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Oscillation of Even Order Nonlinear Functional Differential Equations with Deviating Arguments

By

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1. Introduction

We consider the functional differential equation

$$L_n x(t) + q(t) f(x[g(t)]) = 0, \qquad n \text{ is even}, \tag{1}$$

where

$$L_0 x(t) = x(t),\ L_k x(t) = \frac{1}{a_k(t)}(L_{k-1}x(t))', \qquad 1 \le k \le n, \qquad \left(' = \frac{d}{dt}\right),$$

$a_n = 1$, $a_i: [t_0, \infty) \to (0, \infty)$, $i = 1, 2, \ldots, n-1$, $q, g: [t_0, \infty) \to \boldsymbol{R}$, $f: \boldsymbol{R} \to \boldsymbol{R}$ are continuous, $q(t) \ge 0$ and not identically zero on any ray of the form $[t^*, \infty)$ for some $t^* \ge t_0$, and $\lim_{t\to\infty} g(t) = \infty$.

The following conditions are assumed to hold throughout this paper.

$$\int^{\infty} a_i(s)ds = \infty, \qquad i = 1, 2, \ldots, n-1. \tag{2}$$

There exists a differentiable function $\sigma: [t_0, \infty) \to \boldsymbol{R}$ such that

$$\sigma(t) \le \min\{t, g(t)\}, \quad \dot{\sigma}(t) > 0 \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) = \infty. \tag{3}$$

The domain of L_n, $D(L_n)$ is defined to be the set of all functions $x: [t_0, \infty) \to \boldsymbol{R}$ such that $L_j x(t)$, $j = 0, 1, \ldots, n$ exist and continuous on $[t_0, \infty)$. Our attention is restricted on those solutions $x \in D(L_n)$ of equation (1) which satisfy $\sup\{|x(t)|: t \ge T\} > 0$ for every $T \ge t_0$. A solution of equation (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called non-oscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

In this paper we establish a new oscillation criterion for equation (1). The theorems that we obtain will generalize and improve some of the results in [1]–[4].

2. Main results

To formulate our results we shall use the following shorthand notation. For any continuous functions p_i: $[t_0, \infty) \to \boldsymbol{R}$, $i = 1, 2, \ldots$, we define

$$I_0 = 1$$

$$I_i(t, s; p_i, \ldots, p_1) = \int_s^t p_i(u) I_{i-1}(u, s; p_{i-1}, \ldots, p_1) du, \qquad i = 1, 2, \ldots .$$

It is easy to verify that for $i = 1, 2, \ldots, n-1$,

$$I_i(t, s; p_1, \ldots, p_i) = (-1)^i I_i(s, t; p_i, \ldots, p_1)$$

and

$$I_i(t, s; p_1, \ldots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \ldots, p_{i-1}) du .$$

The following two lemmas will be needed in the proofs of our results.

Lemma 1. *If $x \in D(L_n)$, then for $t, s \in [t_0, \infty)$ and $0 \le i < k \le n$*

(i) $$L_i x(t) = \sum_{j=i}^{k-1} I_{j-i}(t, s; a_{i+1}, \ldots, a_j) L_j x(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \ldots, a_{k-1}) a_k(u) L_k x(u) du.$$

(ii) $$L_i x(t) = \sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \ldots, a_{i+1}) L_j x(s) + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-1}, \ldots, a_{i+1}) a_k(u) L_k x(u) du.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

Lemma 2. *Suppose condition* (2) *holds. If $x \in D(L_n)$ is of constant sign and not identically zero for all large t, then there exist a $t_x \ge t_0$ and an integer l, $0 \le l \le n$ with $n + l$ even for $x(t)L_n x(t)$ nonnegative or $n + l$ odd for $x(t)L_n x(t)$ nonpositive and such that for every $t \ge t_x$*

$$l > 0 \quad \textit{implies} \quad x(t)L_k x(t) > 0, \qquad (k = 0, 1, \ldots, l)$$

and

$$l \le n - 1 \textit{ implies } (-1)^{l-k} x(t) L_k x(t) > 0, \qquad (k = l, l+1, \ldots, n) .$$

This formula generalizes a well-known lemma of Kiguradze and can be proved similarly.

It will be convenient to make use of the following notation in the remainder of this paper. For any $T \geq t_0$ and all $t \geq T$ we let

$$w_1[t, T] = \min \left\{ a_1(t) \int_T^t I_{n-l-1}(t, s; a_{n-1}, \ldots, a_{l+1}) a_l(s) \right.$$

$$\left. \times I_{l-2}(t, s; a_2, \ldots, a_{l-1}) ds : l = 3, 5, \ldots, n-3 \right\} \text{ if } n > 2$$

$$= a_1(t) \text{ if } n = 2 ;$$

$$w_2[t, s] = a_1(s) I_{n-2}(t, s; a_{n-1}, \ldots, a_2) ;$$

$$w_3[t, T] = a_1(t) I_{n-2}(t, T; a_2, \ldots, a_{n-1}) ;$$

$$w[t, T] = \min \left\{ w_1\left[\frac{t}{2}, T\right]; w_2\left[t, \frac{t}{2}\right]; w_3\left[\frac{t}{2}, T\right] \right\}, \; t \geq 2T.$$

Theorem 1. *Let conditions* (2) *and* (3) *hold, and*

$$\frac{f(x)}{x} \geq k > 0 \qquad \text{for } x \neq 0 . \tag{4}$$

Furthermore, we assume that there exists a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that

$$\limsup_{t \to \infty} \int_{T_1}^t \left[\rho(s) q(s) - \frac{[\dot{\rho}(s)]^2}{2k\rho(s)\dot{\sigma}(s) w[\sigma(s), T]} \right] ds = \infty , \tag{5}$$

for all large T *with* $\sigma(T_1) > 2T$ *for some* $T_1 \geq T \geq t_0$. *Then equation* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0$, since the substitution $u = -x$ transforms equation (1) into an equation of the same form subject to the assumptions of the theorem.

By Lemma 2, there exists a $t_1 \geq t_0$ and $l \in \{1, 3, \ldots, n-1\}$ such that

$$\begin{cases} L_i x(t) > 0 & \text{on } [t_1, \infty) \text{ for } i = 0, 1, \ldots, l \\ (-1)^{i-l} L_i x(t) > 0 & \text{on } [t_1, \infty) \text{ for } i = l, l+1, \ldots, n . \end{cases} \tag{6}$$

From (6) we observe that

$$\dot{x}(t) > 0 \quad \text{and} \quad L_{n-1} x(t) > 0 \quad \text{for all } t \geq t_1 . \tag{7}$$

Next, we claim that

$$\dot{x}\left[\frac{t}{2}\right] \geq w[t, T]L_{n-1}x(t) \qquad \text{for all large } T \text{ and } \frac{t}{2} \geq T \geq t_1 . \tag{8}$$

Assume first that $l \in \{3, 5, \ldots, n-3\}$. From Lemma 1(ii) we obtain

$$L_l x(s) = \sum_{j=l}^{n-2} (-1)^{j-l} I_{j-l}(t, x; a_j, \ldots, a_{l+1}) L_j x(t)$$
$$+ (-1)^{n-l-1} \int_s^t I_{n-l-2}(u, s; a_{n-2}, \ldots, a_{l+1}) a_{n-1}(u) L_{n-1}x(u)du .$$

Using the facts that $L_{n-1}x(t)$ is non-increasing on $[t_1, \infty)$ and $n-l$ is odd, we have

$$L_l x(s) \geq \int_s^t a_{n-1}(u) I_{n-l-2}(u, s; a_{n-2}, \ldots, a_{l+1})du \cdot L_{n-1}x(t)$$
$$= I_{n-l-1}(t, s; a_{n-1}, \ldots, a_{l+1}) L_{n-1}x(t) \text{ for } t \geq s \geq t_1 .$$

On the other hand, from Lemma 1 (i), we get

$$\dot{x}(t) = a_1(t)\left\{\sum_{j=1}^{l-1} I_j(t, t_1; a_2, \ldots, a_j) L_j x(t_1) + \int_{t_1}^t I_{l-2}(t, s; a_2, \ldots, a_{l-1}) a_l(s) L_l x(s) ds\right\}$$
$$\geq a_1(t) \int_{t_1}^t I_{l-2}(t, s; a_2, \ldots, a_{l-1}) a_l(s) L_l x(s) ds, \; t \geq t_1 .$$

Combining the above inequalities, we obtain

$$\dot{x}(t) \geq a_1(t) \int_{t_1}^t I_{l-2}(t, s; a_2, \ldots, a_{l-1}) a_l(s) I_{n-l-1}(t, s; a_{n-1}, \ldots, a_{l+1}) ds \cdot L_{n-1}x(t) ,$$

or

$$\dot{x}(t) \geq w_1[t, t_1] L_{n-1}x(t) \qquad \text{for } t \geq t_1 . \tag{9}$$

Next, we assume $l = 1$. From Lemma 1 (ii) we obtain

$$\dot{x}(s) = a_1(s)\left\{\sum_{j=1}^{n-2} (-1)^{j-1} I_{j-1}(t, s; a_j, \ldots, a_2) L_j x(t)\right.$$
$$\left. + (-1)^{n-2} \int_s^t I_{n-3}(u, s; a_{n-2}, \ldots, a_2) a_{n-1}(u) L_{n-1}x(u)du\right\} .$$

Since $L_{n-1}x$ is non-increasing on $[t_1, \infty)$ and n is even, we have

$$\dot{x}(s) \geq a_1(s) \int_s^t I_{n-3}(u, s; a_{n-2}, \ldots, a_2) a_{n-1}(u) du \cdot L_{n-1} x(t)$$

$$= a_1(s) I_{n-2}(t, s; a_{n-1}, \ldots, a_2) L_{n-1} x(t)$$

or

$$\dot{x}(s) \geq w_2[t, s] L_{n-1} x(t) \qquad \text{for } t \geq s \geq t_1 . \tag{10}$$

Finally, we let $l = n - 1$. From Lemma 1 (i) we get

$$\dot{x}(s) = a_1(s) \left\{ \sum_{j=1}^{n-2} I_{j-1}(s, t_1; a_2, \ldots, a_j) L_j x(t_1) \right.$$

$$\left. + \int_{t_1}^{s} I_{n-3}(s, u; a_2, \ldots, a_{n-2}) a_{n-1}(u) L_{n-1} x(u) du \right\} .$$

Thus,

$$\dot{x}(s) \geq a_1(s) I_{n-2}(s, t_1; a_2, \ldots, a_{n-1}) L_{n-1} x(s) ,$$

or

$$\dot{x}(s) \geq w_3[s, t_1] L_{n-1} x(s) \qquad \text{for } s \geq t_1 . \tag{11}$$

Combining (9), (10) and (11) yields (8) as claimed.

Choose a $T_1 \geq T \geq t_1$ so that $\sigma(t) > 2T$ for all $t \geq T_1$ and

$$\dot{x}\left[\frac{\sigma(t)}{2}\right] \geq w[\sigma(t), T] L_{n-1} x(t) \qquad \text{for every } t \geq T_1 . \tag{12}$$

Define

$$W(t) = \frac{\rho(t) L_{n-1} x(t)}{k x\left[\dfrac{\sigma(t)}{2}\right]} \qquad \text{for } t \geq T_1 .$$

Thus,

$$\dot{W}(t) \leq -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} W(t) - \frac{1}{2} \dot{\sigma}(t) \frac{\dot{x}\left[\dfrac{1}{2}\sigma(t)\right]}{x\left[\dfrac{\sigma(t)}{2}\right]} W(t) . \tag{13}$$

Using (4) and (12) in (13) we have

$$\dot{W}(t) \leq -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} W(t) - \frac{k}{2} \frac{\dot{\sigma}(t)}{\rho(t)} w[\sigma(t), T] W^2(t) \qquad \text{for } t \geq T_1 .$$

$$= -\rho(t)q(t) + \frac{[\dot{\rho}(t)]^2}{2k\dot{\sigma}(t)\rho(t)w[\sigma(t), T]}$$

$$- \left[\sqrt{\frac{1}{2}k\frac{\dot{\sigma}(t)}{\rho(t)}w[\sigma(t), T]}\,W(t) - \frac{\dfrac{\dot{\rho}(t)}{\rho(t)}}{2\sqrt{\dfrac{1}{2}k\dfrac{\dot{\sigma}(t)}{\rho(t)}w[\sigma(t), T]}}\right]^2$$

$$\leq -\rho(t)q(t) + \frac{[\dot{\rho}(t)]^2}{2k\rho(t)\dot{\sigma}(t)w[\sigma(t), T]} \qquad \text{for all } t \geq T_1 .$$

Integrating the above inequality from T_1 to t we obtain

$$\int_{T_1}^{t}\left[\rho(s)q(s) - \frac{[\dot{\rho}(s)]^2}{2k\rho(s)\dot{\sigma}(s)w[\sigma(s), T]}\right]ds \leq W(T_1) - W(t)$$

$$\leq W(T_1) < \infty ,$$

a contradiction to (5). This completes the proof.

Remark 1. In Theorem 1, the function f is not required to be monotonic. In case when f is a monotonic function and it satisfies the condition

$$xf(x) > 0 \quad \text{and} \quad f'(x) \geq \gamma > 0 \quad \text{for } x \neq 0, \quad \left(' = \frac{d}{dx}\right), \tag{14}$$

we observe that (14) is stronger than (4):

The condition (14) implies, in the case $x > 0$

$$f(x) - f(\varepsilon) = \int_{\varepsilon}^{x} f'(u)du \geq \gamma(x - \varepsilon), \qquad 0 < \varepsilon < x , \tag{15}$$

whence, letting $\varepsilon \to 0$ in (15), we have $f(x) \geq \gamma x$ $(x > 0)$.

Similarly, we have $f(x) \leq \gamma x$ $(x < 0)$.

Consequently, if we put $\gamma = k$, we get condition (4).

Now, the following corollaries are immediate consequence of Theorem 1.

Corollary 1. *Let conditions* (2), (3) *and* (14) *hold. Moreover, we assume that there exists a differentiable function*

$$\rho: [t_0, \infty) \to (0, \infty)$$

such that condition (5) *holds with* γ *replaces* k. *Then equation* (1) *is oscillatory.*

Corollary 2. *Let condition* (5) *of Theorem* 1 *be replaced by*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \rho(s)q(s)ds = \infty \tag{16}$$

and

$$\lim_{t\to\infty} \int_{T_1}^{t} \frac{[\dot{\rho}(s)]^2}{\dot{\sigma}(s)\rho(s)w[\sigma(s), T]} ds < \infty\,, \tag{17}$$

for all large T *with* $\sigma(T_1) > 2T$ *for* $T_1 \geq T \geq t_0$. *Then equation* (1) *is oscillatory.*

Remark 2. The results of this paper are of high degree of generality. Theorem 1 extends Theorem 2 in [1], Theorem 2 in [2] and Theorem 3 in [1]. Moreover, Theorem 1 improves and unifies Theorems 1 and 2 in [3] and Theorem 5.1(i) in [4]. To illustrate our results we give the following example.

Example 1. Consider the differential equation

$$L_n x(t) + ct^{1-2n} f(x[g(t)]) = 0,\ n \text{ is even, and } t > 0\,, \tag{18}$$

where

$$L_0 x(t) = x(t),\ L_k x(t) = \frac{1}{t}(L_{k-1}x(t))^{\cdot}\,,\ k = 1, 2, \ldots, n,\ a_n = 1$$

and $a_i(t) = t$, $i = 1, 2, \ldots, n-1$, the functions f and g are as defined in equation (1) and $c > 0$.

Let the function f be any one of the following:

(I) $f(x) = x$, $x \in R$.

(II) $f(x) = x + |x|^{\lambda} \operatorname{sgn} x$, $x \in \boldsymbol{R}$ for $\lambda > 0$ and $\lambda \neq 1$.

(III) $f(x) = \sinh x$, $x \in \boldsymbol{R}$.

(IV) $f(x) = ex \exp \sin x$, $x \in \boldsymbol{R}$.

Also, we let the function g to be one of the following:

(i) $g(t) = t^{\alpha}$, $t \geq t_0$ and $\alpha \geq 1$.

(ii) $g(t) = t + \tau \sin t$, $t \geq t_0$ and $\tau > 0$.

(iii) $g(t) = t - \tau$, $t \geq t_0$ and $\tau > 0$.

Here we take $\sigma(t) = t$ in case (i) and $\sigma(t) = t - \tau$ in case (ii) or case (iii). Now, for $t \geq T$ we have

$$w_1[t, T] = \min_{l} \left\{ \frac{t(t^2 - T^2)^{n-2}}{[2^{n-2}(n-2)](n-l-1)!(l-2)!} : l = 3, 5, \ldots, n-3 \right\};$$

$$w_2[t, s] = \frac{s(t^2 - s^2)^{n-2}}{2^{n-2}(n-2)!}$$

and $w_3[t, T] = \dfrac{t(t^2 - T^2)^{n-2}}{2^{n-2}(n-2)!}$. We let $T = 1$ and $T_1 = 3$. Then

$$w[t, 1] = \frac{t^{2n-3}}{2^{3n-5}(n-2)!}\left(1 - \frac{4}{t^2}\right)^{n-2}, \qquad t \geq 3 .$$

All the hypotheses of Theorem 1 are satisfied with $\rho(t) = t^{2n-1}$ and $c > (2n-1)^2 2^{3n-6}(n-2)!$ and hence equation (18) is oscillatory.

Remark 2. Theorems 1 and 3 in [3] fail to apply to equation (18) because of presence of the restrictive condition on the functions $a_i(t)$, $i = 2, \ldots, n$, namely, condition (2) in [3]. Also, Theorem 5.1 (i) in [4] is not applicable to equation (18) since condition (4.12; l) in [4] is violated.

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