



On the Competition-Diffusion Equations for Closely Competing Species

Kahane, S. Charles

(Citation)

Funkcialaj Ekvacioj, 35(1):51-64

(Issue Date)

1992-04

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100499743>

(URL)

<https://hdl.handle.net/20.500.14094/0100499743>



On the Competition-Diffusion Equations for Closely Competing Species

By

Charles S. KAHANE

(Vanderbilt University, U.S.A.)

1. Introduction

The purpose of this paper is to study the asymptotic behavior of the solutions of a system of parabolic equations which models the growth of two competing species that are very much alike, compete on roughly equal terms and migrate over a geographic region Ω . With $u(x, t)$ and $v(x, t)$ ($(x, t) \in \mathbf{R}^n \times [0, \infty)$) denoting the population densities of the two species, the system of equations in question is

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - a\Delta u = ju(m - u - \alpha v) \\ \frac{\partial v}{\partial t} - a\Delta v = jv(m - v - \alpha u) \end{cases} \quad \text{in } \Omega \times (0, \infty)$$

where Ω denotes a bounded domain in $\mathbf{R}^n$ and a, j, m and α are all positive constants.

The Laplacian terms on the left are designed to model the idea that each species migrates over Ω in a more or less random fashion.

The growth laws on the right are of logistic type. To see what they mean, we multiply out the first of them, obtaining

$$jmu - ju^2 - j\alpha uv;$$

the leading term, jmu , represents the so-called “natural” growth rate for the species (the growth rate if the food supply was unlimited and in the absence of any competition); the next term, $-ju^2$, is the death rate due to competition between the members of the same species; while the last term, $-j\alpha uv$, represents the death rate due to competition between the two species. There is, of course, a similar interpretation for the right side of the second equation. The fact that the constants a, j and m are the same for both equations is then interpreted to mean that the migration and growth characteristics for the two species are exactly the same; the fact that the constant α is the same in both

equations is interpreted to mean that the two species compete with each other on roughly equal terms. (For a discussion of the biological background see Okubo [10], Pielou [13], Rescigno and Richardson [15] and Smith [16].)

In connection with (1.1) it is natural to consider the boundary-initial value problem corresponding to the boundary conditions

$$(1.2) \quad \frac{\partial u}{\partial n} = 0 \quad \text{and} \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

which mean that both species are confined to Ω ; together with the initial conditions

$$(1.3) \quad u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x) \quad \text{for } x \in \Omega.$$

Of course, since u_0 and v_0 represent population densities, we assume that they are both non-negative; by the maximum principle [14] it will then follow that $u(x, t)$ and $v(x, t)$ are automatically non-negative for all subsequent times $t > 0$.

There is no difficulty in showing that the solution of (1.1)–(1.3) for non-negative bounded initial data exists for all time. The question of interest is then: How does this solution behave asymptotically? Or, in biological terms: What are the fates of u and v ?

The answer depends on α which measures the intensity of the competition between the two species. When $\alpha < 1$, we can show that “peaceful co-existence” prevails; meaning that (u, v) converges to the stable equilibrium solution $(m/(1 + \alpha), m/(1 + \alpha))$, i.e.

$$(u, v) \rightarrow (m/(1 + \alpha), m/(1 + \alpha)) \quad \text{as } t \rightarrow \infty$$

uniformly in Ω , independent of the initial data (u_0, v_0) .

The situation becomes much more complicated as the competition intensifies, i.e. when $\alpha > 1$. In this case one expects that the asymptotic behavior will depend on the initial data. In particular, it is reasonable to conjecture that when one species initially dominates the other, in the sense that, say

$$(1.4) \quad u_0(x) > v_0(x) \quad \text{in } \Omega,$$

then the initially dominant species will win the competition by extinguishing the other species and growing to its maximally sustainable size, i.e.

$$(1.5) \quad v(x, t) \rightarrow 0 \quad \text{and} \quad u(x, t) \rightarrow m \quad \text{as } t \rightarrow \infty,$$

uniformly in Ω .

In some earlier work [5] the writer has shown that (1.5) follows from (1.4) under a further hypothesis regarding the initial data (u_0, v_0) ; namely the data

was assumed to satisfy the “smallness” condition:

$$u_0(x) + v_0(x) \leq m \quad \text{in } \Omega.$$

Since this condition does not appear to be intrinsically necessary, one would like to establish (1.5) solely in the basis of the hypothesis (1.4). And it is the goal of the present paper to accomplish this. More precisely, our objective will be to prove

Theorem 1. *For the solution (u, v) of (1.1)–(1.3) with non-negative bounded initial data (u_0, v_0) satisfying the condition*

$$(1.6) \quad u_0(x) \geq v_0(x) \text{ in } \Omega \text{ with } u_0 \not\equiv v_0 \text{ in } \Omega,$$

we have

$$(1.7) \quad v \rightarrow 0 \quad \text{and} \quad u \rightarrow m \quad \text{as } t \rightarrow \infty,$$

uniformly in Ω .

The proof will be based on comparison theorems derived from the maximum principle; and the remainder of the paper will be devoted to a description of this proof together with the supporting material needed for it.

We have arranged the discussion as follows: Section 2 contains preparatory material needed in the sequel; in particular, we will obtain some preliminary estimates for the solutions of (1.1)–(1.3) independent of the nature of the initial data. In Section 3 we will turn our attention to the solutions of (1.1)–(1.3) for which the initial data satisfies the condition (1.6): $u_0(x) \geq v_0(x)$ in Ω with $u_0 \not\equiv v_0$ in Ω . The principal result of that section will be the proof that under condition (1.6) v is extinguished, i.e. $v \rightarrow 0$ as $t \rightarrow \infty$ uniformly in Ω ; in fact we will show that $v \rightarrow 0$ as an exponential rate. Having established this, it is a relatively easy matter to show that $u \rightarrow m$ as $t \rightarrow \infty$, uniformly in Ω ; and this will complete the proof of Theorem 1.

The study of the asymptotic behavior of solutions of competition diffusion equations has in recent years attracted a great deal of attention. In particular we mention the work of Brown [1], Levin [7], Leung [8], Matano and Mimura [9], Pao [11] and Pao and Zhou [12].

2. Preliminaries

In this section we gather together various comparison and positivity results for parabolic inequalities that we will need in the sequel and then apply them to obtain preliminary estimates for the solutions of (1.1)–(1.3).

We begin with the following positivity result whose proof can be found in [3] or [14].

Theorem 2.1. *Let $w(x, t)$ be a classical solution of*

$$(2.1) \quad \begin{cases} \frac{\partial w}{\partial t} - c\Delta w \geq qw & \text{in } \Omega \times (0, T], \\ \frac{\partial w}{\partial n} \geq 0 & \text{on } \partial\Omega \times (0, T] \quad \text{and} \\ w|_{t=0} = w_0(x) & \text{initially for } x \in \Omega; \end{cases} \quad \text{with}$$

where c is a positive constant and $q(x, t)$ is assumed to be bounded in $\Omega \times (0, T]$. Suppose that initially we have

$$(2.2) \quad w_0(x) \geq 0 \quad \text{in } \Omega,$$

then subsequently we will have

$$(2.3) \quad w(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times (0, T].$$

Furthermore, if w_0 is not identically zero in Ω , then w will actually be strictly positive at all subsequent times, i.e.

$$(2.4) \quad w(x, t) > 0 \quad \text{for } (x, t) \in \bar{\Omega} \times (0, T].$$

Remark. For the proof of this we require that Ω have the inside sphere property. Namely, for any point p of $\partial\Omega$, there should exist a closed ball B containing p on its boundary such that the ball B except for the point p lies entirely in Ω . This is in the nature of a smoothness requirement on $\partial\Omega$, and we will assume that it holds throughout the remainder of the paper.

By a classical solution w of (2.1) we mean a function in $C(\bar{\Omega} \times [0, T])$ which has continuous derivatives up to the second order with respect to the space variables x in $\Omega \times (0, T]$, as well as a continuous t derivative in that region; the inequality $\frac{\partial w}{\partial t} - c\Delta w \geq qw$ is then understood to be satisfied pointwise in $\Omega \times (0, T]$. In regards to the satisfaction of the boundary condition, we assume that for each fixed $t \in (0, T]$, w is a C^1 function with respect to the space variables in $\Omega \times \{t\}$, up to and including the boundary $\partial\Omega \times \{t\}$ with the normal derivative there assumed to be non-negative pointwise: $\frac{\partial w}{\partial n} \geq 0$ at each point of $\partial\Omega \times \{t\}$.

The preceding positivity theorem leads to the following comparison result.

Theorem 2.2. Suppose that u and $v \in C(\bar{\Omega} \times [0, T])$ are solutions of

$$(2.5) \quad \begin{cases} \frac{\partial u}{\partial t} - c\Delta u \geq F(x, t, u) \\ \frac{\partial v}{\partial t} - c\Delta v \leq G(x, t, v) \end{cases} \quad \text{in } \Omega \times (0, T]$$

with

$$(2.6) \quad F(x, t, w) \geq G(x, t, w) \quad \text{in } \Omega \times (0, T] \times \mathbf{R};$$

and where F is assumed to have a derivative $F_u(x, t, u)$ which is continuous in $\bar{\Omega} \times [0, T] \times \mathbf{R}$. Assume also that

$$(2.7) \quad \frac{\partial u}{\partial n} \geq \frac{\partial v}{\partial n} \quad \text{on } \partial\Omega \times (0, T].$$

Denoting the initial values $u|_{t=0}$ and $v|_{t=0}$ by $u_0(x)$ and $v_0(x)$, respectively, suppose now that initially we have

$$(2.8) \quad u_0(x) \geq v_0(x) \quad \text{for } x \in \Omega,$$

then this implies that subsequently we will have

$$(2.9) \quad u(x, t) \geq v(x, t) \quad \text{for } (x, t) \in \bar{\Omega} \times (0, T].$$

Proof. Subtracting the equation (2.5) satisfied by v from that satisfied by u , we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(u - v) - c\Delta(u - v) &\geq F(x, t, u) - G(x, t, v) \\ &= [F(x, t, u) - F(x, t, v)] + [F(x, t, v) - G(x, t, v)] \\ &\geq F(x, t, u) - F(x, t, v) \quad \text{in } \Omega \times (0, T] \end{aligned}$$

because of (2.6).

Using the mean value theorem the difference on the right may be written in the form

$$F(x, t, u) - F(x, t, v) = q(x, t)(u - v) \quad \text{in } \Omega \times (0, T]$$

where in view of the assumptions made on F , together with the assumed continuity of u and v in $\bar{\Omega} \times [0, T]$, the coefficient function q is bounded in $\Omega \times (0, T]$. Since by (2.7), we also have $\frac{\partial}{\partial n}(u - v) \geq 0$ on $\partial\Omega \times (0, T]$, it follows that $u - v$ satisfies inequalities of precisely the form considered in Theorem

2.1. Invoking that Theorem, we conclude from the assumption (2.8)—that u is initially greater than or equal to v —that subsequently u will also be greater than or equal to v :

$$u(x, t) \geq v(x, t) \quad \text{for } (x, t) \in \bar{\Omega} \times (0, T];$$

and this completes the proof.

We are now going to apply the preceding results to obtain some preliminary estimates for the solutions of (1.1)–(1.3). For this purpose as well as for all our subsequent considerations of (1.1)–(1.3), it will be convenient to assume that the positive constant j figuring on the right of the system of equations (1.1) is equal to 1. We can always reduce to this situation by changing variables as follows:

$$\bar{u}(x, t) = u(x, j^{-1}t) \quad \text{and} \quad \bar{v}(x, t) = v(x, j^{-1}t);$$

the functions $\bar{u}$ and $\bar{v}$ will then satisfy exactly the same kind of equations that u and v satisfied, but with j now equal to 1 and the original diffusion constant a replaced by $j^{-1}a$ (the other constants m and α remaining the same).

Accordingly, from now on $(u(x, t), v(x, t))$ will be assumed to be solutions of

$$(2.10) \quad \begin{cases} \frac{\partial u}{\partial t} - a\Delta u = u(m - u - \alpha v) \\ \frac{\partial v}{\partial t} - a\Delta v = v(m - v - \alpha u) \end{cases} \quad \text{in } \Omega \times (0, \infty);$$

satisfying the boundary conditions

$$(2.11) \quad \frac{\partial u}{\partial n} = 0 \quad \text{and} \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty);$$

together with the initial condition

$$(2.12) \quad u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x) \quad \text{in } \Omega.$$

In what follows we will of course assume that u_0 and v_0 are non-negative as well as being bounded. The basic estimates for solutions of (2.10)–(2.12) which start out from such bounded, non-negative initial data are described in

Theorem 2.3. *Consider solutions (u, v) of the boundary-initial value problem (2.10)–(2.12). Suppose the initial data is non-negative and bounded from above in Ω :*

$$(2.13) \quad \begin{cases} 0 \leq u_0(x) \leq \sup_{\Omega} u_0(x) < \infty, \\ 0 \leq v_0(x) \leq \sup_{\Omega} v_0(x) < \infty \end{cases} \quad \text{for } x \in \Omega$$

Then the solution (u, v) of (2.10)–(2.12) will subsequently also be non-negative and bounded from above. More precisely we will have

$$(2.14) \quad \begin{cases} 0 \leq u(x, t) \leq \max[m, \sup_{\Omega} u_0(x)], \\ 0 \leq v(x, t) \leq \max[m, \sup_{\Omega} v_0(x)] \end{cases} \quad \text{for } (x, t) \in \bar{\Omega} \times (0, \infty).$$

Proof. The non-negativity of $u(x, t)$ and $v(x, t)$ for subsequent t in (2.14) follows by applying Theorem 2.1 separately to u and v ; after having observed that the equations and side conditions (2.10)–(2.12) satisfied by u and v are precisely of the form considered in Theorem 2.1.

To obtain the upper bound for u in (2.14), we note that, in view of the non-negativity of u and v , u will satisfy the inequality

$$(2.15) \quad \begin{aligned} \frac{\partial u}{\partial t} - a\Delta u &= u(m - u) - \alpha uv \\ &\leq u(m - u) \quad \text{in } \Omega \times (0, \infty). \end{aligned}$$

We now compare $u(x, t)$ with the solution $z(t)$ of the ordinary differential equation initial value problem

$$(2.16) \quad \begin{cases} \frac{dz}{dt} = z(m - z) & \text{for } t > 0 \\ z(0) = \sup_{\Omega} u_0(x) \end{cases}$$

Since we can regard z as a space independent solution of the parabolic boundary-initial value problem

$$\begin{cases} \frac{\partial z}{\partial t} - a\Delta z = z(m - z) & \text{in } \Omega \times (0, \infty); \\ \frac{\partial z}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ z|_{t=0} = \sup_{\Omega} u_0(x) & \text{in } \Omega, \end{cases}$$

it follows from Theorem 2.2, taking (2.15) into account, that

$$(2.17) \quad u(x, t) \leq z(t) \quad \text{for } (x, t) \in \bar{\Omega} \times (0, \infty).$$

But, for the solution $z(t)$ of the logistic equation problem (2.16), it is easily verified that (since $z(0) = \sup_{\Omega} u_0(x) \geq 0$)

$$z(t) \leq \max[m, z(0)] = \max[m, \sup_{\Omega} u_0(x)]$$

for $t > 0$; and combining this with (2.17) we obtain the desired upper estimate

for u in (2.14). The upper estimate for v in (2.14) is obtained in exactly the same way.

3. The extinction estimate

In this section we are going to study the asymptotic behavior of those solutions (u, v) of (2.10)–(2.12) with non-negative bounded initial data (u_0, v_0) satisfying the condition

$$(3.1) \quad u_0(x) \geq v_0(x) \geq 0 \text{ in } \Omega \quad \text{but with } u_0 \not\equiv v_0 \text{ in } \Omega.$$

Our aim will be to show that under this condition $v \rightarrow 0$ as $t \rightarrow \infty$. As a preliminary to doing so we will need the estimate describe in the following

Theorem 3.1. *Consider solutions (u, v) of (2.10)–(2.12) whose initial data satisfies (3.1). Then this same relation will persist subsequently; even better we will have*

$$(3.2) \quad u(x, t) > v(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times (0, \infty).$$

Moreover, we will then be able to bound u from below as follows:

$$(3.3) \quad u(x, t) \geq \min[m/(1 + \alpha), \inf_{\Omega} u_0(x)] \quad \text{in } \bar{\Omega} \times (0, \infty).$$

Proof. Subtracting the equations (2.10) that u and v satisfy, we find that $u - v$ satisfies

$$\frac{\partial}{\partial t}(u - v) - a\Delta(u - v) = (u - v)[m - (u + v)] \quad \text{in } \Omega \times (0, \infty).$$

In addition, $\frac{\partial}{\partial n}(u - v) = 0$ on $\partial\Omega \times (0, \infty)$ and $u - v$ is initially non-negative in Ω without being identically zero (because of our hypotheses (3.1)). We may therefore apply the “strong” part of Theorem 2.1 to conclude that $u - v$ will subsequently be strictly positive in $\bar{\Omega} \times (0, \infty)$; and this is the desired result (3.2).

To obtain the lower bound (3.3), we first insert the estimate (3.2) into the equation (2.10) satisfied by u . This leads to the inequality

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u &= u(m - u) - \alpha uv \\ &\geq u(m - u) - \alpha u^2 \\ &= (1 + \alpha)u[m/(1 + \alpha) - u] \end{aligned}$$

satisfied by u in $\Omega \times (0, \infty)$. Now, with this in mind, compare u with the

solution $z(t)$ of the ordinary differential equation initial value problem

$$(3.4) \quad \begin{cases} \frac{dz}{dt} = (1 + \alpha)z[m/(1 + \alpha) - z] & \text{for } t > 0 \\ z(0) = \inf_{\Omega} u_0(x) \end{cases}$$

Then (exactly as in the proof of Theorem 2.3), by regarding $z(t)$ as a space independent solution of a suitable parabolic boundary initial value problem, and invoking Theorem 2.2 we arrive at

$$u(x, t) \geq z(t) \quad \text{for } (x, t) \in \bar{\Omega} \times (0, \infty).$$

But for the solution $z(t)$ of (3.4) we can easily show (in view of $z(0) = \inf_{\Omega} u_0(x) \geq 0$) that

$$z(t) \geq \min[m/(1 + \alpha), z(0)] = \min[m/(1 + \alpha), \inf_{\Omega} u_0(x)]$$

for $t > 0$; and combining this with the preceding, (3.3) follows.

Let us now note that as a consequence of (3.2) we will have

$$(3.5) \quad \inf_{\Omega} u(x, t) > 0 \quad \text{and} \quad \sup_{\Omega} \left(\frac{v(x, t)}{u(x, t)} \right) < 1$$

for $t > 0$. This allows us to assume, without loss of generality, that the conditions

$$(3.6) \quad \inf_{\Omega} u_0(x) > 0 \quad \text{and} \quad \sup_{\Omega} \left(\frac{v_0(x)}{u_0(x)} \right) < 1$$

obtain for the initial data u_0 and v_0 . Otherwise, if these conditions do not hold, we merely replace u_0 and v_0 by the values of u and v at some later time $t > 0$ and use these as our initial data. Because of (3.5) one sees that the desired conditions corresponding to (3.6) for the replacement initial data will then hold.

An immediate implication of (3.6) is that u will be positively bounded away from zero, uniformly, for all subsequent times $t > 0$. Namely by (3.3) we have that

$$(3.7) \quad u(x, t) \geq \beta \quad \text{in } \bar{\Omega} \times (0, \infty)$$

where

$$(3.8) \quad \beta = \min[m/(1 + \alpha), \inf_{\Omega} u_0(x)] > 0$$

on account of (3.6).

We are now in position to show that for the solutions (u, v) of (2.10)–(2.12) under condition (3.1), $v \rightarrow 0$ as $t \rightarrow \infty$. This is a consequence of the principal

estimate for the solutions of our boundary initial value problem which is described in the following

Theorem 3.2. *Consider solutions (u, v) of (2.10)–(2.12) whose initial values (u_0, v_0) satisfy condition (3.1). Assume also, as we may, that u_0 and v_0 satisfy the additional conditions (3.6) and consequently that β as defined in (3.8) is positive. Then when $\alpha > 1$, the following estimate holds for $v(x, t)$:*

$$(3.9) \quad v(x, t) \leq (\sigma e^{-(\alpha-1)\beta(1-\sigma)t})u(x, t) \quad \text{for } (x, t) \in \bar{\Omega} \times (0, \infty),$$

where σ is the non-negative constant defined by

$$(3.10) \quad \sigma = \sup_{\Omega} \left(\frac{v_0(x)}{u_0(x)} \right),$$

so that $\sigma < 1$ according to (3.6).

As an immediate consequence of the estimate (3.9) we have

Corollary 3.3. *Assume that (u, v) is a solution of (2.10)–(2.12) with non-negative, bounded initial data (u_0, v_0) in Ω , satisfying condition (3.1), then*

$$v(x, t) \rightarrow 0 \quad \text{uniformly in } \Omega \text{ as } t \rightarrow \infty;$$

in fact $v \rightarrow 0$ at an exponential rate.

This follows by noting that as a result of the assumed boundedness of the initial data (u_0, v_0) in Ω , Theorem 2.3 gives us a uniform upper bound for u , namely

$$u(x, t) \leq \max[m, \sup_{\Omega} u_0(x)] \quad \text{for } (x, t) \in \bar{\Omega} \times (0, \infty).$$

Hence, because of the positivity of the constant $(\alpha - 1)\beta(1 - \sigma)$ in the exponential on the right of the estimate (3.9), that estimate implies that $v \rightarrow 0$ uniformly in Ω as $t \rightarrow \infty$ (at an exponential rate).

Proof of Theorem 3.2. Set

$$(3.11) \quad \theta = \theta(t) = \sigma e^{-(\alpha-1)\beta(1-\sigma)t}$$

and note that θ is a solution of the ordinary differential equation

$$(3.12) \quad \theta'(t) = -(\alpha - 1)\beta(1 - \sigma)\theta(t).$$

We are going to show that the function $\theta u - v$ satisfies a parabolic inequality of the form

$$(3.13) \quad \frac{\partial}{\partial t}(\theta u - v) - a\Delta(\theta u - v) \geq (\theta u - v)p \quad \text{in } \Omega \times (0, \infty)$$

with

$$(3.14) \quad p = m - (u + v) + u(\alpha - 1)(\theta - 1).$$

Since $\theta u - v$ clearly has a vanishing normal derivative on $\partial\Omega \times (0, \infty)$:

$$\frac{\partial}{\partial n}(\theta u - v) = \theta \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = 0;$$

and, as we will show in a moment, $\theta u - v$ is initially non-negative:

$$(3.15) \quad (\theta u - v)|_{t=0} \geq 0 \quad \text{in } \Omega;$$

it will then follow from an application of Theorem 2.1, that $\theta u - v$ is subsequently non-negative as well:

$$\theta(t)u(x, t) - v(x, t) \geq 0 \quad \text{in } \bar{\Omega} \times (0, \infty);$$

and this is precisely the content of (3.9).

To verify (3.15), note that it is equivalent to the assertion

$$\theta(0)u_0(x) - v_0(x) \geq 0 \quad \text{in } \Omega,$$

i.e.

$$\frac{v_0(x)}{u_0(x)} \leq \theta(0) = \sigma \quad \text{in } \Omega;$$

which, in the light of our definition (3.10) of σ , is clearly correct.

We now turn to the proof of (3.13). In the argument that follows it is to be understood, without any further mention, that the point (x, t) always lies in the set $\Omega \times (0, \infty)$.

We begin by multiplying the equation in (2.10) that u satisfies by θ and then adding $\theta'u$ to both sides:

$$\theta \frac{\partial u}{\partial t} + \theta'u - \theta a \Delta u = \theta u(m - u - \alpha v) + \theta'u.$$

This may be re-written as

$$\frac{\partial}{\partial t}(\theta u) - a \Delta(\theta u) = \theta u(m - u - \alpha v) + \theta'u.$$

Now subtract from this the equation in (2.10) satisfied by v , i.e.

$$\frac{\partial v}{\partial t} - a \Delta v = v(m - v - \alpha u).$$

This results in

$$(3.16) \quad \frac{\partial}{\partial t}(\theta u - v) - a\Delta(\theta u - v) = (\theta u - v)m - (\theta u^2 - v^2) + \alpha uv(1 - \theta) + \theta' u.$$

Noting that

$$\theta u^2 - v^2 = (\theta u - v)(u + v) + (1 - \theta)uv;$$

and inserting this into the right side of (3.16) we then obtain

$$(3.17) \quad \begin{aligned} \frac{\partial}{\partial t}(\theta u - v) - a\Delta(\theta u - v) \\ &= (\theta u - v)[m - (u + v)] + \alpha uv(1 - \theta) - (1 - \theta)uv + \theta' u \\ &= (\theta u - v)[m - (u + v)] + u[v(1 - \theta)(\alpha - 1) + \theta']. \end{aligned}$$

Next, we shall re-work the expression $v(1 - \theta)(\alpha - 1) + \theta'$ which appears inside the brackets on the extreme right of the last display. Making use of the differential equation (3.12) that θ satisfies in conjunction with β being an over-all lower bound for u (see (3.7)), we find that

$$\theta' = -(\alpha - 1)\beta(1 - \sigma)\theta \geq -(\alpha - 1)u(1 - \sigma)\theta.$$

Hence

$$\begin{aligned} v(1 - \theta)(\alpha - 1) + \theta' &\geq v(1 - \theta)(\alpha - 1) - (\alpha - 1)u(1 - \sigma)\theta \\ &= (\alpha - 1)[v(1 - \theta) - u(1 - \sigma)\theta] \\ &= (\alpha - 1)[(v - \theta u) + \theta(\sigma u - v)]. \end{aligned}$$

But as $\theta(t)$ is a decreasing function of t and $\theta(0) = \sigma$, replacing σ by θ in the expression $\sigma u - v$ on the right of the preceding display decreases this expression: $\sigma u - v \geq \theta u - v$; and so making this replacement in the preceding we arrive at

$$v(1 - \theta)(\alpha - 1) + \theta' \geq (\alpha - 1)[(v - \theta u) + \theta(\theta u - v)] = (\alpha - 1)[(\theta u - v)(\theta - 1)].$$

Finally, inserting this into the right side of (3.17), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\theta u - v) - a\Delta(\theta u - v) \\ &\geq (\theta u - v)[m - (u + v)] + u(\alpha - 1)(\theta u - v)(\theta - 1) \\ &= (\theta u - v)[m - (u + v) + u(\alpha - 1)(\theta - 1)]; \end{aligned}$$

which is the desired result (3.13).

With Theorem 3.2 established, it follows immediately (as we pointed out in Corollary 3.3) that $v \rightarrow 0$ uniformly in Ω at $t \rightarrow \infty$. Finally, to show that

$u \rightarrow m$ uniformly in Ω as $t \rightarrow \infty$, we re-write the equation (2.10) that u satisfies in the form

$$\frac{\partial u}{\partial t} - a\Delta u = u(m - u + g(x, t))$$

where

$$g(x, t) = -\alpha v(x, t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

uniformly in Ω . The asymptotic behavior of solutions to such equations under homogeneous Neumann boundary conditions was considered in [5], Theorem 4.2, p. 209. Applying that Theorem we then obtain the desired result: $u \rightarrow m$ uniformly in Ω as $t \rightarrow \infty$.

References

- [1] Brown, P. N., Decay to uniform states in ecological interactions, *SIAM J. Appl. Math.*, **38**(1980), 22–37.
- [2] Fife, P. C., *Mathematical Aspects of Reacting and Diffusing Systems*, Springer Verlag, Berlin, 1979.
- [3] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall Inc., Englewood Cliffs, 1964.
- [4] Il'in, A. M., Kalashnikov A. S. and Oleinek, O. A., Second order linear equations of parabolic type, *Russian Math. Surveys*, **17**, No. 3, (1962), 1–143.
- [5] Kahane, C. S., On the asymptotic behavior of solutions of parabolic equations under homogeneous Neumann boundary conditions, *Funkcial. Ekvac.*, **32**(1989), 191–213.
- [6] Ladyzhenskaja, O. A., Solonnikov V. A. and Ural'ceva, N. N., *Linear and Quasilinear Equations of Parabolic type*, Amer. Math. Soc., Providence, 1968.
- [7] Levin, S. A., Models of population dispersal, differential equations and applications, *Ecology, Epidemics and Population Problems*, edited by S. Busenberg and K. Cooke, Academic Press, 1981, 1–18.
- [8] Leung, A. W., Nonlinear density dependent diffusion for competing species interactions: Large time asymptotic behavior, *Proc. Edinburgh Math. Soc.*, (2), **27**(1984), 131–144.
- [9] Matano, H. and Mimura, M., Pattern formation in competition-diffusion systems in non-convex domains, *Publ. RIMS, Kyoto Univ.*, **19**(1983), 1049–1079.
- [10] Okubo, A., *Diffusion and Ecological Problems*, Spring Verlag, Berlin, 1980.
- [11] Pao, C. V., Coexistence and stability of a competition-diffusion system in population dynamics, *J. Math. Anal. Appl.*, **83**(1981), 54–76.
- [12] Pao, C. V. and Zhou, L., Asymptotic behavior of a competition-diffusion system in population dynamics, *Nonlinear Anal.*, **6**(1982) 1163–1184.
- [13] Pielou, E. C., *An Introduction to Mathematical Ecology*, Wiley-Interscience, New York, 1969.
- [14] Protter, M. H. and Weinberger, H. F., *Maximum Principles in Differential Equations*, Prentice-Hall Inc., Englewood Cliffs, 1967.
- [15] Rescigno A. and Richardson, I., The struggle for life: I Two species, *Bull. Math. Biophys.*, **29**(1967), 377–388.

- [16] Smith, J. M., *Mathematical Ideas in Biology*, Cambridge University Press, Cambridge, 1968.

nuna adreso:
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37235
U.S.A.

(Ricevita la 13-an de februaro, 1990)