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On Well-Posedness of Boundary Value Problems Involving Deviating Arguments

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1. Introduction

Boundary value problems (B.V.P. for short) constitute a very interesting class of problems in the qualitative theory of ordinary differential equations. The books of Bernfeld and Lakshmikantham [2], Gaines and Mawhin [5] and Mawhin [14] gave a reasonably complete view of this field.

Especially, boundary value problems for differential equations with deviating arguments were the subject in several mathematical articles (cf. El'sgol'ts [4], Myshkis and El'sgol'ts [15], Kamenskii [11], Grimm and Schmitt [7, 8], Heikkilä, Mooney and Seikkala [9], Heikkilä [10]).

Usually, in boundary value problems for differential equations with deviating arguments, we search for functions (solutions) which satisfy the differential equation in a closed interval I and identify with an a priori given function outside of I . In such a problem, the point boundary conditions (as they are known from the boundary value problems for ordinary equations) are replaced by boundary conditions given via a function. This fact creates some difficulties in the posedness of the boundary value problems and in the proofs of the existence theorems.

In this paper we deal with the well posedness of a boundary value problem involving equation with deviating arguments. The results obtained here extend some results in [10].

2. The boundary value problem

Consider the differential equation

$$(E) \quad x''(t) = f(t, x(\sigma_1(t)), \dots, x(\sigma_k(t)), x'(\sigma_{k+1}(t)), \dots, x'(\sigma_{k+m}(t))), \quad t \in I$$

where $t \in I = [a, b]$, ($a < b$), $f: I \times (\mathbb{R}^n)^{k+m} \rightarrow \mathbb{R}^n$ is a continuous function and σ_i , $i = 1, \dots, k + m$ are continuous real valued functions defined on I , so that the set $\{t \in I: \sigma_i(t) = a \text{ or } \sigma_i(t) = b, \quad i = k + 1, \dots, k + m\}$ is finite. Also we suppose that

$$-\infty < r_a = \min_{1 \leq i \leq k+m} \min_{t \in I} \sigma_i(t) < a$$

and

$$b < r_b = \max_{1 \leq i \leq k+m} \max_{t \in I} \sigma_i(t) < +\infty$$

and set

$$E(a) = [r_a, a], \quad E(b) = [b, r_b], \quad J = [r_a, r_b].$$

The prescribed boundary conditions for equation (E) are of the general form

$$\begin{aligned} \text{(B.C.)} \quad & -\alpha_0 x(t) + \alpha_1 x'(t) = q_1(t), \quad t \in E(a) \\ & \beta_0 x(t) + \beta_1 x'(t) = q_2(t), \quad t \in E(b) \end{aligned}$$

where $\alpha_i, \beta_i, i = 0, 1$ are nonnegative real constants such that

$$\alpha_0 + \alpha_1 > 0 \quad \text{and} \quad \beta_0 + \beta_1 > 0,$$

and q_1, q_2 are continuous $\mathbf{R}^n$ -valued functions defined on $E(a)$ and $E(b)$ respectively, which are differentiable if $\alpha_1 = 0 = \beta_1$.

If A is an interval of the real line and for notational purposes we introduce the following symbol

$C^i(A, \mathbf{R}^n)$, $i = 0, 1, 2$, for the set of all $\mathbf{R}^n$ -valued defined and continuous on A functions which are i -times continuously differentiable on A .

Recently, in [20] we gave some results concerning the existence of solutions of the B.V.P. (E)–(B.C.). The derivative of solutions in [20] have in general discontinuities at the ends a, b of the interval I . This fact occurs because the equation (E) is an equation with deviating arguments. Usually, for boundary value problems involving this kind of equations smoothness of solutions at the points a, b is not required (see El'sgol'ts [4]). However, it is interest to find conditions under which the B.V.P. (E)–(B.C.) has solutions which are smooth at a, b .

The following definition illustrates the concept “smooth solution”.

Definition 2.1. A function x is called *smooth solution* of the B.V.P. (E)–(B.C.) if $x \in C^1(J, \mathbf{R}^n) \cap C^2(I, \mathbf{R}^n)$, x satisfies the equation (E) for $t \in I$ and the boundary condition (B.C.) for $t \in E(a) \cup E(b)$.

Clearly, if x is a smooth solution of the B.V.P. (E)–(B.C.), then we have $x'_-(a) = x'_+(a)$ and $x'_-(b) = x'_+(b)$ (in addition to the obvious relations $x_-(a) = x_+(a)$, $x_-(b) = x_+(b)$). B.V.P.'s which have smooth solutions have been posed

by El'sgol'ts [4] and Kamenskii [11]. In [17] Schmitt has obtained some results of this type of solutions, for differential equations with retarded arguments, and particular boundary conditions.

When the function f depends on the arguments σ_i , $i = k + 1, \dots, k + m$, in x' , then, because of the nonsmoothness of the solutions x at a, b , we must accept that the second derivative of the solutions will be in general piecewise continuous on I .

With this in mind we give the following.

Definition 2.2. A function x is called a solution of the B.V.P. (E)–(B.C.) if $x \in C(J, \mathbb{R}^n) \cap C^1(I, \mathbb{R}^n) \cap C^1(E(a) \cup E(b), \mathbb{R}^n)$, x is piecewise twice differentiable on I and satisfies the equation (E) for $t \in I$ and the boundary conditions (B.C.) for $t \in E(a) \cup E(b)$.

In general, if the derivative x' of the solution x has discontinuities at $t = a$ or $t = b$ then x is called *non-smooth solution* of the B.V.P. (E)–(B.C.).

We remark that, we can have only one side smooth solution. Especially, if $x \in C(J, \mathbb{R}^n) \cap C^1(E(a) \cup I, \mathbb{R}^n) \cap C^1(E(b), \mathbb{R}^n)$ then we say that x is a left-side smooth solution of the B.V.P. (E)–(B.C.) and if $x \in C(J, \mathbb{R}^n) \cap C^1(E(a), \mathbb{R}^n) \cap C^1(I \cup E(b), \mathbb{R}^n)$ then we say that x is a right-side smooth solution of the B.V.P. (E)–(B.C.).

A discussion concerning the formulation of B.V.P.'s of this type may be found in [7], [8], [17].

3. Existence results

In the next we consider the normed space $C^1(J, \mathbb{R}^n)$ endowed with the norm

$$\|x\| = \max \left\{ \max_{t \in J} |x(t)|, \max_{t \in J} |x'(t)| \right\},$$

Our existence theorem is based on the next lemma, which is a version of the well known topological transversality theorem of Granas [3].

Lemma 3.1. (Lloyd [13]). Suppose that X is a normed space (possibly incomplete). Let S be a bounded, closed, convex subset of X , containing the origin 0 in its interior D . Let $H: [0, 1] \times S \rightarrow X$ be a homotopy of compact transformations such that $H(0, \partial S) \subset S$ and $H(t, x) \neq x$ ($0 \leq t \leq 1, x \in \partial S$). Then $\varphi = H(1, \cdot)$ has a fixed point in S .

Theorem 3.2. Let $f: I \times (\mathbb{R}^n)^{k+m} \rightarrow \mathbb{R}^n$ be a continuous function. Assume that there exists a constant M such that $\|x\| \leq M$ for every solution x of the B.V.P. (E) _{λ} –(B.C.), where (E) _{λ} stands for the equation

$$(E) \quad x''(t) = \lambda f(t, x(\sigma_1(t)), \dots, x(\sigma_k(t)), x'(\sigma_{k+1}(t)), \dots, x'(\sigma_{k+m}(t))), \quad t \in I.$$

Then if $\alpha_1 \neq 0 \neq \beta_1$ the B.V.P. (E)–(B.C.) has at least one smooth solution.

Proof. Consider the subspace X of $C^1(J, \mathbb{R}^n)$

$$X = \{u \in C^1(J, \mathbb{R}^n) : u \text{ is continuously twice differentiable on } I\}.$$

Now, for $x \in X$ we define

$$T_\lambda x = \lambda Lx + w, \quad \lambda \in [0, 1]$$

where

$$Lx(t) = \begin{cases} \int_a^b G(t, s) f(s, x(\sigma_1(s)), \dots, x(\sigma_k(s)), x'(\sigma_{k+1}(s)), \dots, x'(\sigma_{k+m}(s))) ds, & t \in I \\ e^{\alpha_0/\alpha_1(t-a)} Lx(a), & t < a, \quad \alpha_1 \neq 0 \\ e^{-\beta_0/\beta_1(t-b)} Lx(b), & t > b, \quad \beta_1 \neq 0, \end{cases}$$

$$w(t) = \begin{cases} \left\{ w(a) - \frac{1}{\alpha_1} \int_a^t q_1(s) \exp\left(-\frac{\alpha_0}{\alpha_1}(s-a)\right) ds \right\} \exp\frac{\alpha_0}{\alpha_1}(t-a), & t < a, \quad \alpha_1 \neq 0 \\ \frac{1}{l} [\beta_0(b-t)q_1(a) + \beta_1 q_1(a) - \alpha_1 q_2(b) - \alpha_0(t-a)q_2(b)], & t \in I \\ \left\{ w(b) + \frac{1}{\beta_1} \int_b^t q_2(s) \exp\left(\frac{\beta_0}{\beta_1}(s-b)\right) ds \right\} \exp\left(-\frac{\beta_0}{\beta_1}(t-b)\right), & t > b, \quad \beta_1 \neq 0, \end{cases}$$

$$l = -\alpha_0\beta_0(b-a) - \alpha_0\beta_1 - \beta_0\alpha_1$$

and $G(t, s)$ is the Green's function for the homogeneous B.V.P. [1],

$$x''(t) = 0$$

$$-\alpha_0 x(a) + \alpha_1 x'(a) = 0$$

$$\beta_0 x(b) + \beta_1 x'(b) = 0.$$

Now, it is obvious that for every $\lambda \in [0, 1]$ we have

$$T_\lambda x_-(a) = T_\lambda x_+(a) \quad \text{and} \quad T_\lambda x_-(b) = T_\lambda x_+(b).$$

Also, since $\alpha_1 \neq 0$ and $\beta_1 \neq 0$, by differentiating $T_\lambda x$ followed by some manipulation we can easily verify that:

$$(T_\lambda x)'_-(a) = (T_\lambda x)'_+(a),$$

$$(T_\lambda x)'_-(b) = (T_\lambda x)'_+(b)$$

and

$$(T_\lambda x)''(t) = f(t, x(\sigma_1(t)), \dots, x(\sigma_k(t)), x'(\sigma_{k+1}(t)), \dots, x'(\sigma_{k+m}(t))), \quad t \in I.$$

From these we conclude that for every $\lambda \in [0, 1]$ the formula T_λ defines a mapping $T_\lambda: X \rightarrow X$. Moreover if

$$U = \{u \in X : \|u\| < M + \|w\| + 1\}$$

and

$$H(\lambda, u) = T_\lambda u,$$

the formula H defines a mapping $H: [0, 1] \times \bar{U} \rightarrow X$.

Following the same arguments as in Theorem 2.4 in [20] we can prove that H is a compact homotopy such that $H(0, \partial\bar{U}) \subset X$ and $H(\lambda, x) \neq x$ for any $x \in \partial\bar{U}$ and $\lambda \in [0, 1]$.

Therefore, there exists at least one $x \in \bar{U}$ such that

$$x = H(1, x) = T_1 x = Lx + w.$$

But the equality $x = Lx + w$ ensures that x is a solution of the B.V.P. (E)–(B.C.) which proves the Theorem.

Below we give some corollaries of the above Theorem 3.2. The proofs are omitted since they are derived immediately by a combination of the Theorem 3.2, and the Theorem 2.4. in [20].

Corollary 3.3. *Suppose that the assumptions of Theorem 3.2 are satisfied and, moreover, for every $i = 1, \dots, k + m$ we have either $\sigma_i(I) \subseteq I$ or $\sigma_i(I) \subseteq E(a) \cup E(b) \setminus \{a, b\}$. Then the B.V.P. (E)–(B.C.) has at least one solution which is continuously twice differentiable on I .*

In the theory of differential equations the case where it happens $\sigma(I) \subseteq I$ for one or more arguments σ , except that includes the ordinary case, is very usual. Especially an interest case of such arguments is when the arguments are involutions [16], [21].

Corollary 3.4. *Suppose that the assumptions of Theorem 3.2 are satisfied and $\alpha_1 \neq 0$ (resp. $\beta_1 \neq 0$). Then the B.V.P. (E)–(B.C.) has at least one left (resp. right) side smooth solution. Moreover, if $\sigma_i(I) \subseteq E(a) \cup I$ (resp. $\sigma_i(I) \subseteq I \cup E(b)$), $i = k + 1, \dots, k + m$, this solution is continuously twice differentiable on I .*

We remark that if the arguments σ_i , $i = k + 1, \dots, k + m$ are of retarded type we have $\sigma_i(I) \subseteq E(a) \cup I$ while if the arguments σ_i , $i = k + 1, \dots, k + m$ are of advanced type we have $\sigma_i(I) \subseteq I \cup E(b)$.

Remark 3.5. If the function f is not depending on x' , i.e. $f = f(t, x(\sigma_1(t)), \dots, x(\sigma_k(t)))$, then, obviously, the results of Corollary 3.4. remain valid. More-

over, in this case, if $\alpha_1 \neq 0$ (resp. $\beta_1 \neq 0$) the B.V.P. (E)–(B.C.) has at least one left (resp. right) side smooth solution, which is also continuously twice differentiable on I .

The applicability of Theorem 3.2. depends upon the existence of an a priori bound for solutions and its derivative of the B.V.P. (E)–(B.C.), which are independent of λ . This can be done by various methods. See for example [6], [12].

Now, we give the following two examples which illustrate the above

Example 3.6. Consider the B.V.P.

$$\begin{aligned}
 x''(t) &= x(t-1) + x'(t+1), & t \in [0, 1] \\
 (*) \quad x(t) - x'(t) &= 0, & t \in [-1, 0] \\
 x(t) + x'(t) &= t, & t \in [1, 2].
 \end{aligned}$$

Also, we consider the function

$$x(t) = \begin{cases} -\frac{e^{t-2}}{2D}, & t \in [-1, 0] \\ k_1 e^{t-1} - k_2 e^{-t-1} + \frac{t^2}{2} + k_3 t + k_4, & t \in [0, 1] \\ \frac{1}{2D}(2e^{-t} - e^{-t-1}) + t - 1, & t \in [1, 2], \end{cases}$$

where $D = -3e^{-3} + 4e^{-2} + 2e^{-1}$ and the relations

$$\begin{aligned}
 (1 - e^{-1})k_1 + e^{-1}k_2 - k_4 &= 0 & (i) \\
 k_1 - (e^{-2} + e^{-1})k_2 + k_3 + k_4 &= -\frac{1}{2} & (ii) \\
 (1 - e^{-1})k_1 - e^{-1}k_2 - k_3 &= 0 & (iii) \\
 k_1 + (e^{-2} + e^{-1})k_2 + k_3 &= 0 & (iv).
 \end{aligned}$$

Then it is easy to verify the following

a) If k_3, k_4 are arbitrary real constants and k_1, k_2 satisfy the relations (i) and (ii), then the function x is a solution (in general nonsmooth) of the B.V.P. (*).

b) If k_4 (resp. k_3) is arbitrary real constant and k_1, k_2, k_3 (resp. k_1, k_2, k_4) satisfy the relations (i), (ii) and (iii) (resp. (i), (ii) and (iv)), then the function x is a left side (resp. right side) smooth solution of the B.V.P. (*).

c) If k_1, k_2, k_3, k_4 satisfy the relations (i), (ii), (iii) and (iv) (i.e. $k_1 = -\frac{1}{2D}e^{-2}$, $k_2 = -\frac{1}{2D}(e^{-1} - 2)$, $k_3 = -\frac{1}{2D}(2e^{-1} - e^{-3})$, $k_4 = -\frac{1}{2D}(-e^{-3} + 2e^{-2} - 2e^{-1})$), then the function x is a smooth solution of the B.V.P. (*).

The above example ensures that a B.V.P. of the form (E)–(B.C.) may be have more than one solution, since Theorem 3.2 and Theorem 2.4 in [20] not assure the uniqueness. Moreover, we observe that it possible to have smooth and nonsmooth solutions for such a problem.

Example 3.7. The function

$$x(t) = \begin{cases} 1, & t \in [-2, 1] \\ \frac{t^2}{2} - t + \frac{3}{2}, & t \in [1, 2] \end{cases}$$

is a smooth solution of the B.V.P.

$$x''(t) = x(t-2) - x'(t-1), \quad t \in [1, 2]$$

$$x(t) = 1, \quad t \in [-2, 1]$$

$$16x(2) - 9x'(2) = 15.$$

This shows that the assumption $\alpha_1 \neq 0$ is not necessary for the existence of left side smooth solutions of a B.V.P. of the form (E)–(B.C.). By analogous example we can show that the assumption $\beta_1 \neq 0$ is not necessary for the existence of right side smooth solutions of a B.V.P. of the form (E)–(B.C.).

4. Some remarks on the domain of the boundary conditions

In the case where the arguments σ_i , $i = 1, \dots, k+m$, and the interval I satisfy the condition

$$(*) \quad \sigma_i(I) \subseteq I \quad \text{for every } i = 1, \dots, k+m,$$

the consideration of the B.V.P. (E)–(B.C.) is essentially similar to that of ordinary case. This happen, since in this case we have $E(a) = \{a\}$, $E(b) = \{b\}$ and so, the boundary conditions (B.C.) are reduced to the following point boundary conditions

$$\begin{aligned} (bc) \quad & -\alpha_0 x(a) + \alpha_1 x'(a) = q_1(a) \\ & \beta_0 x(b) + \beta_1 x'(b) = q_2(b). \end{aligned}$$

For the sake of brevity such a problem, i.e. a B.V.P. of the form (E)–(bc), where the arguments in the equation (E) satisfy the condition (*), will be mentioned in the next as (*)–B.V.P.

Obviously, in a (*)–B.V.P. the term smooth solution becomes meaningless.

Condition (*) is in general a very usual condition in the theory of differential equations with deviating arguments. Among numerous studies on this type equations we refer to the papers by Shah and Wiener [16], Wiener and Aftabzadeh [21], Staikos and Tsamatos [18], Tsamatos [19] and the references in these papers.

In the following we will examine some cases, where the solvability of a B.V.P. of the form (E)–(B.C.) it depends immediately on the solvability of a (*)–B.V.P.

Proposition 4.1. *Suppose that the arguments σ_i , $i = 1, \dots, k + m$, and the interval I in the B.V.P. (E)–(B.C.) are such that*

$$(H) \quad \left\{ \begin{array}{l} \text{for every } i \in \{1, \dots, k + m\} \text{ we have either} \\ \sigma_i(I) \subseteq I \quad \text{or} \quad \sigma_i(I) \subseteq E(a) \cup E(b). \end{array} \right.$$

Then there exists a ()–B.V.P. $(\hat{E})$ –(bc) such that if x is a solution of the B.V.P. (E)–(B.C.) the function $x|I$ is a solution of the B.V.P. $(\hat{E})$ –(bc) and, conversely, for every solution $x: I \rightarrow \mathbf{R}^n$ of $(\hat{E})$ –(bc) there exists a solution y of (E)–(B.C.) such that $y|I = x$.*

Proof. Because of (H) and the continuity of σ_i , $i = 1, \dots, k + m$ the sets $T = \{i \in \{1, \dots, k + m\} : \sigma_i(I) \subseteq I\}$, $T_a = \{i \in \{1, \dots, k + m\} : \sigma_i(I) \subseteq E(a)\}$, $T_b = \{i \in \{1, \dots, k + m\} : \sigma_i(I) \subseteq E(b)\}$ constitutes a partition of the set $\{1, \dots, k + m\}$.

Now, we consider the function $w_a: E(a) \rightarrow \mathbf{R}^n$ and $w_b: E(b) \rightarrow \mathbf{R}^n$ by

$$w_a(t) = \begin{cases} w_a(t, c_1) = \left[c_1 + \frac{1}{\alpha_1} \int_a^t q_1(s) \exp\left(\frac{\alpha_0}{\alpha_1}(s - a)\right) ds \right] \exp\left(-\frac{\alpha_0}{\alpha_1}(t - a)\right), & \text{if } \alpha_1 \neq 0 \\ -\frac{1}{\alpha_0} q_1(t), & \text{if } \alpha_1 = 0 \end{cases}$$

and

$$w_b(t) = \begin{cases} w_b(t, c_2) = \left[c_2 + \frac{1}{\beta_1} \int_b^t q_2(s) \exp\left(\frac{\beta_0}{\beta_1}(s - b)\right) ds \right] \exp\left(-\frac{\beta_0}{\beta_1}(t - b)\right), & \text{if } \beta_1 \neq 0 \\ -\frac{1}{\beta_0} q_2(t), & \text{if } \beta_1 = 0. \end{cases}$$

Also, we set

$$T \cap \{1, \dots, k\} = \{i_1, \dots, i_s\}, \quad T \cap \{k+1, \dots, k+m\} = \{i_{s+1}, \dots, i_{s+r}\}$$

($s \geq 0$, $r \geq 0$) and for every $x \in C^1(I, \mathbb{R}^n)$ and $t \in I$

$$\begin{aligned} \hat{f}(t, x(\sigma_{i_1}(t)), \dots, x(\sigma_{i_s}(t)), x'(\sigma_{i_{s+1}}(t)), \dots, x'(\sigma_{i_{s+r}}(t)), c_1, c_2) \\ = f(t, u_1(t), \dots, u_{k+m}(t), c_1, c_2), \end{aligned}$$

where

$$u_i(t) = \begin{cases} x(\sigma_i(t)), & \text{if } i \in T \cap \{1, \dots, k\} \\ x'(\sigma_i(t)), & \text{if } i \in T \cap \{k+1, \dots, k+m\} \\ w_a(\sigma_i(t)), & \text{if } i \in T_a \cap \{1, \dots, k\} \\ w'_a(\sigma_i(t)), & \text{if } i \in T_a \cap \{k+1, \dots, k+m\} \\ w_b(\sigma_i(t)), & \text{if } i \in T_b \cap \{1, \dots, k\} \\ w'_b(\sigma_i(t)), & \text{if } i \in T_b \cap \{k+1, \dots, k+m\}. \end{cases}$$

Then it is clear that, if x is a solution of the B.V.P. (E)–(B.C.), the function $x|I$ is solution of the B.V.P.

$$(\hat{E}) \quad x''(t) = \hat{f}(t, x(\sigma_{i_1}(t)), \dots, x(\sigma_{i_s}(t)), x'(\sigma_{i_{s+1}}(t)), \dots, x'(\sigma_{i_{s+r}}(t)), x(a), x(b)), \quad t \in I,$$

$$\begin{aligned} & -\alpha_0 x(a) + \alpha_1 x'(a) = q_1(a) \\ (\text{bc}) \quad & \beta_0 x(b) + \beta_1 x'(b) = q_2(b) \end{aligned}$$

Also, if x is a solution of $(\hat{E})$ –(bc) then it is obvious that the function

$$y(t) = \begin{cases} w_a(t, x(a)), & \text{if } t \in E(a) \text{ and } \alpha_1 \neq 0 \\ -\frac{1}{\alpha_0} q_1(t), & \text{if } t \in E(a) \text{ and } \alpha_1 = 0 \\ x(t), & \text{if } t \in I \\ w_b(t, x(b)), & \text{if } t \in E(b) \text{ and } \beta_1 \neq 0 \\ \frac{1}{\beta_0} q_2(t), & \text{if } t \in E(b) \text{ and } \beta_1 = 0 \end{cases}$$

is a solution of (E)–(B.C.)

The following example is an immediate application of the above proposition.

Example 4.2. Let us consider the B.V.P.

$$\begin{aligned} x''(t) &= x(t-2) + x(t/2) + x'(t+2), & t \in [0, 1] \\ (A) \quad x(t) - x'(t) &= 0, & t \in [-2, 0] \\ 2x(t) + x'(t) &= t, & t \in [1, 3]. \end{aligned}$$

For every solution x of the above boundary problem the function $x|_{[0, 1]}$ is a solution of the following B.V.P.

$$\begin{aligned} x''(t) &= x(0)e^{t-2} + x\left(\frac{t}{2}\right) - \left(2x(1) - \frac{1}{4}\right)e^{-2(t+1)} + \frac{1}{2}, & t \in [0, 1] \\ x(0) - x'(0) &= 0 \\ 2x(1) + x'(1) &= 1. \end{aligned}$$

Conversely if x is a solution of the last B.V.P. then the function

$$y(t) = \begin{cases} x(0)e^t, & \text{if } t \in [-2, 0] \\ x(t), & \text{if } t \in [0, 1] \\ \left(x(1) - \frac{1}{4}\right)e^{-2(t-1)} + \frac{1}{2}t - \frac{1}{4}, & t \in [1, 3] \end{cases}$$

is a solution of the B.V.P. (A).

On the other hand for the B.V.P.

$$\begin{aligned} x''(t) &= x(t) + x'(2t), & t \in [-1, 1] \\ x(t) - x'(t) &= 0, & t \in [-2, -1] \\ 2x(t) + x'(t) &= t, & t \in [1, 3] \end{aligned}$$

the condition (H) is not valid. Hence in this example we cannot apply Proposition 4.1.

Remark 4.3. If the equation $(\hat{E})$ in the above Proposition 4.1. is a reducible equation (see [16]) then it is obvious that the problem (E)–(B.C.) can be reduced to a B.V.P. in which the involving differential equations is an ordinary one.

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