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On Global Solutions of Some Degenerate Quasilinear Hyperbolic Equations with Dissipative Terms

By

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1. Introduction

This paper is concerned with the global solvability of the initial boundary value problem for degenerate quasilinear hyperbolic equations:

$$\frac{\partial^2 u}{\partial t^2} - a\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)\Delta u + 2\gamma\frac{\partial u}{\partial t} = 0 \qquad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u|_{t=0} = u_0\,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \qquad \text{in } \Omega\,, \tag{1.3}$$

where $a(r)$ is a nonnegative C^1-function on $[0,\infty)$, γ is a positive constant, Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$, $|\nabla u|^2 = \sum_{i=1}^N |\partial u/\partial x_i|^2$ and $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the Laplace operator. Equation (1.1) has its origin in the nonlinear vibration of an elastic string (cf.Carrier [3], Narasimha [10], Nishida [11]):

$$\rho h\frac{\partial^2 u}{\partial t^2} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right)\frac{\partial^2 u}{\partial x^2}\,, \qquad 0 \leqq x \leqq L,\ t > 0\,.$$

Here u is the lateral deflection, ρ is the mass density, h is the cross-section area, L is the length, E is the Young's modulus and P_0 is the initial axial tension. We are interested in the case $P_0 = 0$, which is corresponding to $a(0) = 0$ in (1.1). When $a(r)$ has zeros, we say that (1.1) is degenerate.

For degenerate equations (1.1) without dissipative terms ($\gamma = 0$), the existence of global solutions has been established by Nishida [11], Arosio and Spagnolo [1] when initial data u_0 and u_1 are analytic, while Yamazaki [18] and Yamada [21] have recently proved the local solvability for non-analytic initial data which belong to suitable Sobolev spaces. See also Ebihara, Medeiros and Milla Miranda [5]. (For non-degenerate equations of the form (1.1), there is a lot of literature; see Dickey [4], Lions [8], Nishihara [12], Perla Menzala [15], Pohozaev [16], Rivera [17], Yamada [20] and the references therein.)

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Our purpose of the present paper is to study the global solvability of (1.1)–(1.3) for non-analytic initial data. To make our arguments clear, let

$$a(r) = r^{\alpha} \qquad \text{with } \alpha \geqq 1 .$$

In the presence of dissipative terms, we will show the global existence, uniqueness and asymptotic decay of solutions of (1.1)–(1.3) provided that u_0 ($\neq 0$) and u_1 are small and u_1 is much smaller than u_0 in some sense. The decay rate of the solutions is not so fast in the degenerate case, although exponential decay properties have been obtained for the non-degenerate case by Yamada [20], Brito [2], Nishihara [13, 14].

The content of this paper is as follows. In Section 2 we reformulate (1.1)–(1.3) to the initial value problem for an abstract evolution equation in a Hilbert space and state our theorem. The proof consists of several steps and will be given in Section 3.

2. Formulation of problem and results

We will discuss (1.1)–(1.3) in an abstract framework.

Let H be a real, separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. Let A be a densely defined linear operator in H with domain $D(A)$. We assume that

(H) A is a self-adjoint, positive definite operator with discrete spectrum and A^{-1} is compact.

We consider the initial value problem for the following evolution equation in H

$$u''(t) + \|A^{1/2}u(t)\|^{2\alpha}Au(t) + 2\gamma u'(t) = 0 , \qquad t \geqq 0 , \tag{2.1}$$

with initial conditions

$$u(0) = u_0 , \qquad u'(0) = u_1 \tag{2.2}$$

where $' = d/dt$, $'' = d^2/dt^2$ and γ is a positive constant. We note that initial value problem (2.1)–(2.2) includes (1.1)–(1.3) by taking $H = L^2(\Omega)$ and $Au = -\Delta u$ for $u \in D(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

We will look for a weak solution of (2.1)–(2.2) which satisfies

(i) $$u \in \bigcap_{i=0}^{2} C_w^i([0, \infty); D(A^{(2-i)/2})) ,$$

where the subscript 'w' means the weak continuity with respect to t,

(ii) $$(u''(t) + \|A^{1/2}u(t)\|^{2\alpha}Au(t) + 2\gamma u'(t), w) = 0$$

for all $t \in (0, \infty)$ and $w \in H$,

(iii) $$u(0) = u_0 \qquad \text{and} \qquad u'(0) = u_1 .$$

Our main theorem is as follows.

Theorem. *Suppose* (*H*) *and* $(u_0, u_1) \in D(A) \times D(A^{1/2})$ *with*

(2.3) $$u_0 \neq 0 .$$

Then there exists a positive number ε_0 *depending on* γ, $\|A^{1/2}u_0\|$ *and* $\|u_1\|$ *with the following property: if*

(2.4) $$\|Au_0\|^2 + \|A^{1/2}u_1\|^2/\|A^{1/2}u_0\|^{2\alpha} \leqq \varepsilon_0 ,$$

then there exists a unique weak solution u of (2.1)–(2.2) *which satisfies*

(2.5) $$\|A^{1/2}u(t)\| > 0 \qquad \textit{for every } t \in [0, \infty) ,$$

and decay estimates

(2.6) $$\|A^{1/2}u(t)\| \leqq C(1 + t)^{-1/2\alpha}$$

(2.7) $$\|u'(t)\| \leqq C(1 + t)^{-(\alpha+1)/2\alpha}$$

(2.8) $$\|A^{1/2}u'(t)\| \leqq C(1 + t)^{-1/2}$$

(2.9) $$\|u''(t)\| \leqq C(1 + t)^{-(\alpha+1)/2\alpha}$$

for every $t \in [0, \infty)$ *with some constant C independent of t.*

Remark 1. When $\alpha = 1$, we may take ε_0 (say, e.g. $\varepsilon_0 < \gamma^2$) depending only on γ.

Remark 2. Our theorem asserts that, if (u_0, u_1) satisfies (2.3) and (2.4), then (2.1)–(2.2) has a unique weak solution u on $[0, \infty)$ such that $\|A^{1/2}u(t)\|$ never vanishes on $[0, \infty)$ because of the effect of the dissipative term. Therefore, one may consider that equation (2.1) itself is actually a non-degenerate type. However, if (2.4) is violated, we do not know whether the global result holds. Similarly, although the local existence results are known in the case $u_0 = 0$ (see Yamazaki [18] or Yamada [21]), the global existence is still an open problem.

Remark 3. (Regularity) In addition to the assumptions in Theorem, suppose $(u_0, u_1) \in D(A^{k+1/2}) \times D(A^k)$ $(k \geqq 1/2)$. Then the corresponding regularity results of $u(t)$ can be derived in the standard manner.

Remark 4. If one makes use of the theory of evolution equations, it is

possible to improve the regularity of u with respect to t. Indeed, one can show the global existence of the solution of (2.1)–(2.2) in the class $\bigcap_{i=0}^{2} C^i([0, \infty); D(A^{(2-i)/2}))$ even if (H) is replaced by

(H′) A is a self-adjoint and positive definite operator

(see Yamada [20]).

3. Proof of theorem

It follows from the assumption (H) that A has an infinite number of eigenvalues $\{\lambda_j^2\}$ such that

$$0 < \lambda_1^2 \leqq \lambda_2^2 \leqq \cdots \leqq \lambda_j^2 \leqq \cdots \to \infty \qquad \text{as } j \to \infty$$

and that there exists a complete orthonormal system $\{w_j\}$ in H, each w_j being an eigenvector corresponding to λ_j^2. Therefore, each $u \in H$ has a Fourier expansion in H

$$u = \sum_{j=1}^{\infty} u_j w_j \qquad \text{with} \qquad \|u\| = \left(\sum_{j=1}^{\infty} u_j^2 \right)^{1/2}.$$

We apply the Galerkin procedure. For each m, we take an approximating solution $u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j$ as a solution of the initial value problem for the following system of ordinary differential equations:

$$(u_m''(t) + (m^{-1} + \|A^{1/2} u_m(t)\|^{2\alpha}) A u_m(t) + 2\gamma u_m'(t), w) = 0 \tag{3.1}$$

$$\text{for any} \quad w \in V_m ,$$

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w_j) w_j ,$$

$$u_m'(0) = u_{1m} = \sum_{j=1}^{m} (u_1, w_j) w_j , \tag{3.2}$$

where V_m is an m-dimensional vector space spanned by $\{w_1, \ldots, w_m\}$. By virtue of the theory of ordinary differential equations $u_m(t)$ can be defined on some interval $[0, t_m)$.

i) A priori estimates.

Taking $w = 2u_m'(t)$ in (3.1) we have

$$\frac{d}{dt}\left[\|u_m'(t)\|^2 + m^{-1} \|A^{1/2} u_m(t)\|^2 + \frac{1}{\alpha + 1} \|A^{1/2} u_m(t)\|^{2\alpha+2} \right] + 4\gamma \|u_m'(t)\|^2 = 0 ;$$

so that integration of the above yields

$$\|u_m'(t)\|^2 + \frac{1}{\alpha+1}\|A^{1/2}u_m(t)\|^{2\alpha+2} + 4\gamma\int_0^t \|u_m'(\tau)\|^2 d\tau$$

$$\leqq \|u_1\|^2 + \|A^{1/2}u_0\|^2 + \frac{1}{\alpha+1}\|A^{1/2}u_0\|^{2\alpha+2}.$$

Hence

$$\|u_m'(t)\|,\ \|A^{1/2}u_m(t)\|,\ \int_0^t \|u_m'(\tau)\|^2 d\tau \leqq C_1. \tag{3.3}$$

(Here and after we denote by C or C_i various positive constants independent of m and t). From (3.3) we conclude that all the intervals $[0, t_m)$ can be extended to the whole interval $[0, \infty)$.

We next define

$$F_m(t) = \frac{\|A^{1/2}u_m'(t)\|^2}{m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha}} + \|Au_m(t)\|^2 \tag{3.4}$$

$$= f_m(t) + \|Au_m(t)\|^2.$$

Then

$$\frac{d}{dt}F_m(t) = \frac{(u_m''(t) + (m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha})Au_m(t),\ 2Au_m'(t))}{m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha}}$$

$$- \frac{2\alpha\|A^{1/2}u_m(t)\|^{2\alpha-2}(A^{1/2}u_m(t),\ A^{1/2}u_m'(t))\|A^{1/2}u_m'(t)\|^2}{(m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha})^2}$$

$$\leqq -4\gamma f_m(t) + 2\alpha\|A^{1/2}u_m(t)\|^{\alpha-1}\frac{\|A^{1/2}u_m(t)\|^{\alpha}}{(m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha})^{1/2}} f_m(t)^{3/2}$$

$$\leqq -4\gamma f_m(t) + 2C_2 f_m(t)^{3/2},$$

where $C_2 = \alpha C_1^{\alpha-1}$. Since

$$\frac{d}{dt}F_m(t) + 2(2\gamma - C_2 f_m(t)^{1/2})f_m(t) \leqq 0, \tag{3.5}$$

it is easy to see that $F_m(t) \leqq F_m(0)$ for $0 \leqq t \leqq t^*$ if $f_m(t) \leqq (\gamma/C_2)^2$ for $0 \leqq t \leqq t^*$. Assume $F(0) \leqq (\gamma/C_2)^2/2$. Since

$$F_m(0) \to F(0) = \frac{\|A^{1/2}u_1\|^2}{\|A^{1/2}u_0\|^{2\alpha}} + \|Au_0\|^2 \qquad \text{as } m \to \infty,$$

it follows that $F_m(0) \leqq (\gamma/C_2)^2$ for sufficiently large m and, therefore $f_m(t) \leqq F_m(t) \leqq (\gamma/C_2)^2$. Thus, taking $\varepsilon_0 = (\gamma/C_2)^2/2$ in (2.4) we may get $t^* = \infty$. In-

tegrating (3.5) over $[0, t]$ we obtain

$$F_m(t) + 2\gamma \int_0^t f_m(\tau)d\tau \leqq (\gamma/C_2)^2 ,$$

which implies

$$\|Au_m(t)\| \leqq \gamma/C_2 , \tag{3.6}$$

$$\frac{\|A^{1/2}u_m'(t)\|}{(m^{-1} + \|A^{1/2}u_m(t)\|^{2\alpha})^{1/2}} \leqq \gamma/C_2 , \tag{3.7}$$

$$\|A^{1/2}u_m'(t)\| \leqq (1 + C_1^{2\alpha})^{1/2}\gamma/C_2 \equiv C_3\gamma , \tag{3.8}$$

$$\int_0^t \frac{\|A^{1/2}u_m'(\tau)\|^2}{m^{-1} + \|A^{1/2}u_m(\tau)\|^{2\alpha}}d\tau \leqq \gamma/2C_2^2 . \tag{3.9}$$

Finally substitution of $w = u_m''(t)$ in (3.1) yields

$$\|u_m''(t)\| \leqq C_4\gamma \quad (C_4 = (1 + C_1^{2\alpha})/C_2 + 2C_1) . \tag{3.10}$$

ii) Passage to the limit.

By virtue of a priori estimates (3.3), (3.6), (3.8) and (3.10) we may extract a subsequence which is still denoted by $\{u_m\}$ such that for every $T > 0$

$$u_m \to u \quad \text{in} \quad C([0, T]; D(A^{1/2})) \cap C_w([0, T]; D(A))$$

$$u_m' \to u' \quad \text{in} \quad C([0, T]; H) \cap C_w([0, T]; D(A^{1/2}))$$

and

$$u_m'' \to u'' \quad \text{in the weak-star topology of} \quad L^\infty(0, T; H)$$

with some u, where the compactness of $D(A)$ (resp. $D(A^{1/2})$) in $D(A^{1/2})$ (resp. H) is used. It is standard to see that such u becomes a weak solution of (2.1)–(2.2). We omit the details (see Lions [7], Kato and Lai [6]).

Moreover, the a priori estimates imply

$$\|Au(t)\| \leqq \gamma/C_2 , \tag{3.11}$$

$$\|A^{1/2}u'(t)\|/\|A^{1/2}u(t)\|^\alpha \leqq \gamma/C_2 , \tag{3.12}$$

$$\|A^{1/2}u'(t)\| \leqq C_3\gamma , \tag{3.13}$$

$$\int_0^t \|A^{1/2}u'(\tau)\|^2/\|A^{1/2}u(\tau)\|^{2\alpha}d\tau \leqq \gamma/2C_2^2 , \tag{3.14}$$

and

$$\|u''(t)\| \leqq C_4\gamma . \tag{3.15}$$

iii) $\|A^{1/2}u(t)\| > 0 \quad$ *for* $0 \leqq t < +\infty$.

We first prepare the following lemma.

Lemma. *If* $v: [-T, T] \to D(A)$ *is a weak solution of*

$$(*) \qquad \begin{cases} v''(t) + \|A^{1/2}v(t)\|^{2\alpha}Av(t) + \lambda v'(t) = 0, & -T \leqq t \leqq T \\ v(0) = 0\,, \quad v'(0) = 0\,, \end{cases}$$

where λ *is a constant, then* $v(t) \equiv 0$ *for* $-T \leqq t \leqq T$.

Proof. Applying $2v'(t)$ to (*), we get

$$\frac{d}{dt}\left[\|v'(t)\|^2 + \frac{1}{\alpha+1}\|A^{1/2}v(t)\|^{2\alpha+2}\right] + 2\lambda\|v'(t)\|^2 = 0\,.$$

Since integration of the above identity gives

$$\|v'(t)\|^2 + \frac{1}{\alpha+1}\|A^{1/2}v(t)\|^{2\alpha+2} \leqq 2|\lambda|\int_0^{|t|}\|v'(\tau)\|^2 d\tau\,,$$

Gronwall's inequality assures $v'(t) \equiv 0$ and $v(t) \equiv 0$ for $t \in [-T, T]$. Q.E.D.

We now show (2.5) by the contradiction. Let T be a point such that $A^{1/2}u(T) = 0$. From (3.12), $A^{1/2}u'(T)$ must be zero. Since u is a weak solution of (2.1) with $(u(T), u'(T)) = (0, 0)$, Lemma implies $u(t) \equiv 0$ $(0 \leqq t \leqq T)$, which contradicts $u_0 \neq 0$. Thus we obtain $\|A^{1/2}u(t)\| > 0$ for all $t \geqq 0$.

iv) Uniqueness.

Let v be another weak solution of (2.1)–(2.2). Then $w = u - v$ satisfies

$$(3.16) \quad w''(t) + \|A^{1/2}u(t)\|^{2\alpha}Aw(t) + 2\gamma w'(t) = -(\|A^{1/2}u(t)\|^{2\alpha} - \|A^{1/2}v(t)\|^{2\alpha})Av(t)\,.$$

Let T be any fixed number. Taking inner product of (3.16) with $2w'(t)$ we have for $0 \leqq t \leqq T$

$$\frac{d}{dt}[\|w'(t)\|^2 + \|A^{1/2}u(t)\|^{2\alpha}\|A^{1/2}w(t)\|^2] + 4\gamma\|w'(t)\|^2 \leqq C(\|w'(t)\|^2 + \|A^{1/2}w(t)\|^2)\,.$$

Since $\delta_T = \min\{\|A^{1/2}u(t)\|^{2\alpha}; 0 \leqq t \leqq T\} > 0$ by iii), one can derive

$$\|w'(t)\|^2 + \delta_T\|A^{1/2}w(t)\|^2 \leqq C\int_0^t (\|w'(\tau)\|^2 + \delta_T\|A^{1/2}w(\tau)\|^2)d\tau\,,$$

which, together with Gronwall's inequality, gives $w(t) = 0$ on $[0, T]$. Thus the proof of the uniqueness part is completed.

v) Asymptotic decay properties.

We make use of the results of Nakao [9] or Yamada [19] on the decay properties of solutions for second order evolution. If E is defined by

$$E(u(t)) = \|u'(t)\|^2 + \frac{1}{\alpha+1}\|A^{1/2}u(t)\|^{2\alpha+2},$$

then $E(u(t)) \leqq Ct^{-(\alpha+1)/\alpha}$ is derived from [9] or [19]. Hence (2.6) and (2.7) follow. (2.8) is direct consequence of (2.6) and (3.12). Finally, one can easily show (2.9) with use of (2.1), (2.6), (2.7) and the uniform boundedness of $\|Au(t)\|$ (see (3.11)).

Thus we have completed the proof of Theorem.

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