A.7.1) \quad \int_a^b (\rho(x))^{-2(1+1/\mu)} (S(x))^{2/\mu} dx \equiv \alpha_\mu < \infty,$$

$$(A.7.2) \quad \int_a^b (\rho(x))^{-4(1+1/\nu)} (\rho_x(x))^{2(1+1/\nu)} (S(x))^{4/\nu+2} dx \equiv \alpha_\nu < \infty.$$

Note 2. One can verify that a function $\rho(x) = (x-a)^\alpha (b-x)^\alpha$ ($a \leq x \leq b$) with $0 < \alpha < 1$ satisfies all conditions stated above.

Now we have the final assertion:

Theorem 4.1. Suppose the assumptions (A.1)', (A.2.2), (A.3.2), (A.4), (A.5) and (A.7). Then we have a unique solution for (II) in $[0, T]$.

Proof. Put $u(t)$ as a formal solution of the problem (II). It may suffice to show

$$(4.1) \quad \sup_{t \in [0, T]} \{|u'(t)|_1 + |u(t)|_2\} < \infty.$$

As in the previous manner, we first get

$$\begin{aligned} & \frac{1}{2} \{ |u'(t)|_0^2 + |u(t)|_1^2 \}' + \left(\frac{1}{q+1} \int_a^b \frac{(u(t))^{q+1}}{\rho} dx \right)' \\ &= (g(t), u'(t)) \leq \frac{1}{2} (|g(t)|_0^2 + |u'(t)|_0^2), \end{aligned}$$

then, we have

$$\begin{aligned} (4.2) \quad & |u'(t)|_0^2 + |u(t)|_1^2 + \frac{2}{q+1} \int_a^b \frac{(u(t))^{q+1}}{\rho} dx \\ & \leq e^T \left\{ |u_1|_0^2 + |u_0|_1^2 + \frac{2}{q+1} \int_a^b \frac{(u_0)^{q+1}}{\rho} dx + \int_0^T |g(t)|_0^2 dt \right\} \\ & \equiv E_1 < \infty, \quad t \in [0, T]. \end{aligned}$$

Next, we also have

$$\frac{1}{2} \{ |u'(t)|_1^2 + |u(t)|_2^2 \}' + \left(\frac{(u(t))^{q+1}}{\rho}, -\partial^2 u'(t) \right) = (g(t), u'(t))_1,$$

then

$$\{ |u'(t)|_1^2 + |u(t)|_2^2 \}' + 2 \left(\left(\frac{(u(t))^q}{\rho} \right)_x, u'_x \right) \leq 2 |g(t)|_1 |u'(t)|_1,$$

hence, integrating over $[0, t]$, we get

$$\begin{aligned} & \{ |u'(t)|_1^2 + |u(t)|_2^2 \} + 2 \int_0^t \left(\left(\frac{(u(\tau))^q}{\rho} \right)_x, u'_x \right) d\tau \\ & \leq |u_1|_1^2 + |u_0|_2^2 + \int_0^t |g|_1^2 d\tau + \int_0^t |u'|_1^2 d\tau, \end{aligned}$$

and then,

$$\begin{aligned} & \{ |u'(t)|_1^2 + |u(t)|_2^2 \} + 2 \left(\left(\frac{(u)^q}{\rho} \right)_x, u_x \right) \\ & \leq E_2 + 2 \int_0^t \left(\left(\frac{qu^{q-1}u'}{\rho} \right)_x, u_x \right) d\tau + \int_0^t |u'|_1^2 d\tau, \end{aligned}$$

where

$$E_2 \equiv |u_1|_1^2 + |u_0|_2^2 + 2 \left| \frac{(u_0)^q}{\rho} \right|_1 |u_0|_1 + \int_0^T |g(t)|_1^2 dt.$$

Here, we know that

$$\left(\left(\frac{(u)^q}{\rho} \right)_x, u_x \right) = \left(\frac{qu^{q-1}u_x}{\rho}, u_x \right) - \left(\frac{\rho_x u^q}{\rho^2}, u_x \right).$$

Since the first term on the right-hand side of the above is non-negative, we can obtain

$$\begin{aligned} |u'(t)|_1^2 + |u(t)|_2^2 + 2q \left(\frac{u^{q-1}}{\rho}, u_x^2 \right) &\leq 2 \left| \frac{\rho_x u^q}{\rho^2} \right|_0 |u|_1 + E_2 \\ &+ 2q \int_0^t \left(\left(\frac{u^{q-1} u'}{\rho} \right)_x, u_x \right) d\tau + \int_0^t |u'|_1^2 d\tau, \end{aligned}$$

Now since $(3/v+2)/(2/v+2) < 2$, where v is the number in (A.7.2), we choose $\theta > 0$ such that

$$(4.3) \quad \frac{3/v+2}{2/v+2} < \theta < 2,$$

then we obtain

$$\begin{aligned} \left| \frac{\rho_x u^q}{\rho^2} \right|_0^2 &= \int_a^b \frac{\rho_x^2}{\rho^4} |u|^{2q} dx = \int_a^b \frac{\rho_x^2 (S(x))^{2\theta}}{\rho^4} \left(\frac{|u|}{S(x)} \right)^{2\theta} |u|^{2q-2\theta} dx \\ &\leq c_1^2 \theta |u|_2^{2\theta} \cdot \gamma^{2(q-\theta)} |u|_1^{2(q-\theta)} \int_a^b \frac{\rho_x^2}{\rho^4} (S(x))^{2\theta} dx. \end{aligned}$$

Here

$$\begin{aligned} \int_a^b \frac{\rho_x^2}{\rho^4} (S(x))^{2\theta} dx &= \int_a^b \left(\frac{\rho_x^2}{\rho^4} S^{(4/v+2)/(1/v+1)} \right) S^{2\theta - (4/v+2)/(1/v+1)} dx \\ &\leq \left[\int_a^b \left(\frac{\rho_x^2}{\rho^4} \right)^{1+1/v} S^{4/v+2} dx \right]^{v/(1+v)} \left[\int_a^b S^{(2\theta - (4+2v)/(1+v))(1+v)} dx \right]^{1/(1+v)}, \\ &\leq \alpha_v^{v/(1+v)} \left[\int_a^b S^{(2\theta(1+v) - (4+2v))} dx \right]^{1/(1+v)} \\ &\equiv \beta_v^2 < \infty \end{aligned}$$

from (4.3).

Next, since we have

$$\begin{aligned} \left(\left(\frac{u^{q-1} u'}{\rho} \right)_x, u_x \right) &= - \left(\frac{\rho_x u^{q-1} u'}{\rho^2}, u_x \right) + \frac{1}{2} \left(\frac{u^{q-1}}{\rho}, (u_x^2)' \right) \\ &\quad + (q-1) \left(\frac{u^{q-2} u_x u'}{\rho}, u_x \right), \\ \int_0^t \left(\left(\frac{u^{q-1} u'}{\rho} \right)_x, u_x \right) d\tau &= \frac{1}{2} \left(\frac{u^{q-1}}{\rho}, u_x^2 \right) - \frac{1}{2} \left(\frac{u_0^{q-1}}{\rho}, u_{0x}^2 \right) \\ &\quad + \frac{q-1}{2} \int_0^t \left(\frac{u^{q-2} u'}{\rho}, u_x^2 \right) d\tau - \int_0^t \left(\frac{\rho_x u^{q-1} u'}{\rho^2}, u_x \right) d\tau. \end{aligned}$$

Therefore, we can obtain for small $\hat{\rho}$

$$\begin{aligned}
& |u'(t)|_1^2 + (1 - \hat{\rho})|u(t)|_2^2 + q \left(\frac{(u(t))^{q-1}}{\rho}, u_x^2(t) \right) \\
& \leq c_1(\rho, \beta_v, \gamma, E_1) + E_2 + \int_0^t |u'|_1^2 d\tau + q \left| \left(\frac{u_0^{q-1}}{\rho}, u_{0x}^2 \right) \right| \\
& \quad + q(q-1) \int_0^t \left| \left(\frac{u^{q-2}u'}{\rho}, u_x^2 \right) \right| d\tau + 2q \int_0^t \left| \left(\frac{\rho_x u^{q-1}u'}{\rho^2}, u_x \right) \right| d\tau.
\end{aligned}$$

Here, we get

$$\begin{aligned}
& \left| \left(\frac{u^{q-2}u'}{\rho}, u_x^2 \right) \right| \leq |u'|_0 \left[\int_a^b \frac{u^{2(q-2)}}{\rho^2} u_x^4 dx \right]^{1/2} \\
& \leq |u'|_0 \left[\int_a^b |u_x|^{4(1+\mu)} dx \right]^{1/2(1+\mu)} \left[\int_a^b \left(\frac{u^{2(q-2)}}{\rho^2} \right)^{(1+1/\mu)} dx \right]^{\mu/2(1+\mu)} \\
& \leq |u'|_0 (\gamma |u|_2)^{(2+4\mu)/2(1+\mu)} |u|_1^{1/2(1+\mu)} \\
& \quad \times \left[\int_a^b \left(\frac{|u|}{S} \right)^{2/\mu} \left(\frac{u^{2q-4}}{\rho^2} \right)^{(1+\mu)/\mu} \left(\frac{S}{|u|} \right)^{2/\mu} dx \right]^{\mu/2(1+\mu)} \\
& \leq |u'|_0 (\gamma |u|_2)^{(1+2\mu)/(1+\mu)} |u|_1^{1/2(1+\mu)} (c_1 |u|_2)^{1/(1+\mu)} (\gamma |u|_1)^{((q-2)(1+\mu)-1)/(1+\mu)} \\
& \quad \times \left[\int_a^b S^{2/\mu} \rho^{-2(1+\mu)/\mu} dx \right]^{v/2(1+v)} \\
& \leq C_2(\gamma, \mu, \alpha_\mu, E_1) (|u|_2^2), \\
& \left| \left(\frac{\rho_x u^{q-1}u'}{\rho^2}, u_x \right) \right| \leq |u|_1 \left[\int_a^b \frac{\rho_x^2 u^{2q-2} (u')^2}{\rho^4} dx \right]^{1/2} \\
& \leq |u|_1 \left[\int_a^b |u'|^{2(1+v)} dx \right]^{1/2(1+v)} \left[\int_a^b \left(\frac{\rho_x^2 u^{2q-2}}{\rho^4} \right)^{1+1/v} dx \right]^{v/2(1+v)} \\
& \leq |u|_1 (\gamma |u'|_1)^{v/(1+v)} |u'|_0^{1/2(1+v)} \\
& \quad \times \left[\int_a^b \left(\frac{|u|}{S} \right)^{(4+2v)/v} \left(\frac{\rho_x^2 u^{2q-2}}{\rho^4} \right)^{1+1/v} \left(\frac{S}{|u|} \right)^{(4+2v)/v} dx \right]^{v/2(1+v)} \\
& \leq |u|_1 (\gamma |u'|)^{v/(1+v)} |u'|_0^{1/2(1+v)} (c_1 |u|_2)^{(2+v)/(1+v)} \\
& \quad \times (\gamma |u|_1)^{((q-1)(v+1)-(2+v))/(1+v)} \left[\int_a^b \left(\frac{\rho_x^2}{\rho^4} \right)^{1+1/v} S^{(4+2v)/v} dx \right]^{v/2(1+v)} \\
& \leq c_3(\gamma, v, \alpha_v, E_1) |u'|_1^{v/(1+v)} |u|_2^{(2+v)/(1+v)} \\
& \leq c_4(\gamma, v, \alpha_v, E_1) (|u'|_1^2 + |u|_2^2)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& |u'(t)|_1^2 + (1 - \hat{\rho})|u(t)|_2^2 \\
& \leq \text{Const.} \left\{ 1 + \int_0^t (|u'(\tau)|_1^2 + |u(\tau)|_2^2) d\tau \right\},
\end{aligned}$$

i.e.,

$$|u'(t)|_1^2 + |u(t)|_2^2 \leq \text{Const.} \left\{ 1 + \int_0^t (|u'(\tau)|_1^2 + |u(\tau)|_2^2) d\tau \right\}.$$

This Gronwall inequality implies (4.1).

Q. E. D.

§5. Remarks

Remark 1. One obtains analogous results to the equations

$$u_t - u_{xx} + \frac{|u|^\ell u}{\rho} = f$$

$$u_{tt} - u_{xx} + \frac{|u|^\ell u}{\rho} = g$$

where ℓ is a number with $\ell \geq 2$.

Remark 2. Concerning the global existence we could not have the assertion in the cases $\alpha \geq 5/3$ in Example 1 and $\beta \geq 1$ in Example 2 respectively.

Remark 3 (cf. [3]). Applying Lemma 2, we can construct a local classical solution to a quasilinear wave equation

$$u_{tt} = (u^m)_{xx} \quad a < x < b \quad (m \geq 5).$$

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nuna adreso:
 Department of Applied Mathematics
 Faculty of Science
 Fukuoka University,
 Fukuoka 814-01
 Japan

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