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Perron's Method for Viscosity Solutions Associated with Piecewise-Deterministic Processes

By

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0. Introduction

In this paper, we consider viscosity solutions [2, 4] of integro-differential equations of the following form:

$$(0.1)\qquad F\left(x, u(x), \nabla u(x), \int_{\Omega} u(y)Q(dy, x)\right) = 0\,,$$

where ∇u is the gradient of u and $Q(\cdot, x)$ is a probability measure. The applications for (0.1) considered here are dynamic programming equations associated with the control of piecewise-deterministic (PD) processes.

A PD process follows deterministic dynamics between random jumps, and is determined by its three "local characteristics":

(i) a continuous vector field g, which determines the flow

$$\dot{x}_t = (g_1(x_t), \dots, g_n(x_t))$$

(ii) a jump rate $\lambda\colon \boldsymbol{R}^n \to \boldsymbol{R}_+$, and

(iii) a transition measure $Q\colon \bar{\Omega} \to P(\Omega)$, where $P(\Omega)$ is the set of probability measures on $\bar{\Omega}$.

The extended generator of x_t is

$$(0.2)\qquad g(x)\cdot \nabla u + \lambda(x)\int_{\Omega} (u(y) - u(x))Q(dy, x)\,;$$

thus such terms occur in (0.1). See Davis [5] for background and definition of PD processes. For detailed information about applications and control of PD processes see Davis [6], Soner [22], Vermes [23], Gugerli [9], Hordijk and van Duyn Schouten [11], Pliska [18], Rosberg, Variaga and Walrand [21], Pragarauskas [19, 20], Davis, Dempster, Sethi, and Vermes [7], Lenhart [14], Lenhart and Liao [15, 16], and Gatarek [8].

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In the case that the state space is Ω, a bounded domain in $\boldsymbol{R}^n$, the PD process jumps back into Ω upon hitting the boundary, which leads to the boundary condition to be coupled with equation (0.1),

$$u(x) = \int_{\Omega} u(y)Q(dy, x), \qquad x \in \partial\Omega . \tag{0.3}$$

The main goal of this paper is to prove existence results for equation (0.1) with boundary condition (0.2), using Perron's method. Perron's method for viscosity solutions for Hamilton-Jacobi equations was introduced by Ishii [12]. The key idea is that the supremum of a class of subsolutions is a solution. The main advantage of this method is a weakening of assumptions, especially eliminating the "largeness of the discount factor" (zero$^{\text{th}}$ order coefficient) assumption. The assumption was used by Lenhart [14] and Lenhart and Liao [15, 16] in *a priori* estimates necessary for existence of solutions to integro-differential equations for PD processes. With Perron's method, a priori $W^{1,\infty}$ estimates are no longer necessary for existence.

After proving a general existence result in section 1, we apply this result in section 2 to examples arising in the control of PD processes. We consider the integro-differential equation with operator (0.2) and the dynamic programming equation associated with optimal stopping control of PD processes. The system associated with the switching control problem for PD processes is treated by an iterative scheme using single equation results.

1. General existence result

In this section we consider the existence of viscosity solutions for equation (0.1) with boundary conditions (0.3). We use Perron's method, which is essentially that "the supremum of a set of subsolution is a solution".

Let Ω be a bounded open set in $\boldsymbol{R}^n$ with smooth boundary. We make the following assumptions:

(A.1) $F(x, u, p, r)\colon \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous.

(A.2) If $u_0 \leqq u_1$, then $F(x, u_0, p, r) \leqq F(x, u_1, p, r)$ for any $(x, p, r) \in \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}$.

(A.3) If $r_0 \leqq r_1$, then $F(x, u, p, r_0) \geqq F(x, u, p, r_1)$ for any $(x, u, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$.

(A.4) $Q(\cdot, x)$ is a probability measure on Ω for $x \in \bar{\Omega}$, which satisfies

$$\left| \int_{\Omega} v(y)Q(dy, x) \right| \leqq C\|v\|_{L^1(\Omega)} \qquad \text{for all } v \in L^1(\Omega) .$$

(A.5) The function

$$x \to \int_{\Omega} v(y)Q(dy, x) ,$$

is continuous with respect to $x \in \bar{\Omega}$, uniformly for $v \in L^\infty(\Omega)$.

Notice if we fix $v \in L^\infty(\Omega)$, then the equation

$$F\left(x, u(x), \nabla u(x), \int_\Omega v(y)Q(dy, x)\right) = 0 \qquad \text{in } \Omega$$

is a first order Hamilton-Jacobi equation.

We give some notation necessary to state the definition of viscosity solution. For bounded functions,

$$u^*(x) = \limsup_{r\to 0} \{u(y) \,|\, |x-y| < r\} \qquad \text{upper semicontinuous envelope of } u$$

and

$$u_*(x) = \liminf_{r\to 0} \{u(y) \,|\, |x-y| < r\} \qquad \text{lower semicontinuous envelope of } u\,.$$

Definition. Let u be a bounded function.

(i) u is a viscosity subsolution of (0.1) if

$$F\left(x, u^*(x), \nabla\phi(x), \int_\Omega u^*(y)Q(dy, x)\right) \leqq 0$$

wherever $u^* - \phi$ attains its maximum for $\phi \in C^1(\Omega)$.

(ii) u is a viscosity supersolution of (0.1) if

$$F\left(x, u_*(x), \nabla\phi(x), \int_\Omega u_*(y)Q(dy, x)\right) \geqq 0$$

wherever $u_* - \phi$ attains its minimum for $\phi \in C^1(\Omega)$.

(iii) u is a viscosity solution if u is a viscosity sub- and supersolution.

In the following, "(sub/super) solution" means "viscosity (sub/super) solution".

Assume that there exists a supersolution W of (0.1) such that

$$W(x) \geqq \int_\Omega W_*(y)Q(dy, x) \qquad \text{on } \partial\Omega\,. \tag{1.1}$$

Define

$$\mathscr{S} = \left\{ v \,|\, v \text{ is a subsolution of (0.1) such that } v \leqq W \text{ in } \Omega \text{ and } v(x) \leqq \int_\Omega v^*(y)Q(dy, x) \text{ on } \partial\Omega \right\}.$$

Define $u_0(x) = \sup\{v(x) \,|\, v \in \mathscr{S}\}$.

Perron's method consists of the following two propositions:

Proposition 1.1. *Assume that $\mathscr{S}$ is not empty and F satisfies* (A.1)–(A.5), *then $u_0 \in \mathscr{S}$.*

Proof. Suppose that $\phi \in C^1$, $u_0^* - \phi$ attains its maximum at $y_0 \in \Omega$. Without loss of generality, we assume

$$(u_0^* - \phi)(y_0) = 0\,, \qquad u_0^* - \phi \leqq 0 \qquad \text{in } \Omega\,,$$

and

$$(u_0^* - \phi)(x) \leqq -|x - y_0|^2 \qquad \text{on } B(y_0, r) \text{ for some } r > 0\,.$$

Then there exists $x_n \in B(y_0, r)$ such that $x_n \to y_0$ and $(u_0 - \phi)(x_n) \nearrow 0$. Let $a_n = (u_0 - \phi)(x_n)$. For each n, there exists $u_n \in \mathscr{S}$ such that $(u_n^* - \phi)(x_n) > a_n - 1/n$ and $(u_n^* - \phi)(x) \leqq -|x - y_0|^2$ in $\overline{B(y_0, r)}$. Let $y_n \in \overline{B(y_0, r)}$ be such that

$$(u_n^* - \phi)(y_n) = \max_{x \in \overline{B(y_0, r)}} (u_n^* - \phi)(x)\,.$$

Since the chain of inequalities,

$$\begin{aligned}-|y_n - y_0|^2 &\geqq (u_0^* - \phi)(y_n) \geqq (u_n^* - \phi)(y_n)\\ &\geqq (u_n^* - \phi)(x_n) \geqq a_n - \frac{1}{n}\,,\end{aligned}$$

implies that $y_n \to y_0$, we have

$$\lim_{n\to\infty} u_n^*(y_n) = \lim_{n\to\infty} \phi(y_n) = \phi(y_0) = u_0(y_0)\,.$$

Since u_n is a viscosity subsolution, we have

$$F\left(y_n, u_n^*(y_n), \nabla\phi(y_n), \int_\Omega u_n^*(y) Q(dy, y_n)\right) \leqq 0\,.$$

Since $u_n \leqq u_0$, we can let $n \to \infty$ to get

$$F\left(y_0, u_0^*(y_0), \nabla\phi(y_0), \int_\Omega u_0^*(y) Q(dy, y_0)\right) \leqq 0\,.$$

Also, we have

$$u_0(x) \leqq \int_\Omega u_0^*(y) Q(dy, x) \qquad \text{on } \partial\Omega\,.$$

Thus $u_0 \in \mathscr{S}$. ■

Proposition 1.2. *Assume $\mathscr{S} \neq \emptyset$. If $v \in \mathscr{S}$ is not a supersolution, then there exists $w \in \mathscr{S}$ such that $v(y) < w(y)$ at some $y \in \Omega$.*

Proof. Suppose $v \in \mathcal{S}$ is not a supersolution. Then there exists $y_0 \in \Omega$, $\tilde{\phi} \in C^1$, and $\eta > 0$ such that $v_* - \tilde{\phi}$ attains its minimum at y_0 and

$$F\left(y_0, v_*(y_0), \nabla\tilde{\phi}(y_0), \int_\Omega v_*(y)Q(dy, y_0)\right) < -\eta \,. \tag{1.2}$$

We may assume $v_*(y_0) = \tilde{\phi}(y_0)$.

We claim that $v_*(y_0) < W_*(y_0)$, the supersolution. If not, $v_*(y_0) = W_*(y_0)$ and $W_* - \tilde{\phi}$ attains its minimum at y_0. Since W is a supersolution and $v \leqq W$,

$$F\left(y_0, v_*(y_0), \nabla\tilde{\phi}(y_0), \int_\Omega v_*(y)Q(dy, y_0)\right) \geqq 0$$

which is a contradiction.

There exists $\delta_1 > 0$ such that

$$v_*(y_0) + \delta_1 < W_*(x) \qquad \text{for } x \in B(y_0, \delta_1)\,.$$

Using (1.2) and (A.1), there exists $\delta_2 > 0$ ($\delta_2 < \delta_1$) such that if x, u, p, $\tilde{v}$ satisfies

$$|x - y_0| < \delta_2\,, \qquad |(u - v_*)(y_0)| < \delta_2\,,$$

$$|p - \nabla\tilde{\phi}(y_0)| < \delta_2 \qquad \text{and} \qquad \int_\Omega |\tilde{v}(y) - v_*(y)|Q(dy, y_0) < \delta_2\,,$$

then

$$F\left(x, u, p, \int_\Omega \tilde{v}(y)Q(dy, x)\right) \leqq 0\,.$$

We can find $\delta > 0$ ($\delta < \min\{\delta_2/2, 1\}$) and $\phi \in C^1$ such that

$$\phi(y_0) = \tilde{\phi}(y_0)\,, \qquad \nabla\phi(y_0) = \nabla\tilde{\phi}(y_0)\,,$$

$$|\phi(x) - v_*(y_0)| < \frac{\delta_2}{2} \qquad \text{for } x \in B(y_0, 2\delta)\,,$$

$$|\nabla\phi(y_0) - \nabla\phi(x)| < \delta_2 \qquad \text{for } x \in B(y_0, 2\delta)\,,$$

$$\int_\Omega |\phi(y) - v_*(y)|Q(dy, y_0) < \frac{\delta_2}{2}\,.$$

We can also choose ϕ and δ to satisfy

$$u_*(x) - \phi(x) > |x - y_0|^2 \qquad \text{for } x \in B(y_0, 2\delta)\,.$$

These inequalities imply

$$|\phi(x) + \delta^2 - v_*(y_0)| < \delta_2$$

and

$$\int_{\Omega} |\phi(y) + \delta^2 - v_*(y)| Q(dy, y_0) < \delta_2$$

which give

$$F\left(x, \phi(x) + \delta^2, \nabla\phi(x), \int_{\Omega} (\phi(y) + \delta^2) Q(dy, x)\right) \leqq 0 \qquad \text{for } x \in B(y_0, 2\delta).$$

This means $\phi(x) + \delta^2$ is a subsolution of (0.1) in $B(y_0, 2\delta)$.

Define

$$w(x) = \begin{cases} \max\{\phi(x) + \delta^2, v(x)\} & \text{for } x \in B(y_0, \delta) \\ v(x) & \text{otherwise}. \end{cases}$$

Note that if $|x - y_0| = \delta$, then $v_*(x) \geqq \phi(x) + \delta^2$. By the similar argument in the preceding proposition, we can see that w is a subsolution (note that our equation has a non-local term), and $w \in \mathscr{S}$. But $v(y_0) < w(y_0)$. ■

By Perron's method, we can find a solution u_0, i.e., u_0^* is a subsolution (u.s.c.) and u_{0*} is a supersolution (l.s.c.). If the comparison principle holds in (0.1), then $u_0^* \leqq u_{0*}$. On the other hand, $u_{0*} \leqq u_0^*$ in general. Hence $u_{0*} = u_0 = u_0^*$ and u_0 is continuous. The following theorem gives a sufficient condition to assure that u_0 satisfies the boundary condition (0.3).

Theorem 1.1. *Assume* (A.1)–(A.5). *Suppose that there exists a supersolution W of* (0.1) *satisfying* (1.1), *and a continuous solution u_1 of*

$$F\left(x, u_1, \nabla u_1, \int_{\Omega} u_0^*(y) Q(dy, x)\right) = 0 \qquad \text{in } \Omega \tag{1.3}$$

with

$$u_1(x) = \int_{\Omega} u_0^*(y) Q(dy, x) \qquad \text{on } \partial\Omega.$$

Assume that the comparison principle holds for (0.1), *i.e.*,

$$u_0 \leqq u_1 \text{ on } \partial\Omega \Rightarrow u_0 \leqq u_1 \text{ in } \Omega.$$

If $u_1 \leqq W$, then u_0 is a continuous solution of (0.1) *satisfying the boundary condition* (0.3).

Proof. We claim $u_1 \in \mathscr{S}$. Let $\phi \in C^1$ such that $u_1^* - \phi$ attains its maximum at y_0, then

$$F\left(y_0, u_1(y_0), \nabla\phi(y_0), \int_{\Omega} u_0^*(y) Q(dy, y_0)\right) \leqq 0.$$

Using $u_0 \leqq u_1$ and (A.3) we have

$$F\left(y_0, u_1(y_0), \nabla\phi(y_0), \int_\Omega u_1(y)Q(dy, y_0)\right) \leqq 0 .$$

Also,

$$u_1(x) = \int_\Omega u_0^*(y)Q(dy, x) \leqq \int_\Omega u_1(y)Q(dy, x) \qquad \text{on } \partial\Omega .$$

Hence, we have the claim. By definition of u_0 and $u_0 \leqq u_1$, we have $u_0 \equiv u_1$ in $\bar{\Omega}$. Therefore u_0 is a desired continuous solution. ■

2. Applications

In this section we apply Theorem 1.1 to prove existence results for various equations and systems associated with piecewise-deterministic processes.

Let

$$\text{(2.1)} \qquad Lu(x) = -g(x)\cdot\nabla u(x) + \alpha(x)u(x) - \lambda(x)\int_\Omega (u(y) - u(x))Q(dy, x) ,$$

where $\cdot$ is the usual inner product in $\boldsymbol{R}^n$, ∇u is the gradient vector of u and $Q(\cdot, x)$ is a probability measure.

We assume the following conditions:

(H.1) Ω is a bounded domain with smooth boundary $\partial\Omega$.
(H.2) $g(x)\colon \Omega \to \boldsymbol{R}^n$ is Lipschitz continuous, and, $\alpha(x)$, $\lambda(x)\colon \bar{\Omega} \to \boldsymbol{R}$ are continuous.
(H.3) There exists $\alpha_0 > 0$ such that $\alpha(x) \geqq \alpha_0$ for $x \in \bar{\Omega}$.
(H.4) $\lambda(x) > 0$ for $x \in \Omega$.
(H.5) $Q(\cdot, x)$ satisfies (A.4) and (A.5).
(H.6) $g(x)\cdot\eta(x) > 0$ for $x \in \partial\Omega$, where $\eta(x)$ is the outward unit normal at $x \in \partial\Omega$.

Note that hypothesis (H.2) can be weakened by using a more complicated hypothesis (see [3]), but that hypothesis is not the main point here.

Consider

$$\text{(2.2)} \qquad Lu(x) = f(x) \qquad \text{in } \Omega$$

$$\text{(2.3)} \qquad u(x) = \int_\Omega u(y)Q(dy, x) \qquad \text{on } \partial\Omega .$$

Theorem 2.1. *Assume* (H.1)–(H.6) *and that* $f \in C(\bar{\Omega})$. *Then there exists a unique continuous solution of equation* (2.2) *satisfying boundary condition* (2.3).

Proof. We apply Theorem 1.1 to

$$F(x, u, p, r) = -g(x)\cdot p + (\alpha(x) + \lambda(x))u - \lambda(x)r .$$

Note that (A.1)–(A.3) are satisfied,

$$w(x) = -\frac{\|f\|_\infty}{\alpha_0} \qquad \text{is a subsolution, and}$$

$$W(x) = \frac{\|f\|_\infty}{\alpha_0} \qquad \text{is a supersolution .}$$

By Propositions 1.1 and 1.2, there exists a solution u_0 of (2.2) satisfying

$$u_0(x) \leqq \int_\Omega u_0^*(y)Q(dy, x) .$$

Since $\int_\Omega u_0^*(y)Q(dy, x)$ is continuous with respect to x,

$$F\left(x, u, p, \int_\Omega u_0^*(y)Q(dy, x)\right)$$

satisfies the conditions (A.1)–(A.5).

Consider the equation

$$-g\cdot\nabla u_1 + (\alpha + \lambda)u_1 - \lambda \int_\Omega u_0^*(y)Q(dy, x) = f \qquad \text{in } \Omega$$

with Dirichlet boundary condition

$$u_1(x) = \int_\Omega u_0^*(y)Q(dy, x) \qquad \text{on } \partial\Omega .$$

Comparison principle for this equation is known [2, 3, 4]. By (H.6) and the method of [13], we have the existence of super- and subsolutions. Then there exists a continuous solution u_1 of this equation.

Now applying Theorem 1.1, u_0 is a continuous solution satisfying

$$u_0 = \int_\Omega u_0(y)Q(dy, x) \qquad \text{on } \partial\Omega .$$

The uniqueness is proved in Lenhart [14]. ∎

Now we proceed to apply Theorem 1.1 to the obstacle problem:

$$\min\{Lu - f, u - \psi\} = 0 \qquad \text{in } \Omega , \tag{2.4}$$

$$u(x) = \int_\Omega u(y)Q(dy, x) \qquad \text{on } \partial\Omega . \tag{2.5}$$

Equation (2.4) with boundary condition (2.5) is the dynamic programming equation associated with the optimal stopping problem for PD processes. See [14] for more background.

Theorem 2.2. *Assume* (H.1)–(H.6). *Suppose that* $f, \psi \in C(\bar{\Omega})$. *Then there exists a unique continuous solution of equation* (2.4) *with boundary condition* (2.5).

Proof. Let $W = \max\{\|f\|_\infty/\alpha_0, \|\psi\|_\infty\}$. Then W is a supersolution and $-W$ is a subsolution of the equation (2.4). Hence by Propositions 1.1 and 1.2, there exists a solution u_0 of (2.4) satisfying

$$u_0(x) \leqq \int_\Omega u_0^*(y)Q(dy, x) \qquad \text{on } \partial\Omega . \tag{2.6}$$

To check the hypothesis of Theorem 1.1, consider the following obstacle problem,

$$\min\left\{-g\cdot\nabla u_1 + (\alpha+\lambda)u_1 - \lambda\int_\Omega u_0(y)Q(dy, x) - f, u_1 - \psi\right\} = 0 \qquad \text{in } \Omega \tag{2.7}$$

$$u_1(x) = \int_\Omega u_0^*(y)Q(dy, x) \qquad \text{on } \partial\Omega . \tag{2.8}$$

Note that the comparison principle holds.

Using (2.6) and $u_0 \geqq \psi$ in Ω, the compatibility condition,

$$\psi(x) \leqq \int_\Omega u_0^*(y)Q(dy, x) \qquad \text{on } \partial\Omega$$

is satisfied.

First assume

$$h(x) = \int_\Omega u_0^*(y)Q(dy, x) \in C^1(\Omega) \cap C(\bar{\Omega}) \tag{2.9}$$

and

$$h(x) = \int_\Omega u_0^*(y)Q(dy, x) > \psi(x) \qquad \text{on } \partial\Omega . \tag{2.10}$$

In this case, problem (2.7)–(2.8) is equivalent to

$$\min\{-g\cdot\nabla w_1 + (\alpha+\lambda)w_1 - \tilde{f}, w_1 - \tilde{\psi}\} = 0 \qquad \text{in } \Omega \tag{2.11}$$

$$w_1(x) = 0 \qquad \text{on } \partial\Omega , \tag{2.12}$$

where $\tilde{f}$, $\tilde{\psi}$ satisfy the same properties as f, ψ and $\tilde{\psi} < 0$ on $\partial\Omega$. For simplicity, we write $f = \tilde{f}$, $\psi = \tilde{\psi}$. We show the existence of a solution to (2.11)–(2.12) by Perron's method. Indeed, the solution of the linear equation

$$-g \cdot \nabla w + (\alpha + \lambda) w = f \qquad \text{in } \Omega ,$$

$$w = 0 \qquad \text{on } \partial\Omega$$

is a subsolution of (2.11)–(2.12).

To construct a supersolution, we follow a barrier construction argument from Oleinik-Radkevic [17] as in Ishii and Koike [13]. Since $\psi < 0$ on $\partial\Omega$, there exists a local barrior, ψ_z in $C(\Omega \cap V_z) \cap C^2(\Omega \cap V_z)$ where $z \in \partial\Omega$, V_z is a sufficiently small neighborhood of z,

$$\psi_z(z) = 0 , \qquad \psi_z \geqq 0 \qquad \text{on } \overline{\Omega \cap V_z} ,$$

$$\psi_z \geqq \frac{\|f\|_\infty}{\alpha_0} \qquad \text{on } \bar{\Omega} \cap \partial V_z ,$$

$$-g \cdot \nabla \psi_z + (\alpha + \lambda)\psi_z \geqq f \qquad \text{in } \Omega \cap V_z , \qquad \text{and}$$

$$\psi_z \geqq \psi \qquad \text{in } \Omega \cap V_z .$$

Define

$$\hat{\psi}_z(x) = \begin{cases} \max\left\{\psi_z(x), \max\left\{\dfrac{\|f\|_\infty}{\alpha_0}, \|\psi\|_\infty\right\}\right\} & \text{in } \Omega \cap V_z \\ \max\left\{\dfrac{\|f\|_\infty}{\alpha_0}, \|\psi\|_\infty\right\} & \text{otherwise} , \end{cases}$$

and

$$\hat{\psi}(x) = \inf\{\hat{\psi}_z(x) | z \in \partial\Omega\} .$$

Then $\hat{\psi}$ is a supersolution. This implies that there exists a continuous solution of (2.7)–(2.8).

For general continuous boundary values h, not necessarily satisfying (2.9)–(2.10), we choose an approximating sequence $\{h_n\}$ such that $h_n \in C(\bar{\Omega}) \cap C^1(\Omega)$, $h_n > \psi$ on $\partial\Omega$ and $h_n \to h$ uniformly on $\bar{\Omega}$. Let u_n be a solution of (2.7)–(2.8) associated with boundary value h_n. By standard comparison argument, we have

$$\sup |u_n(x) - u_m(x)| \leqq \sup |h_n(x) - h_m(x)| .$$

Hence $\{u_n\}$ converges to some $u \in C(\bar{\Omega})$ and by stability of viscosity solutions, we have that u is a solution of (2.7)–(2.8).

By the comparison result for obstacle problems, we have $u_1 \leqq W$. Hence by Theorem 1.1, u_0 satisfies the boundary condition (0.3). The uniqueness follows from Lenhart [14]. ∎

As a third example of using Theorem 1.1 to get existence results, we consider a system associated with a switching control of PD processes. We will approximate the system by decoupled equations which we solve by using Theorem 1.1.

A finite collection of PD processes, indexed by $a \in \{1, \dots, m\}$, are controlled in order to minimize the expected value of a cost functional with continuous and switching costs. The associated dynamic programming equation is the following system:

$$u = (u^1, \dots, u^m)$$

$$\max\{L^a u^a - f^a, u^a - M^a u\} = 0 \qquad \text{in } \Omega, \tag{2.13}$$

$$u^a(x) = \int_\Omega u^a(y) Q^a(dy, x), \qquad x \in \partial\Omega, \tag{2.14}$$

where

$$L^a u^a = -g^a \cdot \nabla u^a + \alpha^a u^a - \lambda^a \int_\Omega (u^a(y) - u^a(x)) Q^a(dy, x)$$

and

$$M^a u(x) = \min_{b \neq a} \{u^b(x) + k(a, b)(x)\}.$$

See Lenhart [14] and Lenhart and Liao [15] for more background on this control problem and for $W^{1,\infty}$ solutions to this system under stronger assumptions (in particular, α_0 is sufficiently large and the switching costs are constants). There u^a is the minimum cost functions with initial control setting a, meaning that the state process is initially a PD process with dynamics g^a, jump rate λ^a, and transition measure Q^a.

We make the following assumptions for all $a \in \{1, 2, \dots, m\}$.

(H.2)′ $g^a: \Omega \to \boldsymbol{R}^n$ is Lipschitz continuous and $f^a, \lambda^a, \alpha^a: \Omega \to \boldsymbol{R}$ are continuous.

(H.3)′ There exists $\alpha_0 > 0$ such that $\alpha^a(x) \geqq \alpha_0 > 0$ on Ω.

(H.4)′ $\lambda^a > 0$ on Ω, $f^a \geqq 0$ on Ω.

(H.5)′ $Q^a(\cdot, x)$ is a probability measure on Ω, satisfying (A.4), (A.5), and

$$Q^a(dy, x) = Q^1(dy, x) \qquad \text{for all } x \in \partial\Omega.$$

(H.6)′ $g^a \cdot \eta > 0$ on $\partial\Omega$ where $\eta = \eta(x)$ is the outward normal at $x \in \partial\Omega$.

(H.7)′ The switching costs are continuous, strictly positive on $\bar{\Omega}$ and strictly subadditive, i.e.,

$$k(a, c) < k(a, b) + k(b, c) \qquad \text{for all } a, b, c \in \{1, \dots, m\} \text{ and for all } x \in \bar{\Omega}.$$

The agreement of the measures on $\partial\Omega$ in assumption (H.5)′ is necessary for the compatibility of the boundary condition and the implicit obstacles, $M^a u$. Hereafter, we denote $Q(\cdot, x) = Q^1(\cdot, x)$, $x \in \partial\Omega$ for simplicity.

Theorem 2.3. *Under assumptions* (H.1), (H.2)′–(H.7)′, *there exists a unique solution to system* (2.13) *satisfying boundary condition* (2.14).

Proof. The comparison and uniqueness results in Lenhart [14] can be applied to the case here.

Define two maps,

$$\sigma,\ M: (C(\Omega))^m \to (C(\Omega))^m$$

by

$$M(v) = \left(\min_{b \neq 1} \{v^b + k(1, b)\}, \dots, \min_{b \neq m} \{v^b + k(m, b)\}\right)$$

and $\sigma(v) = w$ where w is the viscosity solution of

$$\max \{L^a w^a - f^a, w^a - v^a\} = 0 \qquad \text{in } \Omega,$$

$$w^a(x) = \int_\Omega w^a(y) Q(dy, x), \qquad x \in \partial\Omega$$

(using Theorem 2.1 to get existence of w). The map $S = \sigma \circ M$ is increasing and concave. Define a supersolution, u_0, to (2.13) as the solution of

$$L^a u_0^a = f^a \qquad \text{in } \Omega$$

$$\text{with} \qquad u_0^a = \int_\Omega u_0^a(y) Q(dy, x) \qquad \text{on } \partial\Omega.$$

There exists $\gamma > 0$ such that

$$\gamma u_0^b \leqq \inf_{a,x} k(a, b)(x) \qquad \text{for all } b \in \{1, \dots, m\}.$$

Each component of $S(0)$ and u_0 satisfies

$$S(0) \geqq \sigma(\gamma u_0) \geqq \gamma\sigma(u_0) = \gamma u_0. \tag{2.15}$$

Thus we have the following property:

$$\begin{aligned} &\text{if } v - w \leqq \beta v \text{ (componentwise) for some } \beta \in [0, 1], \\ &\text{then } Sv - Sw \leqq (1 - \gamma)\beta Sv \text{ (componentwise)}. \end{aligned} \tag{2.16}$$

Property (2.16) is proved by starting with

$$(1 - \beta)v + \beta \cdot 0 \leqq w,$$

applying S to both sides, and using concavity and (2.15). See [1, 10] for similar arguments. Define $u_n = S^n u_0$. Using (2.16), first on

$$u_0 - u_1 \leqq u_0 \qquad \text{to obtain} \qquad u_1 - u_2 \leqq (1-\gamma)u_1$$

and then inductively to obtain

$$u_{n-1} - u_n \leqq (1-\gamma)^{n-1} u_0 . \tag{2.17}$$

By (2.17), we conclude the sequence converges uniformly,

$$u_n^k \searrow u^k \qquad \text{as } n \to \infty .$$

The uniform convergence insures the convergence of the integral terms and the implicit obstacles, and that u is the desired solution to (2.13)–(2.14). ■

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