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Viscosity Solutions for a System of Elliptic Inequalities with Bilateral Obstacles

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§ 1. Introduction

This paper deals with the existence and uniqueness of viscosity solutions for a system of second order elliptic inequalities with bilateral obstacles. Let

$$A^p u = - \sum_{i,j=1}^N a_{ij}^p u_{x_i x_j} + \sum_{i=1}^N b_i^p u_{x_i} + c^p u, \quad p = 1, \dots, m$$

be linear elliptic operators. We consider the following system of inequalities:

$$(1.1) \quad \begin{aligned} m^p u &\leq u^p \leq M^p u && \text{in } \Omega, \quad u^p|_{\partial\Omega} = 0, \\ A^p u^p &= f^p && \text{if } m^p u < u^p < M^p u, \\ A^p u^p &\leq f^p && \text{if } u^p = M^p u, \\ A^p u^p &\geq f^p && \text{if } u^p = m^p u, \end{aligned}$$

for $p=1, \dots, m$. Here $M^p u$ and $m^p u$ are defined as follows:

$$\begin{aligned} M^p u &= \min \{u^q + K(p, q) \mid 1 \leq q \leq m, q \neq p\}, \\ m^p u &= \max \{u^q - k(p, q) \mid 1 \leq q \leq m, q \neq p\}, \end{aligned}$$

where $K(p, q)$ and $k(p, q)$ are given nonnegative constants.

S. M. Lenhart and S. A. Belbas [3] considered a system of equations with unilateral constraint:

$$(1.2) \quad \begin{aligned} \max \{A^p u^p - f^p, u^p - M^p u\} &= 0 \quad \text{in } \Omega, \\ u^p|_{\partial\Omega} &= 0, \quad p = 1, \dots, m \end{aligned}$$

and proved the existence and uniqueness of the solution in $W^{2,\infty}(\Omega)$. In [5], the author have treated the system (1.1) in $W^{2,r}(\Omega)$ in which the bilateral obstacles are of the form

$$M^p u = u^{p+1} + K, \quad m^p u = u^{p+1} - k.$$

As was explained in [3], [5], these systems are closely related to some

stochastic control problems (see [3], [5] or [2] for detail). The system (1.1) is a generalization of systems which were considered in [3] and [5]. However, our argument in this paper presents a new approach which stands on the notion of viscosity solutions.

The notion of viscosity solution was first introduced by M. G. Crandall and P. L. Lions [1] for first order Hamilton-Jacobi equations and by P. L. Lions [4] for second order Hamilton-Jacobi-Bellman equations.

We slightly modify the definition of viscosity solution for our system by noting that (1.1) is equivalent to the following system of mini-max equations:

$$(1.3) \quad \begin{aligned} \min \{ \max \{ A^p u^p - f^p, u^p - M^p u \}, u^p - m^p u \} &= 0 \quad \text{in } \Omega, \\ u^p|_{\partial\Omega} &= 0, \quad p = 1, \dots, m. \end{aligned}$$

In section 2 we shall state our assumptions and main theorem. Section 3 is devoted to a study of a system of penalty equations and we shall prove some a priori estimates. In section 4, we shall prove our main theorem. The proof of existence stands on the stability of viscosity solutions. To prove the uniqueness, we compare an arbitrary viscosity solution with a limit of approximate solutions. The same idea is used in [6] for some Hamilton-Jacobi-Bellman equations to prove the uniqueness of solutions.

§ 2. Assumptions and main theorem

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary. We follow normal usage to denote various function spaces such as $W^{n,r}(\Omega)$ ($1 \leq r \leq \infty$) or $C^n(\Omega)$. We use the summation convention throughout this paper.

Let $m > 1$ be a fixed integer and set $P = \{1, \dots, m\}$. For $p \in P$, we consider the second order elliptic differential operator

$$(2.1) \quad A^p v = -a_{ij}^p(x) v_{x_i x_j} + b_i^p(x) v_{x_i} + c^p(x) v.$$

We make the following assumptions on A^p :

$$(2.2) \quad a_{ij}^p \xi_i \xi_j \geq \theta |\xi|^2$$

for some $\theta > 0$, all $\xi \in \mathbf{R}^N$ and $p \in P$,

$$(2.3) \quad a_{ij}^p, \quad b_i^p, \quad c^p \in C^1(\bar{\Omega})$$

for $p \in P$ and $1 \leq i, j \leq N$.

There exists a sufficiently large constant c_0 such that

$$(2.4) \quad c^p \geq c_0$$

for all $p \in P$. The size of c_0 is determined only by θ and the sizes of a_{ij}^p , b_i^p and their derivatives (see Lemma 3.2).

On given functions f^p , $p \in P$, we assume

$$(2.5) \quad f^p \in C^1(\bar{\Omega}).$$

Let $k(p, q)$, $K(p, q)$ be a family of nonnegative constants parametrized by $p, q \in P$, $p \neq q$. For this family we assume

$$(2.6) \quad \begin{aligned} k(p, r) &\leq k(p, q) + k(q, r), \\ K(p, r) &\leq K(p, q) + K(q, r) \end{aligned}$$

for $p, q, r \in P$ and

$$(2.7) \quad \kappa(p_1, p_2) + \kappa(p_2, p_3) + \cdots + \kappa(p_n, p_1) \neq 0$$

for any choice of a chain of parameters $\{p_1, p_2, \dots, p_n, p_1\} \subset P$, where $\kappa(p_i, p_j)$ stands for $K(p_i, p_j)$ or $-k(p_i, p_j)$.

As was explained in [3], starting with an arbitrary family, we can always construct a family which satisfies (2.6) and works equivalently on the new system of inequalities. On the other hand, (2.7) is a natural extension of the so-called "no loop of zero cost" condition in [3] which was essential in the proof of uniqueness of solutions. We also note that we already introduced an essentially equivalent condition in [5] which was used to prove the $W^{2,r}(\Omega)$ regularity for an analogous system with bilateral constraints.

Now we can state our main result.

Theorem 2.1. *Under the assumptions (2.2)–(2.7) we have:*

(i) *there exists a viscosity solution $u = \{u^1, u^2, \dots, u^m\}$ of the system of inequalities*

$$(2.8) \quad \begin{aligned} \min \{ \max \{ A^p u^p - f^p, u^p - M^p u \}, u^p - m^p u \} &= 0 \quad \text{in } \Omega, \\ u^p|_{\partial\Omega} &= 0, \quad p \in P, \end{aligned}$$

each of which belongs to $W^{1,\infty}(\Omega)$, where

$$\begin{aligned} M^p u &= \min \{ u^q + K(p, q) \mid q \in P, q \neq p \}, \\ m^p u &= \max \{ u^q - k(p, q) \mid q \in P, q \neq p \}. \end{aligned}$$

(ii) *the viscosity solution of (2.8) is unique in $C(\bar{\Omega})$.*

The notion of viscosity solution of (2.8) is defined as follows which is a modification of the original definition in M. G. Crandall and P. L. Lions [1] and P. L. Lions [4].

Definition. A set of functions $u = \{u^1, \dots, u^m\}$, $u^p \in C(\bar{\Omega})$, $p \in P$, is a viscosity

solution of (2.8) if and only if both (2.9) and (2.10) hold:

$$(2.9) \quad m^p u(x) \leq u^p(x) \leq M^p u(x) \quad \text{in } \Omega \quad \text{and} \quad u^p|_{\partial\Omega} = 0, \quad p \in P.$$

$$(2.10) \quad \text{Let } \varphi \in C^2(\Omega),$$

(i) if $u^p - \varphi$ attains its local maximum at x_0 and $m^p u(x_0) < u^p(x_0)$, then

$$-a_{ij}^p \varphi_{ij} + b_i^p \varphi_i + c^p u^p \leq f^p \quad \text{at } x_0,$$

(ii) if $u^p - \varphi$ attains its local minimum at x_0 and $u^p(x_0) < M^p u(x_0)$, then

$$-a_{ij}^p \varphi_{ij} + b_i^p \varphi_i + c^p u^p \geq f^p \quad \text{at } x_0.$$

§3. Approximate systems and a priori estimates

To prove the result we use the penalty method. Let $\beta \in C^\infty(\mathbf{R})$ be a function such that

$$(3.1) \quad \begin{aligned} \beta(t) &= 0 \quad \text{if } t \leq 0, \quad \beta(t) = t - 1 \quad \text{if } t \geq 2 \\ \beta'(t) &\geq 0, \quad \beta''(t) \geq 0 \quad \text{on } \mathbf{R}. \end{aligned}$$

For each $\varepsilon > 0$, we put $\beta_\varepsilon(t) = \beta(t/\varepsilon)$.

We consider the following approximate systems:

$$(3.2) \quad \begin{aligned} A^p u_\varepsilon^p + \sum_{\substack{q \in P \\ q \neq p}} \beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^q - K(p, q)) - \sum_{\substack{q \in P \\ q \neq p}} \beta_\varepsilon(u_\varepsilon^q - k(p, q) - u_\varepsilon^p) &= f^p \quad \text{in } \Omega \\ u_\varepsilon^p|_{\partial\Omega} &= 0, \quad p \in P. \end{aligned}$$

To prove the solvability of (3.2) we apply the Schauder fixed point theorem. For this purpose, we choose the Banach space $E = (C^\delta(\Omega))^m$ ($0 < \delta < 1$) and the convex set

$$K = \{v = (v^1, v^2, \dots, v^m) \mid \|v^i\|_{C^\delta(\Omega)} \leq C, v^i|_{\partial\Omega} = 0, 1 \leq i \leq m\}.$$

Here C is a suitable constant to be selected later. For $v \in E$ define $u = Tv$ as a solution of the system of equations

$$(3.3) \quad \begin{aligned} A^p u^p + \sum_{\substack{q \in P \\ q \neq p}} \beta_\varepsilon(u^p - v^q - K(p, q)) - \sum_{\substack{q \in P \\ q \neq p}} \beta_\varepsilon(v^q - k(p, q) - u^p) &= f^p \quad \text{in } \Omega \\ u^p|_{\partial\Omega} &= 0, \quad p \in P. \end{aligned}$$

By the same argument as in [5, Lemma 3.1] and Lemma 3.1 below we can choose the constant C in such a way that T maps K into K . It follows from well-

known linear elliptic theory that the other conditions in the fixed point theorem are satisfied.

In the following of this section we shall state some a priori estimates on u_ε^p . Since the proofs of these estimates are almost same as in [5], we only mention the outline of their proofs.

First by the same argument as in [5, Lemmas 3.1 and 4.2], we have the following lemma.

Lemma 3.1. *We have*

$$\|u_\varepsilon^p\|_{C(\Omega)} + \|u_\varepsilon^p\|_{W^{1,\infty}(\partial\Omega)} \leq \text{Const.}$$

The next lemma states an interior derivative estimate.

Lemma 3.2. *We have*

$$(3.4) \quad \|u_\varepsilon^p\|_{W^{1,\infty}(\Omega)} \leq \text{Const.}$$

Proof. As in the proof of [5, Lemma 4.3], consider an auxiliary function

$$z_\varepsilon^p(x) = |\text{grad } u_\varepsilon^p|^2.$$

Choose $p_0 \in P$, $x_0 \in \bar{\Omega}$ such that

$$z_\varepsilon^{p_0}(x_0) = \max_{p,x} z_\varepsilon^p(x).$$

It is sufficient to consider the case $x_0 \in \Omega$. In the following of the proof we omit sub and super scripts p_0 , ε and denote u_{x_i} by u_i and etc. By the maximum principle, we get

$$(3.5) \quad \begin{aligned} Az &\leq -2\theta u_{ij}u_{ij} + 2f_i u_i + a_{k\ell,i} u_{k\ell} u_i - 2b_{k,i} u_k u_i \\ &- 2c_i u_i u_i - 2 \sum_{\substack{q \in P \\ q \neq p_0}} \beta'(u - u^q - K(p_0, q))(u_i - u_i^q) u_i \\ &+ 2 \sum_{\substack{q \in P \\ q \neq p_0}} \beta'(u^q - k(p_0, q) - u)(u_i^q - u_i) u_i - c_0 u_i u_i. \end{aligned}$$

Noting that

$$\begin{aligned} &-2\beta'(u - u^q - K(p_0, q))(u_i - u_i^q) u_i \\ &\leq \beta'(u - u^q - K(p_0, q))(u_i^q u_i^q - u_i u_i) \\ &\leq \beta'(u - u^q - K(p_0, q))(z_\varepsilon^q - z_\varepsilon^{p_0}) \leq 0 \end{aligned}$$

and

$$\begin{aligned} &2\beta'(u^q - k(p_0, q) - u)(u_i^q - u_i) u_i \\ &\leq \beta'(u^q - k(p_0, q) - u)(z_\varepsilon^q - z_\varepsilon^{p_0}) \leq 0, \end{aligned}$$

for each $q \in P$, $q \neq p_0$, we get

$$Az \leq -\theta u_{ij}u_{ij} - \frac{c_0}{2} u_i u_i + M u_i u_i + \text{Const.},$$

where M is a constant appearing from the third and fourth terms in (3.5) which depends only on θ and a_{ij}^p , b_i^p and their derivatives.

Hence if c_0 is large enough, say $c_0 > 4M$, we get

$$Az \leq -\frac{c_0}{4} u_i u_i + M.$$

On the other hand, since $z_\varepsilon^{p_0}$ attains its maximum at x_0 , we have by the maximum principle

$$0 \leq A^{p_0} z_\varepsilon^{p_0}(x_0) \leq -\frac{c_0}{4} z_\varepsilon^{p_0}(x_0) + M$$

which completes the proof.

Lemma 3.3. *There exists $u^p \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$, $p \in P$, such that $m^p u(x) \leq u^p(x) \leq M^p u(x)$ in Ω .*

Proof. By a priori estimates obtained in Lemmas 3.1 and 3.2 we can find subsequences (denoted again by u_ε^p) of u_ε^p and $u^p \in W^{1,\infty}(\Omega)$, $p \in P$, such that for every $p \in P$, u_ε^p converges to u^p uniformly in Ω .

Since, by the same argument as in [5, Lemma 5.1], we can prove that

$$\begin{aligned} & \langle \beta'_\varepsilon(u_\varepsilon^p - u_\varepsilon^q - K(p, q)), u_\varepsilon^p - u_\varepsilon^q - K(p, q) \rangle \\ & + \langle \beta'_\varepsilon(u_\varepsilon^q - k(p, q) - u_\varepsilon^p), u_\varepsilon^q - k(p, q) - u_\varepsilon^p \rangle \end{aligned}$$

is bounded for all $q \in P$, $q \neq p$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$, it follows that

$$m^p u(x) \leq u^p(x) \leq M^p u(x) \quad \text{in } \Omega.$$

§4. Proof of the main theorem

In this section we shall prove that the set of limit functions $u = \{u^1, \dots, u^m\}$ which was obtained in Lemma 3.3 is the unique viscosity solution of the problem (2.8).

Lemma 4.1. *The set of functions $u = \{u^1, \dots, u^m\}$ which was obtained in Lemma 3.3 is a viscosity solution of (2.8).*

Proof. From Lemma 3.3 it is obvious that u satisfies (2.9). We shall prove

(2.10). Let $\varphi \in C^2(\Omega)$ and assume that $u^p - \varphi$ takes its local maximum at $x_0 \in \Omega$ and $m^p u(x_0) < u^p(x_0)$. Without loss of generality we may assume that x_0 is a strict maximum point of $u^p - \varphi$ in Ω .

Since u_ε^p converges to u^p uniformly, there exists a sequence $\{x_\varepsilon^p\} \subset \Omega$ such that

$$(4.1) \quad \begin{aligned} & (i) \quad \lim_{\varepsilon \rightarrow 0} x_\varepsilon^p = x_0, \\ & (ii) \quad u_\varepsilon^p - \varphi \text{ attains its local maximum at } x_\varepsilon^p, \\ & (iii) \quad u_\varepsilon^p(x_\varepsilon^p) \geq u_\varepsilon^q(x_\varepsilon^p) - k(p, q), \quad q \in P. \end{aligned}$$

Note that (4.1)–(iii) implies that

$$(4.2) \quad \beta_\varepsilon(u_\varepsilon^q - k(p, q) - u_\varepsilon^p) = 0 \quad \text{at } x_\varepsilon^p$$

and

$$(4.3) \quad \beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^q - K(p, q)) = 0 \quad \text{in } \Omega$$

for each $q \in P$, $q \neq p$. Substituting (4.2) and (4.3) into (3.2), we get

$$(4.4) \quad -a_{ij}^p u_{\varepsilon,i,j}^p + b_i^p u_{\varepsilon,i}^p + c^p u_\varepsilon^p \leq f^p \quad \text{at } x_\varepsilon^p.$$

On the other hand, since (4.1)–(ii) implies

$$a_{ij}^p (u_\varepsilon^p - \varphi)_{ij} \leq 0, \quad (u_\varepsilon^p - \varphi)_i = 0 \quad \text{at } x_\varepsilon^p,$$

we have

$$(4.5) \quad -a_{ij}^p \varphi_{\varepsilon,i,j}^p + b_i^p \varphi_{\varepsilon,i}^p + c^p u_\varepsilon^p \leq f^p \quad \text{at } x_\varepsilon^p.$$

Passing to the limit as $\varepsilon \rightarrow 0$, we can conclude (2.10–i). The opposite inequality (2.10–ii) is obtained by the same argument.

To prove the uniqueness of viscosity solutions, we choose a viscosity solution $u = \{u^1, \dots, u^m\}$ which was obtained in Lemma 4.1 and fix it.

Lemma 4.2. *Let $v = \{v^1, \dots, v^m\}$ be any viscosity solution of (2.8). Then we have $u = v$.*

Proof. We shall prove $v^p \leq u^p$ for all $p \in P$. If not, we can find a convergent sequence $\{x_\varepsilon\}$ from the family of points of $v^p - u_\varepsilon^p$ in $\bar{\Omega}$, which satisfies the following conditions:

There exist $p_0 \in P$, $x_0 \in \Omega$ such that

$$(4.6) \quad \begin{aligned} & (i) \quad \lim_{\varepsilon \rightarrow 0} x_\varepsilon = x_0, \\ & (ii) \quad (v^{p_0} - u^{p_0})(x_0) = \max_{x \in \Omega, p \in P} (v^p - u^p)(x) > 0, \end{aligned}$$

$$(iii) \quad (v^{p_0} - u_\varepsilon^{p_0})(x_\varepsilon) = \max_{x \in \Omega} (v^{p_0} - u_\varepsilon^{p_0})(x).$$

First we shall claim that

$$(4.7) \quad u^{p_0}(x_0) = M^{p_0}u(x_0) \quad \text{or} \quad v^{p_0}(x_0) = m^{p_0}v(x_0).$$

Assume the contrary. Then the convergence and continuity of $u_\varepsilon^{p_0}$, we have

$$(4.8) \quad u_\varepsilon^{p_0}(x_\varepsilon) \leq u_\varepsilon^q(x_\varepsilon) + K(p_0, q), \quad q \in P.$$

Substituting this into (3.2) we get

$$(4.9) \quad -a_{ij}^{p_0}u_{\varepsilon,ij}^{p_0} + b_i^{p_0}u_{\varepsilon,i}^{p_0} + c^{p_0}u_\varepsilon^{p_0} \geq f^{p_0} \quad \text{at } x_\varepsilon.$$

On the other hand, since v is a viscosity solution, (4.6-iii) implies

$$(4.10) \quad -a_{ij}^{p_0}u_{\varepsilon,ij}^{p_0} + b_i^{p_0}u_{\varepsilon,i}^{p_0} + c^{p_0}v^{p_0} \leq f^{p_0} \quad \text{at } x_\varepsilon.$$

Combining (4.9) and (4.10) and letting $\varepsilon \rightarrow 0$, we have $c^{p_0}(u^{p_0} - v^{p_0})(x_0) \geq 0$, which contradicts (4.6-ii). Hence we have (4.7).

Therefore there exists $p_1 \in P$ such that

$$(4.7)' \quad \begin{aligned} u^{p_0}(x_0) &= u^{p_1}(x_0) + K(p_0, p_1) \\ \text{or} \\ v^{p_0}(x_0) &= v^{p_1}(x_0) - k(p_0, p_1). \end{aligned}$$

On the other hand, from (4.6-ii), we get

$$v^{p_0}(x_0) - u^{p_0}(x_0) \geq v^{p_1}(x_0) - u^{p_1}(x_0).$$

Since this implies

$$\begin{aligned} 0 &\geq v^{p_0}(x_0) - v^{p_1}(x_0) - K(p_0, p_1) \\ &\geq u^{p_0}(x_0) - u^{p_1}(x_0) - K(p_0, p_1), \end{aligned}$$

we have $v^{p_0} = v^{p_1} + K(p_0, p_1)$ at x_0 if $u^{p_0} = u^{p_1} + K(p_0, p_1)$. Therefore we can conclude that

$$(4.11) \quad \begin{aligned} v^{p_0}(x_0) &= M^{p_0}v(x_0) = v^{p_1}(x_0) + K(p_0, p_1) \\ \text{or} \\ v^{p_0}(x_0) &= m^{p_0}v(x_0) = v^{p_1}(x_0) - k(p_0, p_1). \end{aligned}$$

Moreover, we have

$$(4.12) \quad (v^{p_0} - u^{p_0})(x_0) = (v^{p_1} - u^{p_1})(x_0) = \max_{x, p} (v^p - u^p)(x).$$

We can repeat this argument starting from the parameter p_1 and can find $p_2 \in P$ such that (4.11) and (4.12) hold. Continuing this procedure, we can find a repeated parameter, say p_0 , and a chain $\{p_0, p_1, \dots, p_n, p_0\} \subset P$ which satisfies (4.11) and (4.12) for each p_i and p_{i+1} . Summing up (4.11) corresponding to this chain, we have

$$\sum_{i=1}^n \kappa(p_i, p_{i+1}) = 0$$

where $\kappa(p_i, p_{i+1}) = K(p_i, p_{i+1})$ or $-k(p_i, p_{i+1})$.

Since this contradicts the assumption (2.7), we have $v^p \leq u^p$ for all $p \in P$.

The opposite inequality $u^p \leq v^p$, $p \in P$, is obtained by the same argument.

These two lemmas complete the proof of Theorem 2.1.

References

- [1] Crandall, M. G. and Lions, P. L., Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, **277** (1983), 1–42.
- [2] Evans, L. C. and Friedman, A., Optimal stochastic switching and the Dirichlet problem for the Bellman equation, *Trans. Amer. Math. Soc.*, **253** (1979), 365–389.
- [3] Lenhart, S. M. and Belbas, S. A., A system of nonlinear partial differential equations arising in the optimal control of stochastic system with switching costs, *SIAM J. Appl. Math.*, **43** (1983), 465–475.
- [4] Lions, P. L., Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part II: Viscosity solutions and uniqueness, *Comm. Partial Differential Equations*, **8** (1983), 1229–1276.
- [5] Yamada, N., A system of elliptic variational inequalities associated with a stochastic switching game, *Hiroshima Math. J.*, **13** (1983), 109–132.
- [6] ———, The Hamilton-Jacobi-Bellman equation with a gradient constraint, to appear in *J. Differential Equations*.

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