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On the Dirichlet Problem of the Equation

$$-\Delta u = Ku^5 \text{ on the Ball in } R^3$$

By

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Dedicated to Professor Noboru Tanaka on his 60th birthday

§ 1. Let us consider the following equation for a given C^∞ function $K > 0$:

$$\begin{aligned} -\Delta u &= Ku^{(n+2)/(n-2)} \quad \text{in } D(R), \\ u &= a \quad \text{on } \partial D(R), \quad u > 0 \quad \text{in } D(R). \end{aligned}$$

where $D(R)$ and a denote the ball of radius R centered at the origin O in R^n ($n \geq 3$) and a small positive constant respectively.

If $K \equiv 1$, then the equation for the unknown function $v = u/a - 1$ becomes

$$-\Delta v = a^{4/(n-2)}(1+v)^{(n+2)/(n-2)}.$$

According to the result of 2.6 in [1], this equation for sufficiently small $a > 0$ has at least two solutions with $v = 0$ on $\partial D(R)$ and $v > 0$ in $D(R)$. Hence we get at least two solutions u , if $K \equiv 1$. Because there is only the trivial solution for $a = 0$ by the result of [5], these solutions may tend to 0 or diverge as $a \rightarrow 0$.

Let us turn to the case for nonconstant K . We can prove by the variational method of [1], the existence of at least two solutions under appropriate conditions on K , one of which converges to 0 as $a \rightarrow 0$. But the behavior of the “large solution” is not clear. The purpose of this paper is to prove the existence of a “large solution” in more exact form for rather restricted K .

If $K \equiv n(n-2)/4$, there exist one parameter family of solutions

$$u_\varepsilon(x) = (2\varepsilon/(\varepsilon^2 + |x|^2))^{(n-2)/2}$$

with $u_{\varepsilon|\partial D(R)} = (2\varepsilon/(\varepsilon^2 + R^2))^{(n-2)/2} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Incidentally, the value $n(n-2)/4$ of the constant is irrelevant, because we get solutions for any constant K by “stretching” this function u_ε . In this paper, we show that a similar phenomenon occurs if $n = 3$ and the given function K is sufficiently close to a constant.

Theorem. Suppose that a C^∞ function K on R^3 depends only on the distance from the origin. If $K(O) = 3/4$, and the second derivatives of K at O

vanishes, then there exists a constant $R > 0$ with the following property: For every sufficiently small $\varepsilon > 0$, there exists a C^∞ solution $u(x) > 0$ of the equation

$$-\Delta u = Ku^5 \quad \text{in } D(R), \quad u = u_\varepsilon \quad \text{on } \partial D(R),$$

which satisfies

$$\|u - u_\varepsilon\|_{1,2} \leq C\varepsilon^{11/8}, \quad \|u - u_\varepsilon\|_\infty \leq C\varepsilon^{7/8},$$

where $\|\cdot\|_{1,2}$ and $\|\cdot\|_\infty$ denote the $W^{1,2}$ -norm and the uniform norm respectively, and the constant C does not depend on ε .

To prove the above theorem, we recall the geometric meaning of the problem. Let us consider Riemannian metrics g and $\phi^{4/(n-2)}g$ ($\phi > 0$), and denote their scalar curvatures by k and k' respectively. Then the following equality holds:

$$-\{4(n-1)/(n-2)\}\Delta\phi + k\phi = k'\phi^{(n+2)/(n-2)},$$

where Δ is the Laplacian in g . Hence our problem is equivalent to finding a conformal metric $u^{4/(n-2)}ds^2$ on $D(R)$ of scalar curvature $\{4(n-1)/(n-2)\}K$. Since $(D(R), u_\varepsilon^{4/(n-2)}ds^2)$ is isometric to a domain D_ε in the standard unit sphere S^n , we may solve the following equation on D_ε :

$$-\{4(n-1)/(n-2)\}\Delta_1 w_\varepsilon + n(n-1)w_\varepsilon = \{4(n-1)/(n-2)\}K_\varepsilon w_\varepsilon^{(n+2)/(n-2)} \quad \text{in } D_\varepsilon, \\ w_\varepsilon > 0 \quad \text{in } D_\varepsilon, \quad w_\varepsilon \equiv 1 \quad \text{on } \partial D_\varepsilon$$

where Δ_1 is the Laplacian on S^n and K_ε is related to K under the isometry mentioned above. Considering the fact that $w_\varepsilon \equiv 1$ is a solution for $K_\varepsilon \equiv n(n-2)/4$, we try to find a solution in the form $w_\varepsilon = 1 + v_\varepsilon$ where v_ε is small in $\|\cdot\|_{1,2}$ and $\|\cdot\|_\infty$. This is done by an iterative method.

§2. In this section, we formulate the problem as a nonlinear equation on a domain of the unit sphere, and investigate the associated linear operator.

Let $P = (0, 0, 0, 1)$ be the north pole of $S^3 \subset \mathbf{R}^4$ and $p: S^3 \rightarrow \mathbf{R}^3$ be the stereographic projection. Denoting by g and ds^2 the standard metric on S^3 and $\mathbf{R}^3$ respectively, we have

$$(p^{-1})^*g = 4/(1 + |x|^2)^2 ds^2 = [\{2/(1 + |x|^2)\}^{1/2}]^4 ds^2.$$

If $\delta_\varepsilon: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is the dilation by ε , i.e., $\delta_\varepsilon(x) = x/\varepsilon$,

$$\delta_\varepsilon^*(p^{-1})^*g = [\{2\varepsilon/(\varepsilon^2 + |x|^2)\}^{1/2}]^4 ds^2 = u_\varepsilon^4 ds^2.$$

Let us define a domain D_ε of S^3 by $p^{-1}\delta_\varepsilon(D(R))$. Then $p^{-1}\delta_\varepsilon$ is an isometry between $(D(R), u_\varepsilon^4 ds^2)$ and (D_ε, g) . The radius of D_ε is

$$\int_0^R \{2\varepsilon/(\varepsilon^2 + r^2)\} dr = 2 \tan^{-1}(R/\varepsilon).$$

Hence, if we set $B_\varepsilon = S^3 \setminus D_\varepsilon$, then B_ε is a geodesic ball centered at the north pole of radius τ with $\tau = O(\varepsilon)$ as $\varepsilon \rightarrow 0$. We want to solve the following equation:

$$(1) \quad \begin{aligned} -8\Delta_1 w_\varepsilon + 6w_\varepsilon &= 8K_\varepsilon w_\varepsilon^5 \text{ in } D_\varepsilon, \\ w_\varepsilon &> 0 \text{ in } D_\varepsilon, \quad w_\varepsilon \equiv 1 \text{ on } \partial D_\varepsilon \end{aligned}$$

where Δ_1 is the Laplacian on S^3 and $K_\varepsilon = K\delta_\varepsilon^{-1}p$. If we set $w_\varepsilon = 1 + v_\varepsilon$, then v_ε has to satisfy the following equation:

$$(2) \quad \begin{aligned} -8\Delta_1 v_\varepsilon + (6 - 40K_\varepsilon)v_\varepsilon &= (8K_\varepsilon - 6) + 8K_\varepsilon\{(1 + v_\varepsilon)^5 - 1 - 5v_\varepsilon\} \text{ in } D_\varepsilon, \\ 1 + v_\varepsilon &> 0 \text{ in } D_\varepsilon, \quad v_\varepsilon \equiv 0 \text{ on } \partial D_\varepsilon, \end{aligned}$$

Let us define the operator L_ε acting on functions which vanish on ∂D_ε by

$$L_\varepsilon(f) = -8\Delta_1 f + (6 - 40K_\varepsilon)f.$$

To investigate the eigenvalues of $-\Delta_1$ on S^3 and on D_ε , we recall some facts concerning the method of separation of variables from [2]. Let us introduce geodesic polar coordinates about the north pole P on S^3 by

$$x(t, u) = \exp(tu)$$

where $0 \leq t \leq \pi$, $u \in S^2 \subset T_P S^3$. For a function $f(x(t, u)) = T(t)\Theta(u)$ on S^3 to satisfy $\Delta_1 f + \lambda f = 0$, there must exist a constant η such that

$$\square \Theta + \eta \Theta = 0,$$

$$(3) \quad \{(\sin t)^2 T'\}' + \{\lambda - \eta(\sin t)^{-2}\}(\sin t)^2 T = 0$$

where $\square$ and the prime denote the Laplacian on $S^2 \subset T_P S^3$ and differentiation with respect to t respectively. One then has $\eta = \ell(\ell + 1)$, $\ell = 0, 1, 2, \dots$. Now write λ as $\lambda = \sigma(\sigma + 3 - 1) = \sigma(\sigma + 2)$ and set $v = \sigma - 1$, $T(t) = S(t)(\sin t)^\ell$. Then (3) becomes

$$(4) \quad \{(\sin t)^{2\ell+2} S'\}' + v(v + 2\ell + 2)(\sin t)^{2\ell+2} S = 0.$$

If we denote a solution by $S_{v,\ell}$, then we get a solution of $\Delta_1 f + \lambda f = 0$ of the form

$$(5) \quad f(x(t, u)) = (\sin t)^\ell S_{v,\ell}(t) \Theta_\ell(u)$$

where Θ_ℓ is in the ℓ -th eigenspace of $\square$. As is well known, one has nontrivial solutions of (4) for the boundary data $S'(0) = S'(\pi) = 0$ only when

$$v = v_j = j - 1, \quad j = 1, 2, 3, \dots,$$

and the collection of solutions thus obtained are complete in $L^2(S^3)$. We write these eigenvalues of $-\mathcal{A}_1$ on S^3 as

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$$

Next we wish to solve (4) with the boundary data $S(\tau) = S'(\pi) = 0$. Again we have a sequence of eigenvalues determined by $v = v_j(\tau)$, $j = 1, 2, 3, \dots$, and

$$v_j(\tau) \downarrow v_j = j - 1 \quad (j = 1, 2, \dots)$$

as $\tau \downarrow 0$, i.e., $\varepsilon \downarrow 0$. The eigenfunctions of the forms (5) are complete in $\{f \in L^2(S^3) | f|_{B_\varepsilon} = 0\}$. We denote the eigenvalues of $-\mathcal{A}_1$ thus obtained by

$$0 < \mu_1(\tau) < \mu_2(\tau) \leq \mu_3(\tau) \leq \dots,$$

and corresponding normalized eigenfunctions by $\phi_1(\tau), \phi_2(\tau), \dots$. Let V_τ denote the subspace of $L^2(D_\varepsilon)$ spanned by $\phi_i(\tau)$ ($2 \leq i \leq 5$).

Proposition 1. *There exists a constant $R > 0$ with the following property: For sufficiently small ε , there exists a constant $C > 0$ which does not depend on ε , such that*

$$\|L_\varepsilon(\phi)\|_2 \geq C\|\phi\|_2$$

for every $\phi \in V_\tau^\perp$ where $\|\cdot\|_2$ denotes the L^2 -norm.

Proof. From the preceding argument, we have $\mu_i(\tau) \downarrow \mu_{i-1}$ for every i as $\tau \downarrow 0$. On the other hand,

$$\mu_0 = 0, \quad \mu_1 = \mu_2 = \mu_3 = \mu_4 = 3, \quad \mu_5 = 8, \dots,$$

which implies for every $\phi \in V_\tau^\perp$

$$\| -8\mathcal{A}_1\phi - 24\phi \|_2 \geq C\|\phi\|_2.$$

If we take R sufficiently small so that $|30 - 40K_\varepsilon|$ is small, then we have

$$\|L_\varepsilon(\phi)\|_2 = \|(-8\mathcal{A}_1 - 24)\phi + (30 - 40K_\varepsilon)\phi\|_2 \geq C\|\phi\|_2$$

for every $\phi \in V_\tau^\perp$, which is the desired result.

From now on we assume that $R \leq 1$ is taken as the preceding proposition holds. Moreover we may also assume that $|K| \leq 1$ on $D(R)$. For $i = 2, 3, 4, 5$, $\|L_\varepsilon(\phi_i(\tau))\|_2 \rightarrow 0$ as $\varepsilon \rightarrow 0$, so we must be more careful. One easily sees that this is the case $\ell = 0$ or $\ell = 1$. If $\ell = 1$, the corresponding eigenfunctions are written as

$$f_i(x(t, u)) = T_\tau(t)\Theta_i(u), \quad 2 \leq i \leq 4$$

where $\Theta_i(u)$ are the eigenfunctions of $\square$ for the eigenvalue 2. These are restrictions to S^2 of linear functions on $\mathbf{R}^3$. If $\ell = 0$, a solution of (4) with $S(\tau) = S'(\pi) = 0$ is $\sin \{2\pi(\pi - t)/(\pi - \tau)\}/\sin t$, and the eigenspace of this Sturm-Liouville problem is one-dimensional. Hence the eigenfunction for $\ell = 0$ is

$$f_5(\tau)(x(t, u)) = C_\tau \sin \{2\pi(\pi - t)/(\pi - \tau)\}/\sin t,$$

where C_τ is the normalization constant for $\|f_5(\tau)\|_2 = 1$.

Since a simple calculation shows

$$v(v+2) = \{2\pi/(\pi - \tau)\}^2 - 1,$$

the corresponding eigenvalue is

$$\lambda = v(v+2) = \{2\pi/(\pi - \tau)\}^2 - 1 = 3 + 8\tau/\pi + O(\tau^2).$$

We summarize the above argument as the following

Proposition 2. *There exists an orthonormal basis $\{f_i(\tau) | 2 \leq i \leq 5\}$ of V_τ with the following properties:*

(i) $f_i(\tau) = F(t)\Theta_i(u)$ ($i = 2, 3, 4$), where $\Theta_i(u)$ are orthonormal eigenfunctions of $\square$ for eigenvalue 2.

(ii) $-\Delta_1 f_5(\tau) = \{3 + \rho(\tau)\}f_5(\tau)$ where $\rho(\tau) = O(\varepsilon)$ as $\varepsilon \downarrow 0$.

§3. In this section, we prepare some lemmas which we use later. We assume that K satisfies the assumption in Theorem throughout this section.

Lemma 1. *For a function ψ on $D(R)$, let us define the function ψ_ε on D_ε by $\psi_\varepsilon = \psi \delta_\varepsilon^{-1} p$.*

(i) *If $|\psi(x)| \leq C|x|^2$, then $\|\psi_\varepsilon\|_2 \leq C\varepsilon^{3/2}$.*

(ii) *If $|\psi(x)| \leq C|x|^4$, then $|\langle f_5(\tau), \psi_\varepsilon \rangle_2| \leq C\varepsilon^3$.*

Here $\langle \rangle_2$ denotes the L^2 -inner product.

Proof. (i) By a straightforward calculation, we get

$$\begin{aligned} \|\psi_\varepsilon\|_2^2 &= \int_{D(R)} \psi^2 \{2\varepsilon/(\varepsilon^2 + |x|^2)\}^3 dx \\ &\leq c \int_0^R r^4 \{2\varepsilon/(\varepsilon^2 + r^2)\}^3 r^2 dr \\ &\leq C\varepsilon^3. \end{aligned}$$

(ii) Note first that the constant C_τ is bounded above as $\tau \rightarrow 0$, i.e., $\varepsilon \rightarrow 0$. To

prove the inequality, we show that $|f_5(\tau)(t)|$ is bounded uniformly in τ for small τ . This is shown, for example, as follows: Let us denote the geodesic distance from the south pole by r , i.e., $r = \pi - t$. Then $0 \leq r \leq \pi - \tau$ and

$$f_5(\tau)(r) = C_\tau \sin \{2\pi r/(\pi - \tau)\} / \sin r.$$

Since $|\sin \{2\pi r/(\pi - \tau)\}| \leq |\sin 2r|$ for r sufficiently close to $\pi - \tau$, we have

$$|f_5(\tau)(r)| \leq C_\tau |\sin 2r| / |\sin r| = 2C |\cos r|.$$

On the other hand, $\sin \{2\pi r/(\pi - \tau)\} / \sin r < \{2\pi r/(\pi - \tau)\} / \sin r$ for small $r > 0$, and the right hand side is bounded uniformly in τ for small τ . This implies as in (i),

$$|\langle f_5(\tau), \psi_\varepsilon \rangle_2| \leq C \int_{D(R)} |x|^4 \{2\varepsilon/(\varepsilon^2 + |x|^2)\}^3 dx \leq C\varepsilon^3.$$

Lemma 2. For a solution f of $L_\varepsilon f = g$, we have

$$\|f\|_\infty \leq C(\|f\|_2 + \|g\|_2), \quad \|f\|_{1,2} \leq C\|f\|_2^{1/2}(\|f\|_2 + \|g\|_2)^{1/2},$$

where C does not depend on ε .

Proof. Integrating both sides of the identity

$$-8(A_1 f)f + (6 - 40K_\varepsilon)f^2 = fg$$

and making use of the integration by parts, we get

$$\begin{aligned} \int 8|Vf|^2 &\leq C \int f^2 + \left(\int f^2 \right)^{1/2} \left(\int g^2 \right)^{1/2} \\ &\leq C\|f\|_2^2 + \|f\|_2 \|g\|_2, \end{aligned}$$

which implies the second inequality. The first inequality is a consequence of Theorem 8.15 in [3]. Its proof is based on the Moser iteration technique which needs Sobolev inequality for the domain D_ε . Because these domains are all conformally equivalent, we may use the same constants in Sobolev inequalities for these domains. Hence the constant which appears in the first inequality does not depend on ε .

Lemma 3. If ε is sufficiently small, then for any C^∞ function g on D_ε , the equation

$$L_\varepsilon(f) = g \text{ in } D_\varepsilon, \quad f = 0 \text{ on } \partial D_\varepsilon$$

has a unique C^∞ solution f . Moreover if g depends only on r and $\|g\|_2 \leq C\varepsilon^p$, then f also depends only on r and $\|f\|_{1,2} \leq C_0 C\varepsilon^{p-1}$, $\|f\|_\infty \leq C_0 C\varepsilon^{p-1}$ where C_0 does not depend on ε .

Proof. Using the fact $\mu_i(\tau) > \mu_{i-1} = 3$ for $2 \leq i \leq 5$, we see $\text{Ker}(L_\varepsilon) = \{0\}$ for small ε in a similar way as the proof of Proposition 1. Hence the first assertion is clear. Assume that the function g depends only on the radial variable r . Then we get a solution f solving an ordinary differential equation because the homogeneous equation of this ordinary differential equation has only the trivial solution. Since the eigenfunctions $f_2(\tau)$, $f_3(\tau)$, $f_4(\tau)$ are orthogonal in L^2 -norm to the radial functions, we can write

$$f = \alpha f_5(\tau) + F, \quad \alpha \in \mathbf{R}, \quad F \in V_\tau^\perp$$

to obtain the following inequality:

$$\|L_\varepsilon(F)\|_2 \leq C\varepsilon^p + |\alpha| \|L_\varepsilon(f_5(\tau))\|_2.$$

On the other hand, calculations using Proposition 2, Lemma 1 and

$$|3/4 - K(x)| \leq C|x|^4 \leq C|x|^2$$

give the following relations:

$$\begin{aligned} (6) \quad & |\langle L_\varepsilon(f), f_5(\tau) \rangle_2| = |\langle g, f_5(\tau) \rangle_2| \leq C_1 \|g\|_2 \leq C_1 C\varepsilon^p, \\ & |\langle L_\varepsilon(F), f_5(\tau) \rangle_2| = C_2 |\langle (3/4 - K_\varepsilon)F, f_5(\tau) \rangle_2| \\ & \leq C_2 \|3/4 - K_\varepsilon\|_2 \|F\|_2 \\ & \leq C_3 \varepsilon^{3/2} \|F\|_2 \\ & \leq C_4 \varepsilon^{3/2} \|L_\varepsilon(F)\|_2, \\ & |\langle L_\varepsilon(\alpha f_5), f_5 \rangle_2| = |\alpha \langle \{8(3 + \rho(\tau)) + (6 - 40K_\varepsilon)\} f_5, f_5 \rangle_2| \\ & = |\alpha \{8\rho(\tau) + 40\langle (3/4 - K_\varepsilon) \rangle f_5, f_5 \rangle_2\}| \\ & \geq 8|\alpha| \{\rho(\tau) - C_5 \varepsilon^{3/2}\}, \\ & \|L_\varepsilon(f_5(\tau))\|_2 \leq 8\{\rho(\tau) + C_5 \varepsilon^{3/2}\}. \end{aligned}$$

Combining these relations with the inequality

$$|\langle L_\varepsilon(\alpha f_5), f_5 \rangle_2| \leq |\langle L_\varepsilon(f), f_5 \rangle_2| + |\langle L_\varepsilon(F), f_5 \rangle_2|,$$

we have

$$8|\alpha| \{\rho(\tau) - C_5 \varepsilon^{3/2}\} \leq C_1 C\varepsilon^p + C_4 \varepsilon^{3/2} [C\varepsilon^p + 8|\alpha| \{\rho(\tau) + C_6 \varepsilon^{3/2}\}].$$

Since $\rho(\tau) = O(\varepsilon)$, this shows $|\alpha| \leq C_7 C\varepsilon^{p-1}$ for sufficiently small ε . Finally we get the following inequalities:

$$\|F\|_2 \leq C_8 \|L_\varepsilon(F)\|_2 \leq C_8 C\varepsilon^p + |\alpha| O(\varepsilon) \leq C_9 C\varepsilon^p,$$

$$\|f\|_2 \leq |\alpha| \|f_5\|_2 + \|F\|_2 \leq C_7 C \varepsilon^{p-1} + C_9 C \varepsilon^p \leq C_0 C \varepsilon^{p-1}.$$

Thus the proof is completed by Lemma 2.

§4. In this section, we prove the theorem. For this purpose, we solve the following equations successively:

$$(7)_1 \quad L_\varepsilon(v_1) = 8K_\varepsilon - 6,$$

$$(7)_2 \quad L_\varepsilon(v_2) = 8K_\varepsilon\{(1+v_1)^5 - 1 - 5v_1\},$$

$$(7)_j \quad L_\varepsilon(v_j) = 8K_\varepsilon\{(1+v_1+\cdots+v_{j-1})^5 - 1 - 5(v_1+\cdots+v_{j-1})\} \\ - 8K_\varepsilon\{(1+v_1+\cdots+v_{j-2})^5 - 1 - 5(v_1+\cdots+v_{j-2})\}$$

for $j \geq 3$. If these equations have solutions, and the sequence $\{v_1 + \cdots + v_j\}_j$ converges in a suitable norm, then the function $\Sigma_j v_j$ will be the desired solution of (2).

Proposition 3. Assume that K depends only on the radial variables, and the second derivatives of K vanishes at O . Then for sufficiently small ε , the equation $(7)_j$ ($j \geq 1$) has a C^∞ solution v_j which depends only on the radial variable and satisfies the following properties:

$$\|v_j\|_{1,2} \leq \varepsilon^{(9+2j)/8}, \quad \|v_j\|_\infty \leq \varepsilon^{(9+2j)/8},$$

$$v_j = 0 \text{ on } \partial D_\varepsilon.$$

Proof. From Lemma 3, we have a radial solution v_1 of $(7)_1$. Since $|(8K-6)(x)| \leq C|x|^4 \leq C|x|^2$ by the assumption on K , the following inequalities hold from Lemma 1:

$$(8) \quad \|8K_\varepsilon - 6\|_2 \leq C\varepsilon^{3/2}, \quad |\langle f_5(\tau), 8K_\varepsilon - 6 \rangle_2| \leq C\varepsilon^3.$$

Taking $p = 3/2$ and using (8) instead of (6) in the proof of Lemma 3, we obtain

$$\|v_1\|_{1,2} \leq C'\varepsilon^{3/2}, \quad \|v_1\|_\infty \leq C'\varepsilon^{3/2},$$

If we take ε so small that $C'\varepsilon^{1/8} < 1$, we get

$$\|v_1\|_{1,2} \leq \varepsilon^{11/8}, \quad \|v_1\|_\infty \leq \varepsilon^{11/8},$$

Next we get a solution v_2 of $(7)_2$. Because we have assumed $|K_\varepsilon| \leq 1$,

$$\begin{aligned} \|8K_\varepsilon\{(1+v_1)^5 - 1 - 5v_1\}\|_\infty &\leq 8\|10v_1^2 + 10v_1^3 + 5v_1^4 + v_1^5\|_\infty \\ &\leq 8(10 + 10\|v_1\|_\infty + 5\|v_1\|_\infty^2 + \|v_1\|_\infty^3)\|v_1\|_\infty^2 \\ &\leq 8 \cdot 26\varepsilon^{11/4} \end{aligned}$$

for $\varepsilon < 1$. Hence we have from Lemma 3,

$$\|v_2\|_{1,2} \leq C_0 \operatorname{vol}(S^3)^{1/2} (8 \cdot 26) \varepsilon^{7/4},$$

$$\|v_2\|_{\infty} \leq C_0 \operatorname{vol}(S^3)^{1/2} (8 \cdot 26) \varepsilon^{7/4}.$$

If we take ε so small that $C_0 \operatorname{vol}(S^3)^{1/2} (8 \cdot 26) \varepsilon^{1/2} < 1$, we obtain

$$\|v_2\|_{1,2} \leq \varepsilon^{13/8}, \quad \|v_2\|_{\infty} \leq \varepsilon^{13/8}.$$

Assume that we have already gotten v_j ($j \geq 2$) with the desired property. Let us denote $v_1 + v_2 + \dots + v_j$ by Σ_j . Then

$$\begin{aligned} & \|8K_{\varepsilon}\{(1 + v_1 + \dots + v_j)^5 - 1 - 5(v_1 + \dots + v_j)\} \\ & - 8K_{\varepsilon}\{(1 + v_1 + \dots + v_{j-1})^5 - 1 - 5(v_1 + \dots + v_{j-1})\}\|_{\infty} \\ & \leq 8\{5\{4\Sigma_{j-1} + 6(\Sigma_{j-1})^2 + 4(\Sigma_{j-1})^3 + (\Sigma_{j-1})^4\}v_j \\ & + \{10(1 + \Sigma_{j-1})^3 + 10(1 + \Sigma_{j-1})^2v_j + 5(1 + \Sigma_{j-1})v_j^2 + v_j^3\}v_j^2\|_{\infty} \\ & \leq 8\{5(8\varepsilon^{11/8} + 24\varepsilon^{11/4} + 32\varepsilon^{33/8} + 16\varepsilon^{11/2})\varepsilon^{(9+2j)/8} \\ & + (20 + 20\varepsilon^{(9+2j)/8} + 10\varepsilon^{(9+2j)/2} + \varepsilon^{(27+6j)})\varepsilon^{(9+2j)/4} \\ & \leq 8\{(5 \cdot 80)\varepsilon^{(20+2j)/8} + 51\varepsilon^{(9+2j)/4}\} \\ & \leq 8(5 \cdot 80 + 51)\varepsilon^{(20+2j)/8}. \end{aligned}$$

We used the properties that $\|\Sigma_{j-1}\|_{\infty} \leq 2\varepsilon^{11/8}$ and $\|1 + \Sigma_{j-1}\|_{\infty}^k \leq 2$ ($k = 1, 2, 3$) for small ε . Consequently from Lemma 3, we obtain a solution v_{j+1} of (7)_{j+1} with

$$\|v_{j+1}\|_{1,2} \leq 8C_0(5 \cdot 80 + 51) \operatorname{vol}(S^3)^{1/2} \varepsilon^{(14+2j)/8},$$

$$\|v_{j+1}\|_{\infty} \leq 8C_0(5 \cdot 80 + 51) \operatorname{vol}(S^3)^{1/2} \varepsilon^{(14+2j)/8}.$$

We take ε so small that

$$8C_0(5 \cdot 80 + 51) \operatorname{vol}(S^3)^{1/2} \varepsilon^{1/8} < 1.$$

Then v_{j+1} satisfies the inequalities

$$\|v_{j+1}\|_{1,2} \leq \varepsilon^{\{9+2(j+1)\}/8}, \quad \|v_{j+1}\|_{\infty} \leq \varepsilon^{\{9+2(j+1)\}/8}.$$

Thus we complete the proof by induction.

Proof of Theorem. By the preceding proposition, $v_{\varepsilon} = \Sigma_j v_j \in W_{1,2}(D_{\varepsilon}) \cap C^0(\operatorname{clos} D_{\varepsilon})$ is well defined and satisfies

$$\|v_{\varepsilon}\|_{1,2} \leq C\varepsilon^{11/8}, \quad \|v_{\varepsilon}\|_{\infty} \leq C\varepsilon^{11/8}, \quad 1 + v_{\varepsilon} > 0.$$

Then $w_{\varepsilon} = 1 + v_{\varepsilon}$ is a weak solution of the equation (1). Hence w_{ε} is C^{∞} by

the elliptic regularity. Now let us define a function u on $D(R)$ by $u(x) = w(x)u_\varepsilon(x)$ where $w(x) = w_\varepsilon(p^{-1}\delta_\varepsilon(x))$. This function $u(x) > 0$ satisfies the equation

$$-\Delta u = Ku^5 \text{ in } D(R), \quad u = u_\varepsilon \text{ on } \partial D(R).$$

From the conformally invariant Sobolev inequality,

$$\begin{aligned} \|u - u_\varepsilon\|_6^6 &= \int_{D(R)} \{v_\varepsilon(p^{-1}\delta_\varepsilon(x))\}^6 u_\varepsilon^6 dx \\ &= \int_{D_\varepsilon} v_\varepsilon^6 \\ &\leq C \|v_\varepsilon\|_{1,2}^6 \\ &\leq C\varepsilon^{33/4}, \end{aligned}$$

which implies by Hölder's inequality,

$$\|u - u_\varepsilon\|_2 \leq C \|u - u_\varepsilon\|_6 \leq C\varepsilon^{11/8}.$$

On the other hand, we get writing $v = v_\varepsilon p^{-1}\delta_\varepsilon$,

$$\begin{aligned} \|\nabla(u - u_\varepsilon)\|_2^2 &= \|\nabla(vu_\varepsilon)\|_2^2 \\ &\leq C \left(\int_{D(R)} |\nabla u_\varepsilon|^2 v^2 dx + \int_{D(R)} |\nabla v|^2 u_\varepsilon^2 dx \right) \\ &\leq C \left(\int_{D(R)} |\nabla u_\varepsilon|^2 \varepsilon^{11/4} dx + \int_{D(R)} |\nabla v|_\varepsilon^2 u_\varepsilon^6 dx \right) \\ &\leq C(\varepsilon^{11/4} + \varepsilon^{11/4}) \\ &\leq C\varepsilon^{11/4} \end{aligned}$$

where $|\cdot|_\varepsilon$ denotes the norm with respect to the metric $u_\varepsilon^4 ds^2$. We used the facts that $\int |\nabla u_\varepsilon|^2 dx = C \int u_\varepsilon^6 dx \leq C$ and $\|(p^{-1}\delta_\varepsilon)_* X\| = u_\varepsilon \|X\|$ for any tangent vector X on $D(R)$. We thus have

$$\|u - u_\varepsilon\|_{1,2} \leq C\varepsilon^{11/8}.$$

Because $\|v\|_\infty \leq C\varepsilon^{11/8}$ and $\|u_\varepsilon\|_\infty \leq C\varepsilon^{-1/2}$, we obtain

$$\|u - u_\varepsilon\|_\infty = \|vu_\varepsilon\|_\infty \leq C\varepsilon^{7/8}$$

which completes the proof of the theorem.

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