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Viscosity Solutions of a Degenerate Parabolic Equation with Measures as Initial Data

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1. Introduction

We consider the initial-boundary value problem

$$(1.1) \quad u_t = uu_{xx} - \lambda |u_x|^2 \quad a < x < b, \quad t > 0$$

$$(1.2) \quad u(a, t) = u(b, t) = 0 \quad t > 0$$

$$(1.3) \quad u(x, 0) = u_0(x) \quad a < x < b$$

where λ is a nonnegative constant and u_0 is a nonnegative Radon measure on $[a, b]$.

We are interested in nonnegative viscosity solutions which are constructed by the well-known viscosity method: We approximate equation (1.1) by adding the viscosity term εu_{xx} for $\varepsilon > 0$ to the right hand side of (1.1) and let ε tend to zero. Bertch, Dal Passo and Ughi [2] established that the Cauchy problem for (1.1) with bounded continuous nonnegative initial function has a viscosity solution and showed that the support of the solution remain constant as time goes. The constancy of support is somewhat unexpected if you consider the interface equation formally (see [5]). It was shown by Ughi [8] and by Dal Passo and Luckhaus [4] that weak solutions of the Cauchy or initial boundary value problem for (1.1) are not uniquely determined by their initial data in general. Using a finite difference approximation, Kaji and the author [6] have constructed weak solutions whose supports shrink as time develops.

Our purpose of this paper is to establish the global existence of viscosity solutions of (1.1)–(1.3) for nonnegative Radon measures and the smoothing property of solutions.

Let X denote the Banach space $\{u \in W^{2,\infty}([a, b]) \mid u(a) = u(b) = 0\}$ with usual norm and X^* is the strong dual of X .

Our main result is as follows:

Theorem 1. *Let $\lambda > \frac{1}{3}$. For any nonnegative Radon measure $u_0 \in X^*$ and any $T > 0$ there exists at least one nonnegative function $u(x, t)$ such that*

$$\begin{aligned}
u &\in L^2((a, b) \times (0, T)) \\
t^{1/2}u_x &\in L^2((a, b) \times (0, T)) \\
tu_{xx} &\in L^2((a, b) \times (0, T)) \\
t^{21/16}u_t &\in L^2((a, b) \times (0, T)) \\
t^{1/2}u &\in L^\infty([0, T]; L^1(a, b)) \\
t^{5/8}u &\in L^\infty((a, b) \times (0, T))
\end{aligned}$$

satisfying

$$u_t = uu_{xx} - \lambda|u_x|^2 \quad \text{a.e. } (x, t) \in (a, b) \times (0, \infty).$$

Moreover for any nonnegative function $\phi \in C^2[a, b]$ with $\phi(a) = \phi'(a) = \phi(b) = \phi'(b) = 0$,

$$\begin{aligned}
\phi u &\in L^\infty([0, T]; L^1(a, b)) \\
\sqrt{\phi}u_x &\in L^2((a, b) \times (0, T))
\end{aligned}$$

and

$$\lim_{t \rightarrow 0} \int_a^b \phi(x)u(x, t)dx = \int_a^b \phi(x)u_0(x)dx.$$

Remark 1. If u_0 is in $L^1(a, b)$, then we have a nonnegative viscosity solution $u(x, t)$ satisfying

$$\begin{aligned}
u &\in L^\infty((0, \infty); L^1(a, b)) \\
u_x &\in L^2((a, b) \times (0, T)) \\
tu_{xx} &\in L^2((a, b) \times (0, T)) \\
t^{1/2}u_t &\in L^2((a, b) \times (0, T))
\end{aligned}$$

and

$$\lim_{t \rightarrow 0} \int_a^b u(x, t)dx = \int_a^b u_0(x)dx.$$

2. Proof of Theorem

For any $\varepsilon > 0$ let $u_\varepsilon(x, t)$ be a classical nonnegative solution of the initial boundary value problem

$$(2.1) \quad u_t = (u + \varepsilon)u_{xx} - \lambda|u_x|^2 \quad a < x < b, \quad t > 0$$

$$\begin{aligned} u(a, t) &= u(b, t) = 0 & t > 0 \\ u(x, 0) &= u_0^\varepsilon(x) & a < x < b \end{aligned}$$

where λ is a nonnegative constant and u_0^ε is a nonnegative smooth function such that

$$u_0^\varepsilon(a) = u_0^\varepsilon(b) = 0$$

and

$$u_0^\varepsilon \rightarrow u_0 \quad \text{weakly* in } X^* \text{ as } \varepsilon \rightarrow 0.$$

Lemma 1. *We have for $0 < \varepsilon < \varepsilon_0$ and $0 < t < T$*

$$(2.2) \quad \int_0^t \int_a^b \{u_\varepsilon(x, s) + \varepsilon\}^2 dx ds \leq C$$

where and in the sequel C depends only on u_0 , ε_0 and T .

Proof. For the sake of simplicity we write u instead of u_ε here and in the sequel in proofs. Set $\varphi(x) = (b - x)(x - a)$. Multipling both side of the equation (2.1) by $\varphi(x)$, integrating by parts with respect to x and integrating in t , we obtain

$$\begin{aligned} & \int_a^b \varphi(x) u(x, t) dx + \int_0^t \int_a^b (u + \varepsilon)^2 dx ds + (\lambda + 1) \int_0^t \int_a^b \varphi u_x^2 dx dt \\ &= \int_a^b \varphi(x) u_0^\varepsilon(x) dx + (b - a) \varepsilon^2 t. \end{aligned}$$

from which the estimate (2.2) is derived.

Lemma 2. *We have*

$$(2.3) \quad \int_a^b u_\varepsilon(x, t) dx \leq C t^{-1/2}$$

and

$$(2.4) \quad \int_0^t s \int_a^b u_{\varepsilon x}^2(x, s) dx ds \leq C t^{1/2}.$$

Proof. Multipling (2.1) by t and integrating in x , we get

$$\frac{d}{dt} \left(t \int_a^b u dx \right) - \int_a^b u dx + (\lambda + 1) t \int_a^b u_x^2 dx = t \varepsilon \{u_x(b, t) - u_x(a, t)\} \leq 0$$

since $u_x(a, t) \geq 0$ and $u_x(b, t) \leq 0$. Integration with respect to t gives

$$\begin{aligned}
t \int_a^b u(x, t) dx + (\lambda + 1) \int_0^t s \int_a^b u_x^2(x, s) dx ds &\leq \int_0^t \int_a^b u(x, s) dx ds \\
&\leq \left(\int_0^t \int_a^b u^2(x, s) dx ds \right)^{1/2} t^{1/2} (b - a)^{1/2} \leq Ct^{1/2}
\end{aligned}$$

from which we deduce (2.3) and (2.4).

Lemma 3. *We have*

$$(2.5) \quad \int_a^b u_\varepsilon^2(x, t) dx \leq Ct^{-1}$$

Proof. We multiply (2.1) by tu and integrate by parts in x to get

$$\frac{d}{dt} \left(t \int_a^b u^2 dx \right) + 2t(\lambda + 2) \int_a^b uu_x^2 dx + 2t\varepsilon \int_a^b u_x^2 dx = \int_a^b u^2 dx.$$

Integrating in t and using (2.2), we obtain (2.5).

Lemma 4.

$$(2.6) \quad \int_a^b u_{\varepsilon x}^2(x, t) dx \leq Ct^{-3/2},$$

$$(2.7) \quad \int_0^t s^2 \int_a^b (u_\varepsilon + \varepsilon) u_{\varepsilon xx}^2 dx ds \leq Ct^{1/2}$$

and

$$(2.8) \quad \|u_\varepsilon(t)\|_{L^\infty} \leq Ct^{-5/8}.$$

Proof. Multiplying (2.1) by $t^2 u_{xx}$ and integrating by parts in x , we have

$$\begin{aligned}
&\frac{d}{dt} \left(t^2 \int_a^b u_x^2 dx \right) + 2t^2 \int_a^b (u + \varepsilon) u_{xx}^2 dx, \\
&= 4t \int_a^b u_x^2 dx + \frac{2\lambda}{3} \{u_x^3(b, t) - u_x^3(a, t)\} t^2 \\
&\leq 4t \int_a^b u_x^2 dx.
\end{aligned}$$

Integrating in t and making use of (2.4), we obtain

$$t^2 \int_a^b u_x^2(x, t) dx + 2 \int_0^t s^2 \int_a^b (u + \varepsilon) u_{xx}^2 dx ds \leq 4 \int_0^t s \int_a^b u_x^2 dx ds \leq Ct^{1/2}$$

from which (2.6) and (2.7) follow. The last assertion follows from

$$\|u\|_{L^\infty} \leq C \|u_x\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2} \leq Ct^{-5/8}$$

Here we have used (2.5) and (2.6).

Lemma 5. *We have*

$$(2.9) \quad u_{\varepsilon t} \geq -Ct^{-13/8}$$

Proof. Put $p = u_t/(u + \varepsilon)$. We see that p satisfies

$$p_t = (u + \varepsilon)p_{xx} + 2(1 - \lambda)u_x p_x + p^2$$

with

$$p(a, t) = p(b, t) = 0$$

and

$$p(x, 0) = u_{0xx}^\varepsilon - \frac{\lambda |u_{0x}^\varepsilon|^2}{u_0^\varepsilon + \varepsilon}.$$

The well-known maximum principle yields that

$$p \geq -\frac{1}{t}$$

(see Aronson-Benilan[1]). Hence we obtain

$$u_t \geq -\frac{u + \varepsilon}{t} \geq -Ct^{-13/8}.$$

Lemma 6.

$$(2.10) \quad \int_0^t s^{21/8} \int_a^b u_{\varepsilon x}^4 dx ds \leq Ct^{1/2},$$

$$(2.11) \quad \int_0^t s^{21/8} \int_a^b u_{\varepsilon t}^2 dx ds \leq Ct^{1/2}(\varepsilon t^{5/8} + 1).$$

Proof. Multiplication of (2.1) by u_x^2 and integration by parts in x gives

$$\begin{aligned} \int_a^b u_t u_x^2 dx &= \int_a^b (u + \varepsilon) u_x^2 u_{xx} dx - \lambda \int_a^b u_x^4 dx \\ &= \frac{1}{3} \varepsilon \{u_x^3(b, t) - u_x^3(a, t)\} - \left(\lambda + \frac{1}{3}\right) \int_a^b u_x^4 dx \\ &\leq -\left(\lambda + \frac{1}{3}\right) \int_a^b u_x^4 dx. \end{aligned}$$

Making use of Lemma 5, we have

$$\left(\lambda + \frac{1}{3}\right) \int_a^b u_x^4 dx \leq - \int_a^b u_t u_x^2 dx \leq Ct^{-13/8} \int_a^b u_x^2 dx.$$

Hence we obtain

$$\int_0^t s^{21/8} \int_a^b u_x^4 dx ds \leq C \int_0^t s \int_a^b u_x^2 dx ds \leq Ct^{1/2}.$$

We have

$$\begin{aligned} u_t^2 &= [(u + \varepsilon)u_{xx} - \lambda u_x^2]^2 \\ &\leq 2(u + \varepsilon)^2 u_{xx}^2 + 2\lambda^2 u_x^4. \end{aligned}$$

Hence, from (2.7), (2.8) and (2.10) we get

$$\begin{aligned} \int_0^t s^{21/8} \int_a^b u_t^2 dx ds &\leq 2 \sup_{0 \leq s \leq t} s^{5/8} \|u(s) + \varepsilon\|_{L^\infty} \int_0^t s^2 \int_a^b (u + \varepsilon) u_{xx}^2 dx ds \\ &\quad + 2\lambda^2 \int_0^t s^{21/8} \int_a^b u_x^4 dx ds \\ &\leq Ct^{1/2}(\varepsilon t^{5/8} + 1). \end{aligned}$$

Lemma 7. *If $\lambda > 0$, then*

$$(2.12) \quad \int_a^b \frac{u_{\varepsilon x}^2(x, t)}{u_\varepsilon(x, t) + \varepsilon} dx \leq Ct^{-1}.$$

If $\lambda > \frac{1}{3}$, then

$$(2.13) \quad \int_0^t s^2 \int_a^b u_{\varepsilon xx}^2 dx ds \leq Ct.$$

Proof. From (2.1) we have

$$\begin{aligned} \int_a^b \frac{u_t}{u + \varepsilon} dx &= \int_a^b u_{xx} dx - \lambda \int_a^b \frac{u_x^2}{u + \varepsilon} dx \\ &= u_x(b, t) - u_x(a, t) - \lambda \int_a^b \frac{u_x^2}{u + \varepsilon} dx \leq -\lambda \int_a^b \frac{u_x^2}{u + \varepsilon} dx. \end{aligned}$$

On the other hand we see that

$$p = \frac{u_t}{u + \varepsilon} \geq -\frac{1}{t}$$

from which it follows that

$$\int_a^b \frac{u_t}{u + \varepsilon} dx \geq -\frac{b-a}{t}.$$

Hence we obtain (2.12).

Multiplying (2.1) by $u_{xx}/(u + \varepsilon)$ and integrating by parts in x , we get

$$\begin{aligned} \int_a^b u_t \frac{u_{xx}}{u + \varepsilon} dx &= \int_a^b u_{xx}^2 dx - \lambda \int_a^b u_x^2 \frac{u_{xx}}{u + \varepsilon} dx \\ &= \int_a^b u_{xx}^2 dx - \frac{\lambda}{3\varepsilon} [u_x^3(b, t) - u_x^3(a, t)] - \frac{\lambda}{3} \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx \\ &\geq \int_a^b u_{xx}^2 dx - \frac{\lambda}{3} \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx. \end{aligned}$$

The left hand side is calculated as

$$\int_a^b u_t \frac{u_{xx}(x, t)}{u(x, t) + \varepsilon} dx = -\frac{1}{2} \frac{d}{dt} \int_a^b \frac{u_x^2}{u + \varepsilon} dx + \frac{1}{2} \int_a^b u_t \frac{u_x^2}{(u + \varepsilon)^2} dx.$$

Hence we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_a^b \frac{u_x^2}{u + \varepsilon} dx + \int_a^b u_{xx}^2 dx &\leq \frac{\lambda}{3} \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx + \frac{1}{2} \int_a^b u_t \frac{u_x^2}{(u + \varepsilon)^2} dx \\ &= \frac{\lambda}{3} \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx \\ &\quad + \frac{1}{2} \left[\int_a^b u_x^2 \frac{u_{xx}}{u + \varepsilon} dx - \lambda \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx \right] \\ &\leq \frac{(1 - \lambda)}{6} \int_a^b \frac{u_x^4}{(u + \varepsilon)^2} dx. \end{aligned}$$

Since

$$\frac{u_t}{u + \varepsilon} \geq -\frac{1}{t},$$

we get

$$u_{xx} + \frac{1}{t} \geq \lambda \frac{u_x^2}{u + \varepsilon}.$$

Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_a^b \frac{u_x^2}{u + \varepsilon} dx + \int_a^b u_{xx}^2 dx &\leq \frac{1 - \lambda}{6\lambda^2} \int_a^b \left(u_{xx} + \frac{1}{t} \right)^2 dx \\ &\leq \frac{1 - \lambda}{6\lambda^2} \left[\int_a^b u_{xx}^2 dx + \frac{b - a}{t^2} \right]. \end{aligned}$$

Multiplying this inequality by t^2 and integrating with respect to t , we finally obtain

$$\begin{aligned} \int_a^b t^2 \frac{u_x^2(x, t)}{u(x, t) + \varepsilon} dx + \frac{(3\lambda - 1)(2\lambda + 1)}{3\lambda^2} \int_0^t s^2 \int_a^b u_{xx}^2 dx ds \\ \leq \frac{1 - \lambda}{3\lambda^2} (b - a)t + 2 \int_0^t s \int_a^b \frac{u_x^2}{u + \varepsilon} dx ds \leq Ct. \end{aligned}$$

Thus we have (2.13) if $\lambda > \frac{1}{3}$.

Lemma 8. For any nonnegative $\phi \in C_c^2([a, b])$ we have

$$(2.14) \quad \int_a^b \phi(x) u_\varepsilon(x, t) dx \leq C,$$

and

$$(2.15) \quad \int_0^t \int_a^b \phi u_{\varepsilon x}^2(x, t) dx dt \leq C.$$

Proof. In the same way as in Lemma 1 we obtain

$$\begin{aligned} (2.16) \quad \int_a^b \phi(x) u(x, t) dx - \int_a^b \phi(x) u_0^\varepsilon(x) dx + \int_0^t \int_a^b \phi''(x) (u + \varepsilon)^2 dx ds \\ + (\lambda + 1) \int_0^t \int_a^b \phi u_x^2 dx dt = 0. \end{aligned}$$

Using Lemma 1 we get (2.14) and (2.15).

Proof of Theorem completed. From Lemma 1, Lemma 2, Lemma 6 and Lemma 7 it follows that $t^{21/16} u_\varepsilon$ is bounded in $H^1([0, T] \times [a, b])$ and in $L^2(0, T; H^2(a, b))$ since $(t^{21/16} u)_t = t^{21/16} u_t + \frac{21}{16} t^{5/16} u$. Hence, by virtue of Aubin's compactness lemma (see Lions [7] p. 58), we see that there is a function u with $t^{21/16} u \in H^1([0, T] \times [a, b]) \cap L^2(0, T; H^2(a, b))$ and a subsequence u_{ε_j} of $\{u_\varepsilon\}$ such that as $j \rightarrow \infty$

$$\begin{aligned} t^{21/16} u_{\varepsilon_j} &\rightarrow t^{21/16} u \quad \text{weakly in } H^1([0, T] \times [a, b]) \cap L^2(0, T; H^2(a, b)), \\ t^{21/16} u_{\varepsilon_j} &\rightarrow t^{21/16} u \quad \text{strongly in } L^2(0, T; H^1(a, b)) \end{aligned}$$

and

$$u_{\varepsilon_j} \rightarrow u \quad \text{and} \quad \frac{\partial}{\partial x} u_{\varepsilon_j} \rightarrow \frac{\partial}{\partial x} u \quad \text{almost everywhere } (0, T) \times (a, b).$$

Therefore the standard argument shows that we may assume that

$$(u_{\varepsilon_j} + \varepsilon_j)u_{\varepsilon_j xx} \rightarrow uu_{xx} \quad \text{weakly in } L^2([\tau, T] \times [a, b]) \text{ for any } \tau > 0.$$

It is easily seen that u satisfies the equation (1.1).

Letting ε tend to 0 in (2.16), we have for any nonnegative $\phi \in C_c^2([a, b])$

$$\begin{aligned} & \int_a^b \phi(x)u(x, t)dx - \int_a^b \phi(x)u_0(x)dx + \int_0^t \int_a^b \phi''(x)u^2 dx ds \\ & + (\lambda + 1) \int_0^t \int_a^b \phi u_x^2 dx dt = 0. \end{aligned}$$

Hence

$$\lim_{t \rightarrow 0} \int_a^b \phi(x)u(x, t)dx = \int_a^b \phi(x)u_0(x)dx$$

for any nonnegative $\phi \in C_c^2([a, b])$. Thus u is a desired solution. This completes the proof.

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