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Gopalsamy, K.

Győri, I.

Ladas, G.

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Oscillations of a Class of Delay Equations with Continuous and Piecewise Constant Arguments

By

K. GOPALSAMY[1], I. GYÖRI[2] and G. LADAS

(The University of Rhode Island, U. S. A.)

1. Introduction and Preliminaries

A primary purpose of this article is to study the oscillatory behavior of solutions of the delay-logistic equation

$$\dot{N}(t) = N(t)\{a - bN(t-\tau) - cN([t-k])\} \tag{1.1}$$

about its positive steady state where b, c, $\tau \in \boldsymbol{R}^+$ with $b + c \neq 0$, $a \in \boldsymbol{R}^+$ and $k \in \boldsymbol{N}$ where $\boldsymbol{N} = \{0, 1, \ldots\}$ and $[\cdot]$ denotes the greatest integer function. The initial conditions for (1.1) are assumed to be as follows:

$$N(s) = \varphi(s)\,, \quad s \in [-\tau, 0]\,, \quad \varphi(0) > 0\,, \quad \varphi \in C([-\tau, 0], \boldsymbol{R}^+) \tag{1.2}$$

$$N(-j) = N_{-j} \geqslant 0 \quad \text{and} \quad N_{-j} = \varphi(-j) \quad \text{if } -j \in [-\tau, 0]\,,\ j = 0, 1, 2, 3, \ldots, k\,.$$

If $b = 0$ and $k = 0$, then it is possible to show by integrating from n to $n+1$ that (1.1) leads to a difference equation of the form

$$N(n+1) = N(n) \exp\{a - cN(n)\}\,, \quad n \in \boldsymbol{N} \tag{1.3}$$

which has been known to have quite complex ("chaotic") behavior for certain values of the parameters a and c (see [8] and [9]). In this article we are interested in the oscillatory nature of (1.1) when both the continuous argument $t - \tau$ and the piecewise constant argument $[t - k]$ are present.

We let

$$N(t) = N^* e^{u(t)} \quad \text{and} \quad N^* = a/(b + c) \tag{1.4}$$

in (1.1) and obtain

$$\dot{u}(t) + bN^*[e^{u(t-\tau)} - 1] + cN^*[e^{u([t-k])} - 1] = 0\,. \tag{1.5}$$

Eq. (1.5) linearized about its trivial solution becomes

$$\dot{y}(t) + bN^* y(t-\tau) + cN^* y([t-k]) = 0\,. \tag{1.6}$$

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[1] On leave from Flinders University, Bedford Park, S.A. 5042, Australia.

[2] On leave from the Computing Centre of the Szeged University of Medicine, 6720 Szeged, Pecsi u. 4/a, Hungary.

We shall first consider in detail the study of the linear equation

$$\dot{x}(t) + px(t-\tau) + qx([t-k]) = 0 \tag{1.7}$$

where

$$p, \tau, q \in \boldsymbol{R}^+ \quad \text{and} \quad k \in \boldsymbol{N}. \tag{1.8}$$

In Section 3 below we will consider a broad class of nonlinear equations of which (1.5) will be a special case. Throughout this paper we will assume without further mention that (1.8) is satisfied.

By a *solution* of Eq. (1.7) we mean a function x which is defined on the set

$$\{-k, \ldots, -1, 0\} \cup [-\tau, \infty)$$

and which satisfies the following properties:

(i) x is continuous on $[-\tau, \infty)$.

(ii) the derivative $\dot{x}(t)$ exists at each point $t \in (0, \infty)$ with the possible exception of the points $t \in \boldsymbol{N}$ where one-sided derivatives exist.

(iii) Eq. (1.7) is satisfied on each interval $[n, n+1]$ for $n \in \boldsymbol{N}$.

Let $\varphi \in C([-\tau, 0], \boldsymbol{R})$ and $A_{-k}, \ldots, A_{-1}, A_0$ be given and such that

$$A_{-j} = \varphi(-j) \quad \text{for} \quad j \leqslant \tau \quad \text{and} \quad j = 0, 1, 2, \ldots, k.$$

Then one can show that Eq. (1.7) has a unique solution x satisfying the initial conditions

$$x(t) = \varphi(t) \quad \text{for} \quad -\tau \leqslant t \leqslant 0$$

and

$$x(-j) = A_{-j} \quad \text{for} \quad j = 0, 1, 2, \ldots, k.$$

As is customary, a solution of Eq. (1.7) is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*. For the case of difference equations, we will say that a solution $\{A_n\}$ is *oscillatory* if the terms A_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution $\{A_n\}$ is called *nonoscillatory*.

It is well known, see for example [6], that when $q = 0$, a necessary and sufficient condition for the oscillation of all solutions of the delay equation

$$\dot{u}(t) + pu(t-\tau) = 0 \tag{1.9}$$

is that its characteristic equation

$$\lambda + pe^{-\lambda\tau} = 0 \tag{1.10}$$

has no real roots. On the other hand, when $p = 0$, it was shown in [5] that a

necessary and sufficient condition for the oscillation of all solutions of the equation with piecewise constant argument

$$\dot{v}(t) + qv[(t-k)] = 0 \tag{1.11}$$

is that its characteristic equation

$$\lambda - 1 + q\lambda^{-k} = 0 \tag{1.12}$$

has no positive roots in (0, 1].

An analysis of the characteristic equations (1.10) and (1.12) shows that the above results can be formulated in terms of the coefficients and the delays as follows:

Lemma 1. *Assume that* $p, \tau \in \boldsymbol{R}^+$. *Then every solution of Eq.* (1.9) *oscillates if and only if*

$$p\tau e > 1\,. \tag{1.13}$$

Lemma 2. *Assume that* $q \in \boldsymbol{R}^+$ *and* $k \in \boldsymbol{N}$. *Then every solution of Eq.* (1.11) *oscillates if and only if*

$$q > \frac{k^k}{(k+1)^{k+1}} \quad \text{if } k > 0 \tag{1.14}$$

and

$$q \geqslant 1 \quad \text{if } k = 0\,. \tag{1.15}$$

It follows from Lemmas 1 and 2 that when

$$p\tau e \leqslant 1 \quad \text{and} \quad q = 0 \tag{1.16}$$

or when

$$p = 0\,, \quad k > 0 \quad \text{and} \quad q \leqslant \frac{k^k}{(k+1)^{k+1}} \tag{1.17}$$

or when

$$p = 0\,, \quad k = 0 \quad \text{and} \quad q < 1\,, \tag{1.18}$$

Eq. (1.7) has nonoscillatory solutions.

Assume that $p\tau e \leqslant 1$ and set

$$F(\lambda) = \lambda + pe^{-\lambda\tau}\,.$$

If $\tau = 0$, then $\lambda_0 = -p \leqslant 0$ is a root of (1.10). If $\tau > 0$, then

$$F\left(-\frac{1}{\tau}\right)F(0) = \frac{p\tau e - 1}{\tau}p \leqslant 0$$

and so Eq. (1.10) has a root $\lambda_0 \in [-1/\tau, 0]$. If $p = 0$ then $\lambda_0 = 0$ while if $p\tau = 1/e$ then $\lambda_0 = (-1/\tau)$. Observe also that when $p\tau e \leqslant 1$,

$$F'(\lambda) = 1 - p\tau e^{-\lambda\tau} \geqslant 0 \quad \text{for} \quad -\frac{1}{\tau} < \lambda \leqslant 0 .$$

When $\tau = 0$, we will use the convention that $1/\tau$ stands for ∞. We summarize the above observations in the following lemma.

Lemma 3. *Assume that*

$$p\tau e \leqslant 1 . \tag{1.19}$$

Then Eq. (1.10) *has a unique root* $\lambda_0 \in [-1/\tau, 0]$.

We will also need the following lemmas:

Lemma 4([7]). *Assume that* $p, \tau \in \boldsymbol{R}^+$ *and suppose that the delay differential inequality*

$$\dot{u}(t) + pu(t - \tau) \leqslant 0 \tag{1.20}$$

has an eventually positive solution. Then the equation

$$\lambda + pe^{-\lambda\tau} = 0$$

has a real root.

Lemma 5([5]). *Assume that* $q \in \boldsymbol{R}^+$, $k \in \boldsymbol{N}$ *and suppose that the difference inequality*

$$B_{n+1} - B_n + qB_{n-k} \leqslant 0 \tag{1.21}$$

has an eventually positive solution. Then the equation

$$\lambda - 1 + q\lambda^{-k} = 0 \tag{1.22}$$

has a root in (0, 1].

The following lemma is interesting in its own right but its proof is straightforward and will be omitted.

Lemma 6. *Assume that* $q > 0$, $k \in \boldsymbol{N}$ *and either* $q \neq 1$ *or* $k \neq 0$. *Suppose also that the equation*

$$G(\lambda) \equiv \lambda - 1 + q\lambda^{-k} = 0$$

has no roots in (0, 1]. *Then there is an* $\varepsilon_0 \in (0, q)$ *such that for every* $\varepsilon \in [0, \varepsilon_0]$ *the equation*

$$\lambda - 1 + (q - \varepsilon)\lambda^{-k} = 0$$

has no roots in (0, 1].

The following lemma whose proof appears in [3] will be used in Section 3 below.

Lemma 7. *Assume that* $p, q > 0$, $\tau \geq 0$, $k \in N$ *and that for all sufficiently small* $\varepsilon > 0$, *the inequality*

$$\dot{x}(t) + (p - \varepsilon)x(t - \tau) + (q - \varepsilon)x([t - k]) \leqslant 0$$

has an eventually positive solution. Then the equation

$$\dot{z}(t) + pz(t - \tau) + qz([t - k]) = 0$$

has also an eventually positive solution.

To the best of our knowledge, this is the first paper dealing with the oscillation of delay equations with both continuous and piecewise constant arguments. For some recent literature on equations with piecewise constant arguments we refer the reader to [1], [2] and [5].

2. Sufficient Conditions for Oscillations

When $p\tau e \leqslant 1$, let λ_0 be as defined in Lemma 3 and set

$$A = \lambda_0(1 + \lambda_0\tau) \quad \text{and} \quad B = e^{-\lambda_0 k}(1 - e^{-\lambda_0}) . \tag{2.1}$$

The main result in this paper is the following:

Theorem 1. *Consider the Eq.* (1.7) *and assume that condition* (1.8) *is satisfied. Let* A *and* B *be as defined by* (2.1). *Then each of the following four conditions implies that every slution of Eq.* (1.7) *is oscillatory*:

(i) $p\tau e > 1$ *and* $q \geqslant 0$.

(ii) $p\tau e = 1$ *and* $q \geqslant 0$.

(iii) $p\tau e < 1$, $p > 0$, $q > 0$ *and the equation*

$$A(\lambda - 1) + Bq\lambda^{-k} = 0 \tag{2.2}$$

with $A \neq Bq$ *or* $k \neq 0$ *has no roots in* $(0, 1]$.

(iv) $$p \geqslant 0 \quad \text{and} \quad q > \frac{k^k}{(k + 1)^{k+1}} \quad \text{if } k > 0$$

or

$$p \geqslant 0 \quad \text{and} \quad q \geqslant 1 \quad \text{if } k = 0 .$$

Proof. Assume that (i) holds and that, for the sake of contradiction, Eq. (1.7) has an eventually positive solution $x(t)$. Then, Eq. (1.7) implies that

$$\dot{x}(t) + px(t - \tau) \leqslant 0 . \tag{2.3}$$

By Lemmas 4 and 1 this implies that $p\tau e \leqslant 1$ and this contradiction completes the proof in this case.

Next assume that (ii) and (iii) hold. Clearly, in these cases $-1/\tau \leqslant \lambda_0 < 0$ where λ_0 is as defined in Lemma 3. Assume for the sake of contradiction, that Eq. (1.7) has an eventually positive solution $x(t)$. Set

$$y(t) = e^{-\lambda_0 t}x(t)$$

and

$$u(t) = y(t) + \lambda_0 \int_{t-\tau}^{t} y(s)ds\,. \tag{2.4}$$

Then for $n \leqslant t < n+1$ and $n = 0, 1, 2, \ldots$

$$\begin{aligned}
\dot{u}(t) &= \dot{y}(t) + \lambda_0 y(t) - \lambda_0 y(t-\tau) \\
&= -\lambda_0 e^{-\lambda_0 t}x(t) + e^{-\lambda_0 t}\dot{x}(t) \\
&\quad + \lambda_0 e^{-\lambda_0 t}x(t) - \lambda_0 e^{-\lambda_0(t-\tau)}x(t-\tau) \\
&= e^{-\lambda_0 t}[\dot{x}(t) - \lambda_0 e^{\lambda_0 \tau}x(t-\tau)] \\
&= e^{-\lambda_0 t}[\dot{x}(t) + px(t-\tau)] \\
&= -e^{-\lambda_0 t}qx([t-k])\,.
\end{aligned}$$

Therefore, for $n \leqslant t < n+1$ and $n = 0, 1, 2, \ldots,$

$$\dot{u}(t) = -e^{-\lambda_0 t}qx([t-k]) < 0 \tag{2.5}$$

and so $u(t)$ is a strictly decreasing function. We claim that $u(t)$ is a bounded function. Otherwise there exists a t_1 such that $u(t_1) < 0$ and

$$y(t_1) = \max_{s \leqslant t_1} y(s)\,.$$

Then (2.4) implies that

$$\begin{aligned}
0 > u(t_1) &= y(t_1) + \lambda_0 \int_{t_1-\tau}^{t_1} y(s)ds \\
&\geqslant y(t_1)(1 + \lambda_0 \tau) \geqslant 0
\end{aligned}$$

which is a contradiction. Thus $u(t)$ is bounded and decreasing and so

$$L \equiv \lim_{t \to \infty} u(t) \tag{2.6}$$

exists and is finite. We claim that $L = 0$. Indeed, integrating (2.5) from t to ∞ we see that

$$\int_t^\infty e^{-\lambda_0 s} q x([s-k])ds = u(t) - L$$

which implies that $y \in L^1[0, \infty)$. Hence $u \in L^1[0, \infty)$ and so L has to be zero. As $u(t)$ decreases to zero, it follows that eventually $u(t) > 0$. Hence (2.4) implies that $y(t) > u(t)$ and by using (2.4) again we find

$$\begin{aligned} y(t) &= u(t) - \lambda_0 \int_{t-\tau}^{t} y(s)ds \\ &> u(t) - \lambda_0 \tau u(t) \\ &= (1 - \lambda_0 \tau)u(t) . \end{aligned}$$

One can show, by induction, that for every $N \in N$ there is a $t_N > 0$ such that

$$y(t) \geqslant c_N u(t) , \quad t \geqslant t_N \tag{2.7}$$

where

$$c_N = \sum_{i=0}^{N-1} (-\lambda_0 \tau)^i = \begin{cases} N , & \text{if } -\lambda_0 \tau = 1 \\ \dfrac{1 - (-\lambda_0 \tau)^N}{1 + \lambda_0 \tau} , & \text{if } -\lambda_0 \tau < 1 . \end{cases}$$

The idea of getting the estimates (2.7) was first used in a similar connection in [4]. By using (2.5) and (2.7) and the definition of y we find that for $t \geqslant t_N$,

$$\dot{u}(t) + q e^{-\lambda_0 t} e^{\lambda_0 (n-k)} c_N u(n-k) \leqslant 0 , \quad n \leqslant t < n+1 . \tag{2.8}$$

By integrating (2.8) from n to t and letting $t \to n+1$ we obtain

$$u(n+1) - u(n) + q c_N e^{-\lambda_0 k} \frac{1 - e^{-\lambda_0}}{\lambda_0} u(n-k) \leqslant 0 , \quad n \geqslant t_N .$$

That is,

$$B_{n+1} - B_n + qQB_{n-k} \leqslant 0 , \quad n \geqslant t_N \tag{2.9}$$

where

$$B_n = u(n) \quad \text{and} \quad Q = e^{-\lambda_0 k} \frac{1 - e^{-\lambda_0}}{\lambda_0} c_N .$$

It follows, by Lemma 5, that the equation

$$\lambda - 1 + qQ\lambda^{-k} = 0 \tag{2.10}$$

has a root in (0, 1].

Now assume that (ii) holds. Choose N so large that

$$qQ \geqslant 1 .$$

Then for $0 < \lambda \leqslant 1$,

$$\lambda - 1 + qQ\lambda^{-k} > -1 + \lambda^{-k} \geqslant 0$$

which contradicts the fact that Eq. (2.10) has a root in (0, 1].

Next, assume that (iii) holds. As Eq. (2.2) has no roots in (0, 1], $q > 0$. By using Lemma 6, choose $\varepsilon \in (0, q)$ such that

$$\lambda - 1 + \frac{B}{A}(q - \varepsilon)\lambda^{-k} > 0 \quad \text{for} \quad 0 < \lambda \leqslant 1 . \tag{2.11}$$

Now observe that

$$\lim_{N\to\infty} c_N = \frac{1}{1 + \lambda_0\tau} > 0 .$$

Choose N so large that

$$qc_N e^{-\lambda_0 k}\frac{1 - e^{-\lambda_0}}{\lambda_0} > (q - \varepsilon)\frac{B}{A} ;$$

that is,

$$c_N > \frac{q - \varepsilon}{q}\frac{1}{1 + \lambda_0\tau} .$$

Then

$$qQ > (q - \varepsilon)\frac{B}{A}$$

and so

$$\lambda - 1 + qQ\lambda^{-k} > \lambda - 1 + (q - \varepsilon)\frac{B}{A}\lambda^{-k} > 0 \quad \text{for} \quad 0 < \lambda \leqslant 1 .$$

This contradicts the fact that Eq. (2.10) has a root in (0, 1] and completes the proof in this case.

Finally, assume that (iv) is satisfied and that, for the sake of contradiction, Eq. (1.7) has an eventually positive solution $x(t)$. Then

$$\dot{x}(t) + qx([t - k]) \leqslant 0 . \tag{2.12}$$

Integrating (2.12) from n to t for $t \in [n, n + 1)$ and then letting $t \to n + 1$, we obtain

$$A_{n+1} - A_n + qA_{n-k} \leqslant 0 \tag{2.13}$$

where

$$A_n = x(n) \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Thus (2.13) has an eventually positive solution. However this, in view of Lemmas 2 and 5 is impossible. The proof of the theorem is complete.

Remark 1. Case (iv) of Theorem 1 can be considered as a limiting case of (iii) of the same theorem. In fact, when $p = 0$ then $\lambda_0 = 0$ and

$$\lim_{\lambda_0 \to 0} \frac{B}{A} = 1 .$$

Therefore, Eq. (2.2) as $\lambda_0 \to 0$ yields

$$\lambda - 1 + q\lambda^{-k} = 0 . \tag{2.14}$$

Finally, from [5] and Lemma 2, Eq. (2.14) has no roots in (0, 1] if and only if (1.14)–(1.15) hold.

3. Linearized Oscillations

Consider the nonlinear equation with continuous and piecewise constant arguments

$$\dot{y}(t) + p_1 F_1(y(t - \tau)) + p_2 F_2(y([t - k])) = 0 , \quad t \geqslant t_0 \tag{3.1}$$

where

$$p_1, p_2, \tau \in \boldsymbol{R}^+ , \quad k \in \boldsymbol{N} \tag{3.2}$$

with

$$|p_2 - 1| + p_1 + k > 0 \tag{3.3}$$

$$F_1 \in C[\boldsymbol{R}, \boldsymbol{R}] , \quad uF_i(u) > 0 \quad \text{for} \quad u \neq 0 \tag{3.4}$$

and

$$\lim_{u \to 0} \frac{F_i(u)}{u} = 1 . \tag{3.5}$$

With Eq. (3.1) we also associate the limiting equation

$$\dot{x}(t) + p_1 x(t - \tau) + p_2 x([t - k]) = 0 . \tag{3.6}$$

As in the linear case one can show that for a given initial function $\varphi \in C[[-\tau, 0], \boldsymbol{R}]$ and constants $A_{-k}, \ldots, A_{-1}, A_0$ such that

$$A_{-j} = \varphi(-j) \quad \text{for} \quad j \leqslant \tau \quad \text{and} \quad j = 0, 1, 2, 3, \ldots, k$$

Eq. (3.1) has a unique solution y satisfying the conditions

$$y(t) = \varphi(t) \quad \text{for} \quad -\tau \leqslant t \leqslant 0$$

and

$$y(-j) = A_{-j} \quad \text{for} \quad j = 0, 1, 2, \ldots, k .$$

The main result of this section is the following:

Theorem 2. *Consider Eq.* (3.1) *and assume that conditions* (3.2)–(3.5) *are satisfied and that every solution of Eq.* (3.6) *oscillates. Then every solution of* (3.1) *also oscillates.*

Proof. Assume, for the sake of contradiction, that Eq. (3.1) has an eventually positive solution $y(t)$. The case where Eq. (3.1) has an eventually negative solution is similar and will be omitted. Then, for n sufficiently large, say $n \geqslant n_0$,

$$\dot{y}(t) \leqslant 0 , \quad n \leqslant t < n + 1$$

which together with the continuity of y implies that $y(t)$ is a nonincreasing function for $t \geqslant n_0$ and so

$$l \equiv \lim_{t \to \infty} y(t)$$

exists and $l \geqslant 0$. Now we claim that $l = 0$. Otherwise, $l > 0$ and so

$$F_i(l) > 0 \quad \text{for} \quad i = 1, 2 .$$

Also p_1 or p_2 is positive for otherwise Eq. (3.6) has a nonoscillatory solution. It then follows from Eq. (3.1) that for n sufficiently large, say $n \geqslant n_1$

$$\dot{y}(t) \leqslant -\beta , \quad n \leqslant t < n + 1 \tag{3.7}$$

where

$$\beta = \tfrac{1}{2}[p_1 F_1(l) + p_2 F_2(l)] > 0 .$$

By integrating both sides of Eq. (3.7) from n to t, for $n \geqslant n_1$ and $n \leqslant t < n + 1$, and by taking limits as $t \to n + 1$ we obtain

$$y(n + 1) - y(n) \leqslant -\beta , \quad n \geqslant n_1 . \tag{3.8}$$

By taking limits as $n \to \infty$ in (3.8) we are lead to a contradiction and so our claim that $l = 0$ has been established.

Now we rewrite Eq. (3.1) in the form

$$\dot{y}(t) + Q_1(t) y(t - \tau) + Q_2(t) y([t - k]) = 0 \tag{3.9}$$

where

$$Q_1(t) = p_1 \frac{F_1(y(t - \tau))}{y(t - \tau)} \quad \text{and} \quad Q_2(t) = p_2 \frac{F_2(y([t - k]))}{y([t - k])}$$

and observe that

$$\lim_{t\to\infty} Q_i(t) = p_i \quad \text{for} \quad i = 1, 2 .$$

We have already noted that one of the p_i's must be positive. In the remaining part of the proof we will assume that both p_1 and p_2 are positive. If $p_1 = 0$ or $p_2 = 0$, the proof can be modified by just deleting the respective term containing this p_i. For any ε satisfying

$$0 < \varepsilon \leqslant \min \{p_1, p_2\} ,$$

Eq. (3.9) yields

$$\dot{y}(t) + (p_1 - \varepsilon)y(t - \tau) + (p_2 - \varepsilon)y([t - k]) \leqslant 0 \tag{3.10}$$

which by Lemma 7 implies the existence of an eventually positive solution of Eq. (3.6) which contradicts the hypothesis of Theorem 2. This completes the proof.

We can now return to the logistic equation (1.1) and its modified form (1.5). It is easy to see that the nonlinearities and the delays in Eq. (1.5) are of the type in (3.1)–(3.5) and that the result of Theorem 2 is applicable for Eq. (1.5). Thus all positive solutions of Eq. (1.1) are oscillatory about its positive steady state provided that all solutions of (1.5) are oscillatory about zero. A set of sufficient conditions for the oscillations of all solutions of Eq. (1.5) can be easily written down using Theorems 1 and 2. For instance each of the following four sets of conditions implies that every positive solution of (1.1) is oscillatory about $N^* = a/(b + c)$;

(i) $bN^*\tau e > 1$ and $c \geqslant 0$,

(ii) $bN^*\tau e = 1$ and $c > 0$,

(iii) $bN^*\tau e < 1$, $a > 0$, $b > 0$, $c > 0$, $\lambda_0 \in [1/\tau, 0]$ with $\lambda_0 + bN^*e^{-\lambda_0\tau} = 0$ and $\lambda_0(1 + \lambda_0\tau) \neq e^{-\lambda_0 k}(1 - e^{-\lambda_0})cN^*$ or $k \neq 0$, and the equation

$$[\lambda_0(1 + \lambda_0\tau)](\lambda - 1) + e^{-\lambda_0 k}(1 - e^{-\lambda_0})cN^*\lambda^{-k} = 0$$

has no roots in $(0, 1]$,

(iv) $$b \geqslant 0 \quad \text{and} \quad cN^* > \frac{k^k}{(k + 1)^{k+1}} \quad \text{if } k > 0$$

or

$$bN^* \geqslant 0 \quad \text{and} \quad cN^* \geqslant 1 \quad \text{if } k = 0 .$$

As a special case assume $b = 0$ and $k = 0$ in Eq. (1.1). By integrating Eq. (1.1) we obtain the difference equation (1.3). It follows by (iv) that if $a \geqslant 1$, every positive solution of (1.3) is oscillatory about a/c.

We conclude with the note that equations with continuous and piecewise constant arguments such as Eq. (1.1) are interesting since these equations are more general than the so-called delay differential equations and difference equations.

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nuna adreso:
Department of Mathematics
The University of Rhode Island
Kingston, Rhode Island 02881
U. S. A.

(Ricevita la 28-an decembro, 1987)
(Reviziita la 27-an de oktobro, 1988)
(Reviziita la 28-an de novembro, 1988)