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Periodic Solutions of Nonlinear Abstract Differential Equations with Infinite Delay*

By

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Introduction

The equation to be studied is

$$w'(t) = Cw(t) + F(t, w(\cdot)), \tag{1}$$

where C is a linear operator (not necessary bounded) in a real Banach space X and $F(t, w(\cdot)) = F(t, w(s), -\infty < s \leq t)$ is a Volterra functional which takes into account all past history. Some concrete examples of equation (1) are: a heat equation with memory

$$u_t = au_{xx}(t, x) + \int_{-\infty}^{t} B(t-s)u(s, x)\,ds + f(u) + F(t) \tag{2}$$

which has application in the theory of heat conduction and viscoelasticity, a telegraph equation

$$u_{tt} = u_{xx}(t, x) - \alpha u_t + h(u) + p(t, x), \tag{3}$$

Volterra integrodifferential equations with infinite delay

$$x'(t) = h(t, x) + \int_{-\infty}^{t} q(t, s, x(s))\,ds. \tag{4}$$

Periodic phenomena are quite apparent throughout biology and astronomy. A number of applications of equation (1) appear in mechanical and electrical systems, nuclear physics, arms race models, problems from economic theory and population control. It is so important to be able to detect periodicity or equilibrium situations with a given structure. Equation (2), with or without memory, has attracted considerable attention in recent years, particularly in the theory of periodic solutions (cf. [9], [14],[23], [28], [29], [33]). Most of the results depend heavily on boundedness of solutions and restrictions on

* This paper is based on the author's dissertation under the direction of Professor T. A. Burton at Southern Illinois University

$f(u)$. A recent work [9] has remedied this deficiency in one aspect. There are a number of papers dealing with the existence of periodic solutions of equation (3) (cf. [10], [11], [19], [23], [25], [31], [36]). Very good results have been obtained. However, the solution to the problem is still far from complete. The difficulty is that the solutions are not smoother than their initial functions. This prevents investigators from constructing a compact set in order to use a fixed point theorem. Thus in some situations only a weak periodic solution can be obtained. Equation (4) has been the subject of extensive investigation as a functional differential equation. For disscussion and history, see Burton [4], Hale [22], Yoshizawa [37]. Great progress has been made in determining the existence of periodic solutions of (4) (cf. [6], [7], [8]). But one still wishes to get more general and verifiable results.

In order to unify and improve the works mentioned above, we study the existence of periodic solutions of equation (1) under very general hypotheses. Two approaches that have been adopted are degree theory and Liapunov techniques. Roughly speaking, the main results of this paper are contained in the following statements: Suppose that

(*a*) C generates a C_0-semigroup of linear operators on X with

$$|e^{Ct}| \leq Ke^{-bt} \quad \text{for all } t \geq 0 \text{ and some } K, b > 0;$$

(*b*) $F(t, \phi(\cdot))$ is T-periodic and Hölder continuous in both t and ϕ;

(*c*) there exists an a priori bound for all T-periodic solutions of the equation

$$w'(t) = Cw(t) + \lambda F(t, w(\cdot)), \qquad \lambda \in [0, 1].$$

Then (1) has a T-periodic solution.

This result generalized the recent work of Burton, Eloe, Islam [7] and Burton, Zhang [9]. It also improved some of the earlier results mentioned above. Our work proceeds as follows. In Section 1, we briefly summarize the degree-theoretic work of Granas and then present a general theorem on the existence of periodic solutions of (1). In Sections 2, 3, 4, the result obtained in Section 1 is applied to heat equations, damped wave equations and functional differential equations, respectively. Many concrete examples are given. As usual, $\boldsymbol{R}^n$ denotes the n-dimensional Euclidean space and $\boldsymbol{R}^-$, $\boldsymbol{R}^+$, $\boldsymbol{R}$ denote intervals $(-\infty, 0]$, $[0, +\infty)$, $(-\infty, +\infty)$, respectively. $|\cdot|$ denotes the Euclidean norm on $\boldsymbol{R}^n$. For any $x = (x_1, x_2, \cdots x_n)$, $y = (y_1, y_2, \cdots y_n) \in \boldsymbol{R}^n$, define

$$x \cdot y = \sum_{j=1}^{n} x_j \cdot y_j.$$

If u, or some derivative of u, is written without its argument, then the argument is (t, x).

§ 1. A general theorem

In this section, we first state the definitions and results from Granas [20], [16]. Let Y be a convex subset of a Banach space E. By a pair (X, A) in Y is meant an arbitrary subset X of Y and an $A \subset X$ closed in X.

Definitions. (1°) A continuous map $\phi: X \to Y$ is compact if $\overline{\phi(X)}$ is compact.
(2°) $h_\lambda := h(\lambda, \cdot): X \to Y$ is a compact homotopy if h is a homotopy and if for each $\lambda \in [0, 1]$, h_λ is compact.
(3°) $\phi: X \to Y$ is admissible with respect to A if ϕ is compact and $\phi|_A$ is fixed point free. Let $M_A(X, Y)$ denote the class of admissible maps with respect to A.
(4°) $\phi \in M_A(X, Y)$ is inessential if there exists ψ in $M_A(X, Y)$ such that $\phi|_A = \psi|_A$ and ψ is without fixed points on X. Otherwise, $\phi \in M_A(X, Y)$ is essential.
(5°) A compact homotopy $h: [0, 1] \times X \to Y$ is admissible if for each $\lambda \in [0, 1]$, h_λ is admissible. Two mappings ϕ, ψ in $M_A(X, Y)$ are homotopic, written $\phi \sim \psi$, if there exists an admissible homotopy $h: [0, 1] \times X \to Y$ such that $h_0 = \psi$ and $h_1 = \phi$.

Theorem G1 (Granas). *Let $X \subset Y$ be closed, and let $A = \partial X$, the boundary of X in Y. If $\phi: X \to Y$ is a constant map ($\phi(x) = p$ for all x in X) and $p \in X \setminus A$, then ϕ is essential.*

Theorem G2 (Granas). *Let (X, A) be a pair in Y. The following conditions on $f \in M_A(X, Y)$ are equivalent:*
(α) $f: X \to Y$ is inessential.
(β) $f \sim g$ in $M_A(X, Y)$ where $g: X \to Y$ is without fixed points.

We consider a companion to (1)

$$u'(t) = Cu(t) + \lambda F(t, u(\cdot)), \qquad 0 \le \lambda \le 1 \tag{1_λ}$$

and the homotopy (on a space to be defined later)

$$h(\lambda, \phi)(t) = \lambda \int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(s))\, ds.$$

The theory of Granas is applied in the following way:
(*a*) An a priori bound B_1 is found for all possible T-periodic solutions of (1_λ).
(*b*) A convex subset Y of a Banach space of T-periodic functions is found and a set $X \subset \{\phi \in Y: |\phi| \le B_1\}$ is constructed with the property that $h: X \to Y$

is compact and any fixed point of $h(\lambda, \cdot)$ satisfies (1_λ).
(*c*) In Theorem G1 we take $\phi = h_0$ so that $h_0: X \to 0 \in X \setminus A$, where $A = \partial X$. Thus, h_0 is essential.
(*d*) By Theorem G2, taking $f = h_0$ and $g = h_1$, if h_1 is without fixed points, then h_0 is inessential, a contradiction.

Let X_j $(j = 1, 2)$ be a Banach space with norm $|\cdot|_{X_j}$ such that X_2 is compactly imbedded in X_1. Recall that X_2 is compactly imbedded in X_1 if X_2 is a vector subspace of X_1, and the identity operator I defined on X_2 into X_1 by $Ix = x$ for all $x \in X_2$ is compact. For any $T > 0$ and a Banach space H, define

$$\begin{cases} \tilde{H} = \{\phi \in C(\boldsymbol{R}, H) \mid \phi(t + T) = \phi(t)\} \\ |\phi|_{\tilde{H}} = \sup\{|\phi(t)|_H \mid 0 \le t \le T\}. \end{cases}$$

Thus $(\tilde{H}, |\cdot|_{\tilde{H}})$ is a Banach space. We also denote $\tilde{H}$ by $C_T(\boldsymbol{R}, H)$.

Next consider the abstract equation in X_1

$$w'(t) = Cw(t) + \lambda F(t, w(\cdot)), \qquad t \ge 0, \tag{1.1}$$

where C is a linear operator on X_1 with domain $D(C)$, $\lambda \in [0, 1]$, $F(t, w(\cdot))$ is a Volterra operator,

$$F(t, w(\cdot)) = F(t, w(s), -\infty < s \le t)$$

and

$$F(t, w(\cdot)) = F(t + T, w(\cdot)) \quad \text{for some} \quad T > 0.$$

For each $\mu > 0$, define

$$S_\mu = \{\phi \in \tilde{X}_2 : |\phi|_{\tilde{X}_2} \le \mu\}. \tag{1.2}$$

We suppose that C is the infinitesimal generator of a C_0-semigroup e^{Ct} for $t \ge 0$ in X_1 and there are positive constants $\tilde{C}$, b and $\beta \in (0, 1]$ such that

$$\begin{cases} |e^{Ct}| \le \tilde{C} e^{-bt} & \text{for } t \ge 0, \\ |(e^{Ct} - I)\phi|_{X_1} \le \tilde{C} t^\beta |\phi|_{X_2} & \text{for } \phi \in X_2. \end{cases} \tag{1.3}$$

Definition 1.1. Let $T > 0$ be a constant. A function $u: \boldsymbol{R} \to X_1$ is called a T-periodic solution of (1.1) if $u(t) = u(t, + T)$ for all $t \in \boldsymbol{R}$, $u: \boldsymbol{R} \to X_1$ is continuously differentiable and (1.1) is satisfied on $\boldsymbol{R}^+$. We denote such a solution by $u(t) = u(t, \phi)$.

Definition 1.2. Let X be a Banach space and I be an interval. A function $f: I \to X$ is called Hölder continuous on I if there exists a constant $L > 0$ and $0 < \theta \le 1$, such that

$$|f(t)-f(s)|_X \le L|t-s|^\theta \qquad \text{for } s, t\in I.$$

Theorem 1.1. *Suppose the following conditions hold*:
(i) *there exists* $B>0$ *such that if* $w(t)=w(t,\phi)$ *is a T-periodic solution of* (1.1), *then* $|w(t)|_{X_2}<B$ *for* $t\ge 0$;
(ii) $F: \boldsymbol{R}\times\tilde{X}_2\to X_1$ *takes bounded sets into bounded sets*;
(iii) $F(\cdot, w(\cdot)): [0, T]\to X_1$ *is continuous whenever* $w\in\tilde{X}_2$ *and* $w: [0, T]\to X_1$ *is Hölder continuous*;
(iv) $\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds\in D(C)$ *for any* $t\ge 0$ *whenever* $w\in\tilde{X}_2$ *and* $w: [0, T]\to X_1$ *is Hölder continuous*;
(v) *for any* $\mu>0$, *there exists an* $M_\mu>0$ *such that* $[w\in S_\mu, t\in\boldsymbol{R}^+]$ *imply that*

$$\left|\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds\right|_{X_2}\le M_\mu;$$

(vi) *for any* $\mu>0$, $\varepsilon>0$, *there exists* $\delta>0$ *such that* $[\phi, \psi\in S_\mu, |\phi-\psi|_{\tilde{X}_1}<\delta, t\in\boldsymbol{R}^+]$ *imply that*

$$\left|\int_{-\infty}^{t} e^{C(t-s)}[F(s, \phi(\cdot))-F(s, \psi(\cdot))]\,ds\right|_{X_2}<\varepsilon.$$

Then (1.1) *has a T-periodic solution for* $\lambda=1$.

Proof. For $B>0$ defined in (i), by (ii) and (v) there exists $M_B>0$ such that

$$\begin{cases} |F(t, w(\cdot))|_{X_1}\le M_B, \\ \left|\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds\right|_{X_2}\le M_B \qquad \text{for } t\in\boldsymbol{R}^+, \end{cases} \tag{1.4}$$

whenever $w\in\tilde{X}_2$ and $|w|_{\tilde{X}_2}\le B$. Let $L=\tilde{C}M_B(1+T^{1-\beta})$ with β given in (1.3) and define

$$\begin{cases} Y=\{w\in\tilde{X}_2: |w(t_1)-w(t_2)|_{X_1}\le L|t_1-t_2|^\beta,\ t_1, t_2\in[0, T]\}, \\ X_B=\{w\in Y: |w|_{\tilde{X}_2}\le B\} \end{cases} \tag{1.5}$$

Lemma 1.1. *If* $h: [0, 1]\times X_B\to Y$ *is defined by*

$$h(\lambda, w)(t)=\lambda\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds \tag{1.6}$$

and if w *is a fixed point of* h, *then* w *is a T-periodic solution of* (1.1).

Proof of Lemma 1.1. Let $w\in X_B$, $t\in\mathbf{R}$, $\delta>0$ and consider

$$(1.7)\quad [h(\lambda, w)(t+\delta) - h(\lambda, w)(t)]/\delta$$
$$= \lambda(1/\delta)\int_t^{t+\delta} e^{C(t+\delta-s)}F(s, w(\cdot))\,ds + \lambda\left(\frac{e^{C\delta}-I}{\delta}\right)\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds.$$

By the continuity condition (iii), we have

$$(1.8)\qquad \lambda(1/\delta)\int_t^{t+\delta} e^{C(t+\delta-s)}F(s, w(\cdot))\,ds \longrightarrow \lambda F(t, w(\cdot))$$

as $\delta \to 0$ in X_1. Since $\int_{-\infty}^{t} e^{C(t-s)}F(s, w(\cdot))\,ds \in D(C)$, it follows that

$$(1.9)\qquad [h(\lambda, w)(t+\delta) - h(\lambda, w)(t)]/\delta \longrightarrow \lambda F(t, w(\cdot)) + Ch(\lambda, w)(t)$$

as $\delta \to 0$. Thus

$$(1.10)\qquad h'(t) = Ch(t) + \lambda F(t, w(\cdot)),$$

where $h(t) = h(\lambda, w)(t)$. This yields that if w is a fixed point of h, then w satisfies (1.1). To complete the proof of Lemma 1.1, we need to show that $h(\lambda, w) \in Y$. Let $t_1, t_2 \in [0, T]$ and $t_2 < t_1$, $w \in X_B$; we have

$$h(\lambda, w)(t_1) - h(\lambda, w)(t_2) = \lambda\int_{t_2}^{t_1} e^{C(t_1-s)}F(s, w(\cdot))\,ds$$
$$+ \lambda[e^{C(t_1-t_2)} - I]\int_{-\infty}^{t_2} e^{C(t_2-s)}F(s, w(\cdot))\,ds.$$

Thus

$$|h(\lambda, w)(t_1) - h(\lambda, w)(t_2)|_{X_1} \le \tilde{C}\int_{t_2}^{t_1} |F(s, w(\cdot))|_{X_1}\,ds$$
$$+ \left|(e^{C(t_1-t_2)} - I)\int_{-\infty}^{t_2} e^{C(t_2-s)}F(s, w(\cdot))\,ds\right|_{X_1}$$
$$\le \tilde{C}M_B T^{1-\beta}|t_1 - t_2|^{\beta} + \tilde{C}|t_1 - t_2|^{\beta}\left|\int_{-\infty}^{t_2} e^{C(t_2-s)}F(s, w(\cdot))\,ds\right|_{X_2}$$
$$\le L|t_1 - t_2|^{\beta}.$$

Thus $h(\lambda, w) \in Y$ and the proof of Lemma 1.1 is complete.

Lemma 1.2. [15] *Let H be a Banach space and Φ be an equicontinuous family of continuous functions in $\tilde{H}$ such that $\{\phi(t) | \forall \phi \in \Phi\}$ is relatively compact in H for each $t \in [0, T]$. Then Φ is relatively compact in $\tilde{H}$.*

Lemma 1.3. *The homotopy $h(\lambda, w)$ is compact.*

Proof of Lemma 1.3. Let $w_j \in X_B$ $(j = 1, 2)$; then

$$|h(\lambda, w_1)(t) - h(\lambda, w_2)(t)|_{X_2} = \lambda \left| \int_{-\infty}^{t} e^{C(t-s)}[F(s, w_1(\cdot)) - F(s, w_2(\cdot))]\, ds \right|_{X_2}. \tag{1.11}$$

By (vi) with $\mu = B$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|w_1 - w_2|_{\tilde{X}_1} < \delta \quad \text{implies that} \quad |h(\lambda, w_1)(t) - h(\lambda, w_2)(t)|_{X_2} < \varepsilon.$$

This, incidentially, shows that $h(\lambda, w)$ is continuous in w for fixed λ. Since $h(\lambda, w)$ is uniformly continuous in λ for fixed w, it follows that h is jointly continuous in (λ, w). Let $\{w_n\}$ be a sequence in X_B; then $\{w_n\}$ is equicontinuous in $\tilde{X}_1$. Moreover, $|w_n|_{\tilde{X}_2} \leq B$ implies that for fixed t, $\{w_n(t)\}$ is relatively compact in X_1. By Lemma 1.2 with $\tilde{H} = \tilde{X}_1$, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $|w_{n_k} - w_{n_m}|_{\tilde{X}_1} \to 0$ as $k, m \to +\infty$. Thus from (1.11) and (vi), it follows that $|h(\lambda, w_{n_k}) - h(\lambda, w_{n_m})|_{\tilde{X}_2} \to 0$ as $k, m \to +\infty$. This means that $\{h(\lambda, w_{n_k})\}$ is a Cauchy sequence in $\tilde{X}_2$, a Banach space, and so h is a compact mapping. The proof of Lemma 1.3 is complete.

Now we are ready to complete the proof of Theorem 1.1. Let X_B and Y be given in (1.5) and define

$$\tilde{A} = \{\phi \in Y \mid |\phi|_{\tilde{X}_2} = B\}.$$

By (i) and Lemma 1.1, $h(\lambda, \cdot)$ is fixed point free on $\tilde{A}$. Y is a convex subset of the Banach space $\tilde{X}_2$. Also X_B is closed in Y and for $\tilde{A}$ defined above, $\tilde{A} = \partial X_B$ in Y. Referring to Granas' Theorem G1, we take $\phi = h_0$ so that $h_0 : X_B \to 0 \in X_B \setminus \tilde{A}$ implies that h_0 is essential. Thus, in Granas' Theorem G2, we take $f = h_0$ and $g = h_1$. If h_1 is without fixed points, then h_0 is inessential, a contradiction. Hence, h_1 has a fixed point in X_B which satisfies (1.1) by Lemma 1.1. This completes the proof of Theorem 1.1.

§ 2. Parabolic equations

Consider the equation

$$u_t + A(x, D)u(t, x) = \lambda F(t, u(\cdot, x)) \tag{2.1}$$

in $L^2(\Omega)$ where Ω is a bounded domain in $\boldsymbol{R}^n$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. We will assume that $\partial\Omega$ is smooth. This will mean that $\partial\Omega$ is of the class C^k for some suitable $k \geq 1$. Recall that $\partial\Omega$ is of the class C^k if for each point $x \in \partial\Omega$ there exists a ball B with center at $x = (x_1, x_2, \cdots x_n)$ such that $\partial\Omega \cap B$ can be represented in the form $x_j = \phi(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ for

some j with ϕ k-times continuously differentiable. Let $H^m(\Omega)$ and $H_0^m(\Omega)$ be the usual Sobolev spaces. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a vector of nonnegative integers and consider the differential operator of order $2m$,

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j \tag{2.2}$$

where the coefficients $a_\alpha(x)$ are real-valued functions of x in $\bar{\Omega}$ with $a_\alpha \in L^\infty(\Omega)$ for all α and $a_\alpha \in C(\bar{\Omega})$ for $|\alpha| = 2m$.

Definition 2.1. The operator $A(x, D)$ defined in (2.2) is strongly elliptic of order $2m$ if there exists a constant $C_0 > 0$ such that

$$(-1)^m \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \geq C_0 |\xi|^{2m} \qquad \text{for all } x \in \bar{\Omega} \text{ and } \xi \in \boldsymbol{R}^n$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$.

With the given strongly elliptic operator $A(x, D)$ on a bounded domain $\Omega \subset \boldsymbol{R}^n$ we associate an unbounded linear operator acting in the Hilbert space $H^0 = L^2(\Omega)$ (cf. [34], p. 206–210).

Definition 2.2. Let $A(x, D)$ be a strongly elliptic operator of order $2m$. Set $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$. For every $u \in D(A)$, define

$$Au = A(x, D)u.$$

Throughout this section, we assume that

$$\begin{cases} A(x, D) \text{ is a strongly elliptic operator of order } 2m \text{ and} \\ \operatorname{Re} \sigma(A) \geq C_0 \text{ for some } C_0 > 0, \text{ where } \sigma(A) \text{ is the spectrum of } A. \end{cases} \tag{2.3}$$

It is well known that $-A$ generates an analytic semigroup on $H^0 = H^0(\Omega)$ and there exists a constant $b > 0$ such that

$$\begin{aligned} &\forall \alpha \geq 0, \exists C_\alpha > 0 \text{ such that } |A^\alpha e^{-At}| \leq C_\alpha t^{-\alpha} e^{-bt} \qquad \text{for } t > 0, \\ &|(e^{-At} - I)\phi|_{H^0} \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha |A^\alpha \phi|_{H^0} \qquad \text{if } 0 < \alpha \leq 1, \phi \in D(A^\alpha). \end{aligned} \tag{2.4}$$

Also, C_α is bounded for α in any compact interval of $(0, +\infty)$.

For any $0 < \alpha \leq 1$, we consider $D(A^\alpha)$ as the Banach space obtained by endowing $D(A^\alpha)$ with the graph norm $|\phi|_{D(A^\alpha)} = |A^\alpha \phi|_{H^0}$ (cf. Henry [24], p. 29, Theorem 1.4.8). The abstract version of (2.1) is

$$u'(t) + Au(t) = \lambda F(t, u(\cdot)), \tag{2.5}$$

where $F(t, w(\cdot))$ is considered as a function $F: \boldsymbol{R} \times \tilde{H}^0 \to H^0$ given by

$$F(t, u(\cdot))(x) = F(t, u(\cdot, x)).$$

We assume that

$$F(t + T, u(\cdot)) = F(t, u(\cdot)) \qquad \text{for some } T > 0. \tag{2.6}$$

Theorem 2.1. *Suppose there exists a* σ, $0 < \sigma < 1$, *with* $D(A^\sigma)$ *being compactly imbedded in* H^0 *such that the following conditions hold*:
(A_1) *there exists* $B > 0$ *such that if* $u(t)$ *is a T-periodic solution of* (2.5), *then* $|A^\sigma u(t)|_{H^0} < B$ *for* $t \geq 0$;
(A_2) $F: \boldsymbol{R} \times D(A^\sigma) \to H^0$ *takes bounded sets into bounded sets*;
(A_3) $F(\cdot, \phi(\cdot)): [0, T] \to H^0$ *is Hölder continuous whenever* $\phi \in C_T(R, D(A^\sigma))$ *and* $\phi: [0, T] \to H^0$ *is Hölder continuous*;
(A_4) *for any* $\alpha > 0$, *there exists* $\beta > 0$ *such that*

$$[\phi, \psi \in C_T(R, D(A^\sigma)),\ |A^\sigma \phi|_{\tilde{H}^0} \leq \alpha,\ |A^\sigma \psi|_{\tilde{H}^0} \leq \alpha,\ t \in \boldsymbol{R}]$$

imply that

$$|F(t, \phi(\cdot)) - F(t, \psi(\cdot))|_{H^0} \leq \beta |\phi - \psi|_{\tilde{H}^0}.$$

Then (2.5) *has a T-periodic solution for* $\lambda = 1$.

Proof. We prove Theorem 2.1 by verifying that all conditions of Theorem 1.1 hold with $C = -A$. Let $X_1 = H^0$ and $X_2 = D(A^\sigma)$. Thus there are positive constants $\tilde{C}$, b such that

$$\begin{cases} |e^{Ct}| \leq \tilde{C} e^{-bt}, \\ |(e^{Ct} - I)\phi|_{X_1} \leq \tilde{C} t^\sigma |\phi|_{X_2} \qquad \text{for } \phi \in X_2, \\ |A^\sigma e^{-At}| \leq \tilde{C} t^{-\sigma} e^{-bt}. \end{cases} \tag{2.7}$$

Let $\phi \in \tilde{X}_2$ and $\phi: [0, T] \to X_1$ be Hölder continuous; then $F(t, \phi(\cdot)): [0, T] \to H^0$ is Hölder continuous by (A_3). From Henry [24, p. 50, Lemma 3.2.1] and Pazy [34, p. 111, Theorem 3.2], it follows that

$$\int_0^t e^{-A(t-s)} F(s, \phi(\cdot))\, ds \in D(C) \qquad \text{for } t \in [0, T].$$

For $t > 0$, as $\int_{-\infty}^0 e^{-A(t-s)} F(s, \phi(\cdot))\, ds$ and $\int_{-\infty}^0 e^{As} F(s, \phi(\cdot))\, ds$ exist in H^0, we have

$$\int_{-\infty}^0 e^{-A(t-s)} F(s, \phi(\cdot))\, ds = e^{-At} \int_{-\infty}^0 e^{As} F(s, \phi(\cdot))\, ds$$

by Friedman [18, p. 94, Lemma 1.2 and p. 101, Theorem 2.1]. Moreover

$$e^{-At}\int_{-\infty}^{0} e^{As}F(s,\phi(\cdot))\,ds\in D(C) \qquad \text{for } t>0.$$

Thus

$$\int_{-\infty}^{t} e^{C(t-s)}F(s,\phi(\cdot))\,ds\in D(C)$$

and condition (iv) of Theorem 1.1 holds.

Next let $\mu>0$ and $\phi\in\tilde{X}_2$ with $|\phi|_{\tilde{X}_2}\le\mu$; then

$$\left|\int_{-\infty}^{t} e^{-A(t-s)}F(s,\phi(\cdot))\,ds\right|_{X_2}=\left|A^{\sigma}\int_{-\infty}^{t} e^{-A(t-s)}F(s,\phi(\cdot))\,ds\right|_{H^0}$$

$$\le\tilde{C}\int_{0}^{+\infty} s^{-\sigma}e^{-bs}\,ds\sup_{\xi\in\mathbf{R}}|F(\xi,\phi(\cdot))|_{H^0}.$$

By (A_2), there exists $M_\mu>0$ such that

$$\left|\int_{-\infty}^{t} e^{-A(t-s)}F(s,\phi(\cdot))\,ds\right|_{X_2}\le M_\mu.$$

Define $E_r=\int_0^{+\infty} s^{-r}e^{-bs}\,ds$ for $r\in[0,1)$. Let $\mu>0$. By (A_4) there exists $\beta>0$ such that

$$[\phi,\psi\in\tilde{X}_2,\ |\phi|_{\tilde{X}_2}\le\mu,\ |\psi|_{\tilde{X}_2}\le\mu,\ s\in\mathbf{R}]$$

imply that

$$|F(s,\phi(\cdot))-F(s,\psi(\cdot))|_{H^0}\le\beta|\phi-\psi|_{\tilde{H}^0}. \tag{2.8}$$

Let $\varepsilon>0$ be given and choose $\delta=\varepsilon/2\tilde{C}\beta E_\sigma$. Suppose that $|\phi-\psi|_{\tilde{X}_1}<\delta$; then

$$\left|\int_{-\infty}^{t} e^{-A(t-s)}[F(s,\phi(\cdot))-F(s,\psi(\cdot))]\,ds\right|_{X_2}$$

$$=\left|A^{\sigma}\int_{-\infty}^{t} e^{-A(t-s)}[F(s,\phi(\cdot))-F(s,\psi(\cdot))]\,ds\right|_{H^0}\le\tilde{C}\beta E_\sigma|\phi-\psi|_{\tilde{X}_1}<\varepsilon.$$

Thus condition (vi) of Theorem 1.1 holds. The continuity condition (iii) of $F(t,\phi(\cdot))$ follows from (A_3) and (A_4). By Theorem 1.1, (2.5) has a T-periodic solution for $\lambda=1$. This completes the proof of Theorem 2.1.

Remark 2.1. One may use the following lemma to find σ in Theorem 2.1 such that $D(A^\sigma)$ is compactly imbedded in H^0 (cf. Hale [23], p. 72; Henry

[24], Theorem 1.4.8).

Lemma 2.0. *Let A be defined in Definition* 2.2. *Then $D(A^\sigma)$ is a dense subspace of $D(A^\beta)$ with continuous inclusion for any $\sigma \geq \beta \geq 0$. If A has compact resovent, then inclusion $D(A^\sigma) \subset D(A^\beta)$ is compact for $\sigma > \beta \geq 0$.*

Let Δ be the Laplacian operator and define $A\phi = -\Delta\phi$ on the space $C_0^{+\infty}$, smooth functions with compact support in Ω. Then one can extend A to a self-adjoint positive definite operator in $H^0(\Omega)$. Moreover, $D(A^{1/2}) = H_0^1(\Omega)$ (cf. Hale [23], p. 124). Since the imbedding $H_0^1(\Omega) \subset H^0(\Omega)$ is compact (cf. Friedman [18], p. 31, Theorem 11.2), it follows that the imbedding $D(A^\sigma) \subset H^0(\Omega)$ is also compact whenever $\sigma \geq 1/2$. It is noted in [23, p. 72] that in the one dimensional case ($\Omega = (0, \pi)$) the resovent operator of A is compact. Thus for $\sigma > 0$, $D(A^\sigma) \subset H^0(0, \pi)$ is compact by Lemma 2.0.

Example 2.1. Consider the one dimensional heat equation

$$u_t = u_{xx} + \lambda\left[f(u) + \int_{-\infty}^{t} D(t-s)\, g(u(s, x))\, ds + p(t, x)\right], \tag{2.9}$$

$u(t, 0) = u(t, \pi) = 0$, $\lambda \in [0, 1]$, where

(2.10)
$$\begin{cases} u\left[f(u) + \displaystyle\int_0^{+\infty} D(s)\, ds\, g(u)\right] \leq ku^2 + M, & \text{for some } k \in [0, 1),\ M > 0, \\ p(t + T, x) = p(t, x) \text{ for some } T > 0, \quad f, g \in C^1, \quad D \text{ is continuous.} \end{cases}$$

We define $P: \boldsymbol{R} \to H^0$ by $P(t)(x) = p(t, x)$ and assume that

$$\begin{cases} |g'(u)| \leq \mu \text{ for some } \mu > 0 \text{ and all } u \in \boldsymbol{R}, \\ \mu \displaystyle\int_0^{+\infty} \left| \int_t^{+\infty} D(s)\, ds \right| dt < 1, \\ P \in \tilde{H}^0 \text{ and } P: [0, T] \longrightarrow H^0 \text{ is Hölder continuous.} \end{cases} \tag{2.11}$$

Theorem 2.2. *Suppose that* (2.10) *and* (2.11) *hold. Then* (2.9) *has a T-periodic solution for $\lambda = 1$.*

Proof. Let $(A\phi)(x) = -\partial^2\phi(x)/\partial x^2$ whenever ϕ is a smooth function on $[0, \pi]$ with $\phi(0) = \phi(\pi) = 0$ and by using Friedrich's theorem, extend A to a self-adjoint, densely defined operator in $H^0(0, \pi)$. Then $D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$ and A satisfies (2.3). We take $\sigma = 1/2$, then $D(A^{1/2}) = H_0^1(0, \pi)$ (cf. Hale [23], p. 75 and Henry [24], p. 18) and $D(A^{1/2})$ is compactly imbedded in $H^0(0, \pi)$.

Now we rewrite (2.9) as an abstract equation in $H^0(0, \pi)$

$$u'(t) + Au(t) = \lambda F(t, u(\cdot)), \tag{2.12}$$

where

$$F(t, u(\cdot)) = f(u) + \int_{-\infty}^{t} D(t-s)g(u(s))\,ds + P(t). \tag{2.13}$$

We prove Theorem 2.2 by verifying that all conditions of Theorem 2.1 are satisfied.

Lemma 2.1. *Suppose that* (2.10) *and* (2.11) *hold. Then there exists a constant* $B_1 > 0$ *such that if* $u(t)$ *is a T-periodic solution of* (2.12), *then* $|u(t)|_{H^1} < B_1$.

Proof of Lemma 2.1. Let $u(t, x)$ be a T-periodic solution of (2.12) and define $V_1(t) = \int_0^\pi u^2(t, x)\,dx$. Then

(2.14)

$$\begin{aligned}
V_1'(t) &= 2\int_0^\pi uu_t\,dx \\
&= 2\int_0^\pi u\left\{u_{xx} + \lambda\left[f(u) + \int_{-\infty}^{t} D(t-s)g(u(s, x))\,ds + P(t)\right]\right\}dx \\
&= -2\int_0^\pi u_x^2\,dx + 2\lambda\int_0^\pi u\left[f(u) + \int_0^{+\infty} D(s)\,ds\; g(u)\right]dx \\
&\quad - 2\lambda\int_0^\pi u\frac{d}{dt}\int_{-\infty}^{t}\int_t^{+\infty} D(\tau - s)\,d\tau\, g(u(s, x))\,ds dx + 2\lambda\int_0^\pi up(t, x)\,dx \\
&\le -2(1-k)\int_0^\pi u_x^2\,dx - 2\lambda\int_0^\pi u\int_{-\infty}^{0}\int_{-s}^{+\infty} D(\tau)\,d\tau\,(g(u(t+s, x)))_t\,ds dx \\
&\quad + 2\lambda\int_0^\pi up(t, x)\,dx + 2M\pi \\
&\le -2(1-k)\int_0^\pi u_x^2\,dx + \delta\mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right|ds\int_0^\pi u^2\,dx + 2M\pi \\
&\quad + \frac{\mu}{\delta}\int_0^\pi\int_{-\infty}^{0}\left|\int_{-s}^{+\infty} D(\tau)\,d\tau\right|u_t^2(t+s, x)\,ds dx + 2\lambda\int_0^\pi up(t, x)\,dx.
\end{aligned}$$

Choosing $\delta > 0$ so small that $k + \delta < 1$, we then have

(2.15)
$$V_1'(t) \leq -\alpha_1 \int_0^\pi u_x^2\,dx + \beta_1 \int_0^\pi \int_{-\infty}^0 \left| \int_{-s}^{+\infty} D(\tau)\,d\tau \right| u_t^2(t+s, x)\,ds dx + \gamma_1$$

for some positive constants $\alpha_1, \beta_1, \gamma_1$. Let

$$V_2(t) = \int_0^\pi u_x^2(t, x)\,dx - 2\lambda \int_0^\pi \int_0^u \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi\, dx.$$

We claim that $d/dt \int_0^\pi u_x^2(t, x)\,dx = -2\int_0^\pi u_t(t, x) u_{xx}(t, x)\,dx$. Indeed, let A be the operator given in (2.12) and $\phi_n(x) = \sqrt{2/\pi} \sin nx$; we then have $A\phi_n = \lambda_n \phi_n$ with $\lambda_n = n^2$ for each n. Moreover, if $u(t)$ is a T-periodic solution of (2.12), then

$$u(t) = \sum_{n=1}^{\infty} C_n(t)\phi_n,\ u_t(t) = \sum_{n=1}^{\infty} C_n'(t)\phi_n$$

where $C_n: \boldsymbol{R} \to \boldsymbol{R}$ is continuously differentiable and T-periodic with

$$|u_t(t)|_{H^0}^2 + |Au(t)|_{H^0}^2 = \sum_{n=1}^{\infty} |C_n'(t)|^2 + \sum_{n=1}^{\infty} \lambda_n^2 |C_n(t)|^2 \leq Q$$

for some constant $Q > 0$.

Notice that $|u_x(t)|_{H^0}^2 = \sum_{n=1}^{\infty} |C_n(t)|^2 \lambda_n$. Thus,

$$\frac{d}{dt}|u_x(t)|_{H^0}^2 = \sum_{n=1}^{\infty} 2C_n(t)C_n'(t)\lambda_n = -2\sum_{n=1}^{\infty} \langle u_{xx}(t), \phi_n\rangle \langle u_t(t), \phi_n\rangle$$
$$= -2\langle u_t(t), u_{xx}(t)\rangle = -2\int_0^\pi u_t(t, x) u_{xx}(t, x)\,dx.$$

Then

(2.16)
$$V_2'(t) = -2\int_0^\pi u_t u_{xx}\,dx - 2\lambda \int_0^\pi u_t \left[f(u) + \int_0^{+\infty} D(\tau)\,d\tau\, g(u) \right] dx$$
$$= -2\int_0^\pi u_t \left[u_{xx} + \lambda \left(f(u) + \int_0^{+\infty} D(\tau)\,d\tau\, g(u) \right) \right] dx$$
$$= -2\int_0^\pi u_t^2\,dx - 2\lambda \int_0^\pi u_t \frac{d}{dt} \int_{-\infty}^t \int_t^{+\infty} D(\tau - s)\,d\tau\, g(u(s, x))\,ds dx$$
$$+ 2\lambda \int_0^\pi u_t p(t, x)\,dx$$

$$\leq -2\int_0^\pi u_t^2\,dx + \mu\int_0^\pi\int_{-\infty}^0 \left|\int_{-s}^{+\infty} D(\tau)d\tau\right| ds u_t^2\,dx$$
$$+\mu\int_0^\pi\int_{-\infty}^0 \left|\int_{-s}^{+\infty} D(\tau)\,d\tau\right| u_t^2(t+s,x)\,dsdx + 2\lambda\int_0^\pi u_t p(t,x)\,dx.$$

As u and V_2 are T-periodic, we have

$$0 = V_2(T) - V_2(0)$$
$$\leq -2\int_0^T\int_0^\pi u_t^2\,dxdt + \mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\int_0^T\int_0^\pi u_t^2\,dxdt$$
$$+\mu\int_0^\pi\int_{-\infty}^0\left|\int_{-s}^{+\infty} D(\tau)\,d\tau\right|\int_0^T u_t^2(t+s,x)\,dtdsdx + 2\lambda\int_0^T\int_0^\pi u_t p(t,x)\,dxdt$$
$$= -2\left(1-\mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\right)\int_0^T\int_0^\pi u_t^2\,dxdt + 2\lambda\int_0^T\int_0^\pi u_t p(t,x)\,dxdt.$$

By (2.11) and the fact that $P\in\tilde{H}^0$, there exists $M_1 > 0$ such that

$$\int_0^T\int_0^\pi u_t^2\,dxdt \leq M_1. \tag{2.17}$$

In the same way we consider V_1 and find that

$$0 = V_1(T) - V_1(0)$$
$$\leq -\alpha_1\int_0^T\int_0^\pi u_x^2\,dxdt + \beta_1\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\int_0^T\int_0^\pi u_t^2\,dxdt + \gamma_1 T.$$

Thus

$$\alpha_1\int_0^T\int_0^\pi u_x^2\,dxdt \leq \beta_1\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\int_0^T\int_0^\pi u_t^2\,dxdt + \gamma_1 T.$$
$$\leq \beta_1\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\,M_1 + \gamma_1 T$$

so that there exists $M_2 > 0$ with

$$\int_0^T\int_0^\pi (u^2 + u_x^2 + u_t^2)\,dxdt \leq M_2. \tag{2.18}$$

Now let $V(t) = V_1(t) + V_2(t)$ and use (2.15) and (2.16) to obtain

$$V'(t) \leq -\alpha\int_0^\pi (u^2 + u_x^2 + u_t^2)\,dx + \beta\int_0^\pi\int_{-\infty}^0\left|\int_{-s}^{+\infty} D(\tau)\,d\tau\right| u_t^2(t+s,x)\,dsdx + \gamma$$

for some positive constants α, β, γ. Now u and u_x are continuous, so by (2.18) there exists $t_1 \in [0, T]$ with

$$T\int_0^\pi [u^2(t_1, x) + u_x^2(t_1, x)]\,dx \le \int_0^T \int_0^\pi (u^2 + u_x^2)\,dx dt \le M_2.$$

Thus, by Sobolev's inequality

$$\sup_{0 \le x \le \pi} |u(t_1, x)| \le C_1 \qquad \text{for some } C_1 > 0.$$

This means that

$$\left| \int_0^{u(t_1,x)} \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi \right| \le \max_{0 \le |u| \le C_1} \left| \int_0^u \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi \right| \stackrel{\text{def}}{=} C_2 \tag{2.19}$$

and

$$V(t_1) \le \int_0^\pi [u^2(t_1, x) + u_x^2(t_1, x)]\,dx + 2\pi C_2 \le 2\pi C_2 + M_2/T \stackrel{\text{def}}{=} C_3.$$

Hence, if $t_1 \le t \le t_1 + T$ we have

$$\begin{aligned} V(t) &= V(t_1) + \int_{t_1}^t V'(s)\,ds \\ &\le V(t_1) + \beta \int_0^\pi \int_{-\infty}^0 \left| \int_{-s}^{+\infty} D(\tau)\,d\tau \right| \left| \int_{t_1}^{t_1+T} u_t^2(t + s, x)\,dt ds dx + \gamma T \right. \\ &\le C_3 + \beta \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)\,d\tau \right| ds \int_0^\pi \int_0^T u_t^2(t, x)\,dt dx + \gamma T \\ &\le C_3 + \beta \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)\,d\tau \right| ds\, M_2 + \gamma T \stackrel{\text{def}}{=} C_4. \end{aligned}$$

From (2.10) we have

$$u\left[f(u) + \int_0^{+\infty} D(s)\,ds\, g(u) \right] \le ku^2 + M, \quad k \in [0, 1). \tag{2.20}$$

Define

$$C^* = \sup_{|u| \le 1} \left| \int_0^u \left[f(\xi) + \int_0^{+\infty} D(s)\,ds\, g(\xi) \right] d\xi \right|$$

and use $V(t) \le C_4$ to obtain

$$\int_0^\pi [u^2 + u_x^2]\,dx \le C_4 + 2\lambda \int_0^\pi \int_0^u \left[f(\xi) + \int_0^{+\infty} D(s)\,ds\, g(\xi) \right] d\xi\, dx. \tag{2.21}$$

Case 1: If $|u| \le 1$, then

$$\int_0^u \left[f(\xi) + \int_0^{+\infty} D(s)\,ds\, g(\xi) \right] d\xi \le C^*.$$

Case 2: Suppose $|u| > 1$.

(a) If $u > 1$, then

$$\begin{aligned}
&\int_0^u \left[f(\xi) + \int_0^{+\infty} D(s)\,ds\, g(\xi) \right] d\xi \\
&= \int_0^1 \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi + \int_1^u \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi \\
&\le C^* + \int_1^u \left\{ s \left[f(s) + \int_0^{+\infty} D(\tau)\,d\tau\, g(s) \right] / s \right\} ds \\
&\le C^* + \int_1^u [ks + (M/s)]\,ds \le C^* + \frac{1}{2}(1+k)u^2 + M^*
\end{aligned}$$

for some $M^* > 0$.

(b) If $u < -1$, then

$$\begin{aligned}
&\int_0^u \left[f(s) + \int_0^{+\infty} D(\tau)\,d\tau\, g(s) \right] ds = -\int_0^{|u|} \left[f(-s) + \int_0^{+\infty} D(\tau)\,d\tau\, g(-s) \right] ds \\
&= -\int_0^1 \left[f(-s) + \int_0^{+\infty} D(\tau)\,d\tau\, g(-s) \right] ds \\
&\quad - \int_1^{|u|} \left[f(-s) + \int_0^{+\infty} D(\tau)\,d\tau\, g(-s) \right] ds \\
&\le C^* + \int_1^{|u|} [ks + (M/s)]\,ds \le C^* + \frac{1}{2}(1+k)u^2 + M^*.
\end{aligned}$$

Thus

$$\lambda \int_0^u \left[f(\xi) + \int_0^{+\infty} D(\tau)\,d\tau\, g(\xi) \right] d\xi \le C^* + \frac{1}{2}(1+k)u^2 + M^*$$

and

$$\lambda \int_0^{\pi} \int_0^{u} \left[f(\xi) + \int_0^{+\infty} D(\tau)\, d\tau g(\xi) \right] d\xi dx \leq \pi C^* + \frac{1}{2}(1+k) \int_0^{\pi} u^2 dx + M^* \pi.$$

From (2.21), we have

$$\int_0^{\pi} [u^2 + u_x^2]\, dx \leq C_4 + 2\pi C^* + (1+k) \int_0^{\pi} u^2 dx + 2\pi M^*.$$

Hence, there exixts $B_1 > 0$ with

$$\sup_{0 \leq t \leq T} \int_0^{\pi} [u^2(t, x) + u_x^2(t, x)]\, dx \leq B_1.$$

This completes the proof of Lemma 2.1.

Lemma 2.2. *Suppose that* (2.10) *and* (2.11) *hold. Then* $F(t, \phi(\cdot)): [0, T] \to H^0$ *given in* (2.13) *is Hölder continuous whenever* $\phi \in \tilde{H}_0^1$ *and* $\phi: [0, T] \to H^0$ *is Hölder continuous.*

Proof of Lemma 2.2. Let $\phi \in \tilde{H}_0^1$ and $\phi: [0, T] \to H^0$ be Hölder continuous. Then there are constants $L_1 > 0$, $\theta \in (0, 1]$, with $|\phi(t_1) - \phi(t_2)|_{H^0} \leq L_1 |t_1 - t_2|^{\theta}$ for $t_1, t_2 \in [0, T]$. By (2.11), we have

$$|P(t_1) - P(t_2)|_{H^0} \leq L_2 |t_1 - t_2|^{\gamma} \qquad \text{for } t_1, t_2 \in [0, T],$$

and some constants $L_2 > 0$, $\gamma \in (0, 1]$. Let $K_1 > 0$ be such that $|f'(\phi)| \leq K_1$. For $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned}
&|F(t_1, \phi(\cdot)) - F(t_2, \phi(\cdot))|_{H^0} \\
&\quad \leq |f(\phi(t_1)) - f(\phi(t_2))|_{H^0} + |P(t_1) - P(t_2)|_{H^0} \\
&\qquad + \int_{-\infty}^{0} |D(-\tau)|\, |g(\phi(t_1 + \tau) - g(\phi(t_2 + \tau))|_{H^0}\, d\tau \\
&\quad \leq K_1 |\phi(t_1) - \phi(t_2)|_{H^0} + |P(t_1) - P(t_2)|_{H^0} \\
&\qquad + \mu \int_{-\infty}^{0} |D(-\tau)|\, |\phi(t_1 + \tau) - \phi(t_2 + \tau)|_{H^0}\, d\tau \\
&\quad \leq K_1 L_1 |t_1 - t_2|^{\theta} + \mu L_1 \int_0^{+\infty} |D(\tau)|\, d\tau\, |t_1 - t_2|^{\theta} + L_2 |t_1 - t_2|^{\gamma}.
\end{aligned}$$

Thus $F(\cdot, \phi(\cdot)): [0, T] \to H^0$ is Hölder continuous and the proof of Lemma 2.2 is complete.

Lemma 2.3. *Suppose that* (2.10) *and* (2.11) *hold. Then* $F: \boldsymbol{R} \times \tilde{H}_0^1 \to H^0$ *takes bounded sets into bounded sets and condition* (A_4) *of Theorem* 2.1 *holds.*

Proof of Lemma 2.3. It is clear from (2.13) that $F: \boldsymbol{R} \times \tilde{H}_0^1 \to H^0$ takes bounded sets into bounded sets. Now let $\alpha > 0$; if $\phi \in \tilde{H}_0^1$, $|\phi|_{\tilde{H}_0^1} \leq \alpha$, then

$$\sup\{|\phi(t, x)|: 0 \leq t \leq T,\ 0 \leq x \leq \pi\} \leq \tilde{\alpha} \qquad \text{for some } \tilde{\alpha} > 0.$$

Let $K = \sup\{|f'(\xi)|: |\xi| \leq \tilde{\alpha}\}$. For $\phi, \psi \in \tilde{H}_0^1$ and $|\phi|_{\tilde{H}_0^1} \leq \alpha$, $|\psi|_{\tilde{H}_0^1} \leq \alpha$, $t \in \boldsymbol{R}$, we have

$$\begin{aligned}
&|F(t, \phi(\cdot)) - F(t, \psi(\cdot))|_{H^0} \\
&\quad \leq |f(\phi) - f(\psi)|_{H^0} + \int_{-\infty}^{0} |D(-\tau)|\,|g(\phi(t+\tau)) - g(\psi(t+\tau))|_{H^0}\, d\tau \\
&\quad \leq K|\phi - \psi|_{\tilde{H}^0} + \mu \int_0^{+\infty} |D(\tau)|\, d\tau\, |\phi - \psi|_{\tilde{H}^0} \\
&\quad \leq \left(K + \mu \int_0^{+\infty} |D(\tau)| d\tau\right) |\phi - \psi|_{\tilde{H}^0}.
\end{aligned}$$

Thus (A_4) of Theorem 2.1 holds. This completes the proof of Lemma 2.3.

Since all conditions of Theorem 2.1 are satisfied, we conclude that (2.12) has a T-periodic solution. The proof of Theorem 2.2 is complete.

Remark 2.2. [9] shows that the methods of establishing the boundedness in Theorem 2.1 do not establish dissipativeness, in fact, the boundedness can be satisfied and there are still unbounded solutions. After a suitable transformation, one may verify that Theorem 2.1 can be applied to the examples given in [9] which are

$$u_t = u_{xx} + f(u) + F(t) + \int_{-\infty}^{t} B(t, s) u_{xx}(s, x)\, ds \tag{2.22}$$

where

$$u(t, 0) = u(t, 1) = 0, \tag{2.23}$$

and

$$u_t = u_{xx} + f(u) + F(t) + \int_{-\infty}^{t} B(t, s) u_t(s, x)\, ds \tag{2.24}$$

with (2.23) holding.

§ 3. The damped wave equations

In this section we give sufficient conditions for the existence of periodic

solutions of damped wave equations. The equations to be studied are the wave equation with linear damping (telegraph equation) and strongly damped wave equations.

3.1. A wave equation with linear damping

Consider the following damped wave equation

$$u_{tt} = \Delta u - 2\alpha u_t + \lambda f(t, u(\cdot)) \text{ in } \Omega \subset \boldsymbol{R}^n, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1.1}$$

where $\alpha > 0$ is a constant, Δ is the Laplacian operator, and Ω is a bounded domain with smooth boundary, $\lambda \in [0, 1]$,

$$f(t, u(\cdot)) = f(t + T, u(\cdot)) \qquad \text{for some } T > 0.$$

We rewrite (3.1.1) as an abstract equation in a Banach space X_1 to be defined.

$$w'(t) = Cw(t) + \lambda F(t, w(\cdot)) \tag{3.1.2}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I \\ -A & -2\alpha I \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{3.1.3}$$

and where

$$\left\{\begin{array}{l} A\phi = -\Delta\phi = -\sum_{j=1}^{n} \partial^2\phi/\partial x_j^2 \text{ on the space } C_0^{+\infty}, \\ \text{smooth functions with compact support in } \Omega. \text{ Then} \\ \text{one can extend } A \text{ to a self-adjoint positive} \\ \text{definite operator in } X = H^0(\Omega) \text{ with} \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega). \end{array}\right. \tag{3.1.4}$$

Moreover $D(A^{1/2}) = H_0^1(\Omega)$ (see, Hale [23], p. 124). Now let $X_1 = H_0^1(\Omega) \times H^0(\Omega)$ with norm $|w|_{X_1} = |u|_{H^1} + |v|_{H^0}$ and $X_2 = D(A) \times H_0^1(\Omega)$ with norm $|w|_{X_2} = |u|_{H^2} + |v|_{H^1}$. By Hale [23, p. 125, Theorem 4.6.7], C with $D(C) = D(A) \times H_0^1(\Omega)$ generates a C_0-semigroup e^{Ct} on X_1. A has a complete countable orthonormal set of eigenfunctions $\phi_1, \phi_2, \cdots$ and eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$, $\lambda_m \to +\infty$ as $m \to +\infty$ and for each m, $\phi_m \in C^{+\infty}(\Omega)$, $\phi_m = 0$ on $\partial\Omega$ and $A\phi_m = \lambda_m\phi_m$.

We employ the method of separation of variables to give the following explicit representation of e^{Ct} (see Hale [23]. p. 126 for the case $n = 1$). We define

$$\langle \phi, \psi \rangle = \int_\Omega \phi(s)\psi(s)\, ds \qquad \text{for } \phi, \psi \in H^0.$$

Let $(\phi, \psi) \in X_1$; then

$$e^{Ct}(\phi, \psi) = (u(t, \phi, \psi), v(t, \phi, \psi)).$$

with

$$u(t, \phi, \psi) = \sum_{n=1}^{\infty} [\langle \phi, \phi_n \rangle U_n(t) + \langle \psi, \phi_n \rangle V_n(t)] \phi_n$$

and

$$v(t, \phi, \psi) = \sum_{n=1}^{\infty} [\langle \phi, \phi_n \rangle U_n'(t) + \langle \psi, \phi_n \rangle V_n'(t)] \phi_n$$

where

$$U_n(t) = \begin{cases} e^{-\alpha t}[\cosh(\alpha^2 - \lambda_n)^{1/2} t + \alpha(\alpha^2 - \lambda_n)^{-1/2} \sinh(\alpha^2 - \lambda_n)^{1/2} t], & \lambda_n < \alpha^2 \\ e^{-\alpha t}(1 + \alpha t), & \lambda_n = \alpha^2, \\ e^{-\alpha t}[\cos(\lambda_n - \alpha^2)^{1/2} t + \alpha(\lambda_n - \alpha^2)^{-1/2} \sin(\lambda_n - \alpha^2)^{1/2} t, & \lambda_n > \alpha^2 \end{cases}$$

and

$$V_n(t) = \begin{cases} e^{-\alpha t}(\alpha^2 - \lambda_n)^{-1/2} \sinh(\alpha^2 - \lambda_n)^{1/2} t, & \lambda_n < \alpha^2, \\ te^{-\alpha t}, & \lambda_n = \alpha^2, \\ e^{-\alpha t}(\lambda_n - \alpha^2)^{-1/2} \sin(\lambda_n - \alpha^2)^{1/2} t, & \lambda_n > \alpha^2. \end{cases}$$

Theorem 3.1.1. *Suppose the following conditions hold*:

(B_1) *there exists a constant* $B > 0$ *such that if* $w(t) = w(t, \phi, \psi)$ *is a T-periodic solution of* (3.1.2) *with* $w = \begin{pmatrix} u \\ v \end{pmatrix}$, *then* $|u(t)|_{H^2} + |v(t)|_{H^1} < B$ *for* $t \geq 0$;

(B_2) $f: \boldsymbol{R} \times \tilde{H}^2 \cap \tilde{H}_0^1 \to H_0^1$ *takes bounded sets into bounded sets*;

(B_3) $f(\cdot, u(\cdot)): [0, T] \to H^0$ *is continuous whenever* $u \in \tilde{H}^2 \cap \tilde{H}_0^1$ *and* $u: [0, T] \to H_0^1$ *is Hölder continuous*;

(B_4) *for any* $\mu > 0$, $\varepsilon \in 0$, *there exists* $\delta > 0$ *such that*

$$[\phi, \psi \in \tilde{H}^2 \cap \tilde{H}_0^1, \ |\phi|_{\tilde{H}^2} \leq \mu, \ |\psi|_{\tilde{H}^2} \leq \mu, \ |\phi - \psi|_{\tilde{H}_0^1} < \delta, \ t \in \boldsymbol{R}]$$

imply that

$$|f(t, \phi(\cdot)) - f(t, \psi(\cdot))|_{H_0^1} < \varepsilon.$$

Then (3.1.2) *has a T-periodic solution for* $\lambda = 1$.

Proof. By a direct computation, we have

$$(3.1.5) \qquad \begin{cases} |e^{Ct}| \leq \tilde{C} e^{-bt} & \text{for } t \geq 0, \\ |(e^{Ct} - I)\phi|_{X_1} \leq \tilde{C} t |\phi|_{X_2} & \text{for } \phi \in X_2 \text{ and } t \geq 0, \end{cases}$$

where b, $\tilde{C}$ are positive constants and C is defined in (3.1.3).

Since $F(t, w(\cdot)) = \begin{pmatrix} 0 \\ f(t, u(\cdot)) \end{pmatrix}$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$, it follows that $F: \boldsymbol{R} \times \tilde{X}_2 \to X_2$ takes bounded sets into bounded sets by (B_2). Thus for any $\mu > 0$, there exists $K_\mu > 0$ such that $[\phi \in \tilde{X}_2, |\phi|_{X_2} \leq \mu, \xi \in \boldsymbol{R}]$ imply that $|F(\xi, \phi(\cdot))|_{X_2} \leq K_\mu$. Moreover, C generates a C_0-semigroup on X_2 (see Hale [23], Theorem 4.6.8. p. 125). We may assume that the same estimate holds as the first inequality of (3.1.5). Thus

$$\int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(\cdot))\, ds \in D(C)$$

and

$$\left| \int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(\cdot))\, ds \right|_{X_2} \leq \int_{-\infty}^{0} |e^{-Cs} F(t+s, \phi(\cdot))|_{X_2}\, ds$$

$$\leq \tilde{C} \int_{-\infty}^{0} e^{bs} |F(t+s, \phi(\cdot))|_{X_2}\, ds \leq \tilde{C} K_\mu / b \overset{\text{def}}{=} M_\mu.$$

Condition (B_3) implies that $F(\cdot, w(\cdot)): [0, T] \to X_1$ is continuous whenever $w \in \tilde{X}_2$ and $w: [0, T] \to X_1$ is Hölder continuous. Also, as $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$, (B_4) implies that condition (vi) of Theorem 1.1 holds. By Theorem 1.1, equation (3.1.2) has a T-periodic solution for $\lambda = 1$. This completes the proof of Theorem 3.1.1.

Example 3.1.1. Consider the scalar damped wave equation

$$u_{tt} = u_{xx} - 2\alpha u_t + \lambda \left[E(u) + \int_{-\infty}^{t} D(t-s) g(u(s, x))\, ds + p(t, x) \right] \tag{3.1.6}$$

where $u(t, 0) = u(t, \pi) = 0$, $\alpha > 0$, $\lambda \in [0, 1]$, E, D, g and p are scalar functions. Assume that

$$u \left[E(u) + \int_{0}^{+\infty} D(\tau) d\tau g(u) \right] \leq ku^2 + M, \qquad k \in [0, 1),\ M > 0,$$

$$p(t + T, x) = p(t, x) \qquad \text{for some } T > 0, \tag{3.1.7}$$

$$E, g \in C^1 \text{ and } D \text{ is continuous, } E(0) = g(0) = 0.$$

Define $P: \boldsymbol{R} \to H^0$ by $P(t)(x) = p(t, x)$ and assume that

$$P \in \tilde{H}_0^1,\ |g'(u)| \le \mu, \qquad \text{for some } \mu > 0,$$
$$\mu \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)\, d\tau \right| ds < 2\alpha. \tag{3.1.8}$$

Let $X_1 = H_0^1(0, \pi) \times H^0(0, \pi)$ and $X_2 = H^2 \cap H_0^1 \times H_0^1$. Write (3.1.6) as an abstract equation in X_1

$$w'(t) = Cw(t) + \lambda F(t, w(\cdot)) \tag{3.1.9}$$

where $w = \begin{pmatrix} u \\ v \end{pmatrix}$, $C = \begin{pmatrix} 0 & I \\ -A & -2\alpha I \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$, A is defined in (3.1.4) and

$$f(t, u(\cdot)) = E(u) + \int_{-\infty}^{t} D(t - s)\, g(u(s))\, ds + P(t). \tag{3.1.10}$$

Theorem 3.1.2. *Suppose that* (3.1.7) *and* (3.1.8) *hold. Then* (3.1.9) *has a T-periodic solution for* $\lambda = 1$.

Lemma 3.1.1. *Suppose that* (3.1.7) *and* (3.1.8) *hold. Then there exists a constant* $B_1 > 0$ *such that if* w *is a T-periodic solution of* (3.1.9), *then* $|w(t)|_{X_2} < B_1$ *for* $t \ge 0$.

Proof of Lemma 3.1.1. Let $w = \begin{pmatrix} u \\ v \end{pmatrix}$ be a T-periodic solution of (3.1.9) and define

$$V_1(t) = \frac{1}{2} \int_0^{\pi} (u_x^2 + v^2)\, dx - \lambda \int_0^{\pi} \int_0^{u} \left[E(\xi) + \int_0^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi dx. \tag{3.1.11}$$

Using $(d/dt) \int_0^{\pi} u_x^2/2\, dx = -\int_0^{\pi} u_t u_{xx}\, dx$, we have

$$\begin{aligned} V_1'(t) = &- \int_0^{\pi} u_{xx} u_t\, dx \\ &+ \int_0^{\pi} v v_t dx - \lambda \int_0^{\pi} u_t \left(E(u) + \int_0^{+\infty} D(\tau)\, d\tau g(u) \right) dx. \end{aligned} \tag{3.1.12}$$

It then follows that

$$\begin{aligned} V_1'(t) &= \int_0^{\pi} u_t \left\{ -u_{xx} + u_{tt} - \lambda \left[E(u) + \int_0^{+\infty} D(\tau) d\tau g(u) \right] \right\} dx \\ &= -2\alpha \int_0^{\pi} u_t^2\, dx - \lambda \int_0^{\pi} u_t \frac{d}{dt} \int_{-\infty}^{t} \int_t^{+\infty} D(\tau - s) d\tau g(u(s, x))\, ds dx \\ &\quad + \lambda \int_0^{\pi} u_t p(t, x) dx \end{aligned}$$

$$\leq -2\alpha \int_0^\pi u_t^2\,dx + \frac{1}{2}\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)\,d\tau\right| ds\mu \int_0^\pi u_t^2\,dx$$
$$+\frac{1}{2}\mu\int_0^\pi\int_{-\infty}^0\left|\int_{-s}^{+\infty} D(\tau)d\tau\right| u_t^2(t+s,x)\,dsdx + \gamma\int_0^\pi u_t^2\,dx + M_1,$$

where $\gamma > 0$ is as small as we like and M_1 is a positive constant. Integrating $V_1'(t)$ from 0 to T, we get

$$0 = V_1(T) - V_1(0)$$
$$\leq -2\alpha\int_0^T\int_0^\pi u_t^2\,dxdt + \frac{1}{2}\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)d\tau\right| ds\mu\int_0^T\int_0^\pi u_t^2\,dxdt$$
$$+\frac{1}{2}\mu\int_0^\pi\int_{-\infty}^0\left|\int_{-s}^{+\infty} D(\tau)\,d\tau\right|\int_0^T u_t^2(t+s,x)dtdsdx + \gamma\int_0^\pi\int_0^T u_t^2\,dtdx + M_1 T$$
$$= -\left[2\alpha - \mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)d\tau\right| ds - \gamma\right]\int_0^T\int_0^\pi u_t^2\,dxdt + M_1 T.$$

Choosing $\gamma = \left(2\alpha - \mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)d\tau\right| ds\right)/2$, we have

$$\int_0^T\int_0^\pi u_t^2(t,x)\,dxdt \leq C_1 \qquad \text{for some } C_1 > 0. \tag{3.1.13}$$

Define

$$V_2(t) = \frac{1}{2}\int_0^\pi [u_t + 2\alpha u]^2\,dx + \frac{1}{2}\int_0^\pi u_x^2\,dx$$
$$-\lambda\int_0^\pi\int_0^u\left[E(\xi) + \int_0^{+\infty} D(\tau)d\tau g(\xi)\right]d\xi dx. \tag{3.1.14}$$

(3.1.15)

$$V_2'(t) = \int_0^\pi [u_t + 2\alpha u]\left[u_{xx} + \lambda E(u) + \lambda\int_{-\infty}^t D(t-s)g(u(s,x))ds\right.$$
$$\left. + \lambda p(t,x)\right]dx - \int_0^\pi u_{xx}u_t dx - \lambda\int_0^\pi u_t\left[E(u) + \int_0^{+\infty} D(\tau)d\tau g(u)\right]dx$$
$$= -2\alpha\int_0^\pi u_x^2\,dx + \lambda\int_0^\pi [u_t + 2\alpha u]\left[E(u) + \int_0^{+\infty} D(\tau)d\tau g(u)\right]dx$$
$$+\lambda\int_0^\pi [u_t + 2\alpha u]\left[-\frac{d}{dt}\int_{-\infty}^t\int_t^{+\infty} D(\tau - s)g(u(s,x))\,d\tau ds + p(t,x)\right]dx$$

$$
\begin{aligned}
&-\lambda\int_0^\pi u_t\left[E(u)+\int_0^{+\infty}D(\tau)d\tau g(u)\right]dx\\
&\quad=-2\alpha\int_0^\pi u_x^2\,dx+\lambda\int_0^\pi 2\alpha u\left[E(u)+\int_0^{+\infty}D(\tau)d\tau g(u)\right]dx\\
&-\lambda\int_0^\pi [u_t+2\alpha u]\left[\frac{d}{dt}\int_{-\infty}^t\int_t^{+\infty}D(\tau-s)\,d\tau g(u(s,x))\,ds\right]dx\\
&+\lambda\int_0^\pi [u_t+2\alpha u]\,p(t,x)\,dx\\
&\quad\le-2\alpha\int_0^\pi u_x^2\,dx+2\alpha\int_0^\pi (ku^2+M)\,dx\\
&-\lambda\int_0^\pi [u_t+2\alpha u]\left\{\int_{-\infty}^0\int_{-s}^{+\infty}D(\tau)\,d\tau g'(u(t+s,x))u_t(t+s,x)ds+p(t,x)\right\}dx
\end{aligned}
$$

Integrating (3.1.15) from 0 to T, we get

$$
\begin{aligned}
V_2(T)-V_2(0)\le&-2\alpha\int_0^T\int_0^\pi u_x^2\,dxdt+2\alpha k\int_0^T\int_0^\pi u_x^2\,dxdt+2\alpha\pi TM\\
&+\int_0^\pi\int_0^T|u_t+2\alpha u|\int_{-\infty}^0\left|\int_{-s}^{+\infty}D(\tau)d\tau\right|\mu|u_t(t+s,x)|\,dsdtdx\\
&+\int_0^\pi\int_0^T|u_t+2\alpha u||p(t,x)|\,dtdx.
\end{aligned}
$$

By (3.1.13) and (3.1.15), there exists $M_1>0$, such that

$$
0=V_2(T)-V_2(0)\le-\alpha(1-k)\int_0^T\int_0^\pi u_x^2\,dxdt+M_1.
$$

This implies that

$$
\int_0^T\int_0^\pi u_x^2\,dxdt\le M_1/\alpha(1-k). \tag{3.1.16}
$$

Thus, there exists a $t_1\in[0,T]$ such that

$$
\int_0^\pi (u_t^2(t_1,x)+u_x^2(t_1,x))\,dx\le C_3\qquad\text{for some } C_3>0. \tag{3.1.17}
$$

Now by Sobolev's inequality, we have

$$
\sup_{0\le x\le\pi}|u(t_1,x)|\le C_4\qquad\text{for some } C_4>0. \tag{3.1.18}
$$

Using an argument similar to the one following (2.19) we conclude that there exists $B_0 > 0$ with

$$\sup_{t\in[0,T]} \int_0^\pi [u^2(t, x) + u_x^2(t, x) + u_t^2(t, x)]\, dx \le B_0 \tag{3.1.19}$$

and $B_1 > 0$ such that

$$|w(t)|_{X_1} < B_1. \tag{3.1.20}$$

Moreover by (3.1.9), we have

$$w(t) = e^{Ct}w(0) + \lambda \int_0^t e^{C(t-s)}F(s, w(\cdot))\, ds \qquad \text{for } t\in[0, T].$$

Since $w(t)$ is T-periodic, we have

$$w(t) = w(t + nT) = e^{C(t+nT)}w(0) + \lambda \int_0^{t+nT} e^{C(t+nT-s)}F(s, w(\cdot))\, ds$$

$$= e^{C(t+nT)}w(0) + \lambda \int_{-nT}^t e^{C(t-s)}F(s, w(\cdot))\, ds$$

in X_1 for any positive integer n. Thus

$$\left| w(t) - \lambda \int_{-nT}^t e^{C(t-s)}F(s, w(\cdot))\, ds \right|_{X_1}$$
$$= |e^{C(t+nT)}w(0)|_{X_1} \le \tilde{C}e^{-b(t+nT)}|w(0)|_{X_1}$$
$$\le \tilde{C}e^{-bnT}|w(0)|_{X_1} \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

This yields

$$w(t) = \lambda \int_{-\infty}^t e^{C(t-s)}F(s, w(\cdot))\, ds \in X_1. \tag{3.1.21}$$

By (3.1.10) and (3.1.20), there exists $\tilde{M} > 0$ such that

$$|f(t, u(\cdot))|_{H_0^1} < \tilde{M} \qquad \text{for } t\in \boldsymbol{R}.$$

Thus (3.1.21) implies that

$$|w(t)|_{X_2} \le \int_{-\infty}^0 |e^{-Cs}F(t + s, w(\cdot))|_{X_2}\, ds$$
$$\le \tilde{C} \int_{-\infty}^0 e^{bs}|F(t + s, w(\cdot))|_{X_2}\, ds = \tilde{C} \int_{-\infty}^0 e^{bs}|f(t + s, u(\cdot))|_{H_0^1}\, ds$$

$$\leq \tilde{C} \int_{-\infty}^{0} e^{bs} ds \sup_{\xi \in \boldsymbol{R}} |f(\xi, u(\cdot))|_{H_0^1} < \tilde{C}\tilde{M}/b \stackrel{\text{def}}{=} B_1.$$

This completes the proof of Lemma 3.1.1.

Lemma 3.1.2. *Suppose that* (3.1.7) *and* (3.1.8) *hold. Then* $f(\cdot, u(\cdot))$: $[0, T] \to H^0$ *given in* (3.1.10) *is continuous whenever* $u \in \tilde{H}^2 \cap \tilde{H}_0^1$ *and* u: $[0, T] \to H_0^1$ *is Hölder continuous.*

We omit the proof of Lemma 3.1.2 which is essentially the same as that of Lemma 2.2.

Lemma 3.1.3. *Suppose that* (3.1.7) *and* (3.1.8) *hold. Then* f: $\boldsymbol{R} \times \tilde{H}^2 \cap \tilde{H}_0^1 \to H_0^1$ *takes bounded sets into bounded sets and condition* (B_4) *of Theorem* 3.1.1 *holds.*

Proof of Lemma 3.1.3. It is clear from (3.1.7), (3.1.8) that f: $\boldsymbol{R} \times \tilde{H}^2 \cap \tilde{H}_0^1 \to H_0^1$ takes bounded sets into bounded sets. Let $\eta > 0$ and $u \in \tilde{H}^2 \cap \tilde{H}_0^1$ with $|u|_{\tilde{H}^2} \leq \eta$; then

$$\sup \{|u(t, x)|: t \in [0, T], x \in [0, \pi]\} \leq \tilde{\eta} \qquad \text{for some } \tilde{\eta} > 0.$$

Let

$$K = \sup \{|E'(\xi)|: |\xi| \leq \tilde{\eta}\}.$$

For $\phi, \psi \in \tilde{H}^2 \cap \tilde{H}_0^1$ with $|\phi|_{\tilde{H}^2} \leq \eta$, $|\psi|_{\tilde{H}^2} \leq \eta$, $t \in \boldsymbol{R}$, we have

$$\begin{aligned}
|(f(t, \phi(\cdot)))_x - (f(t, \psi(\cdot)))_x|_{H^0} &\leq |E'(\phi(t))(\phi_x(t) - \psi_x(t))|_{H^0} \\
&+ |(E'(\phi(t)) - E'(\psi(t)))\psi_x(t)|_{H^0} \\
&+ \int_{-\infty}^{0} |D(-s)|\, |g'(\phi(t+s))(\phi_x(t+s) - \psi_x(t+s))|_{H^0}\, ds \\
&+ \int_{-\infty}^{0} |D(-s)|\, |(g'(\phi(t+s)) - g'(\psi(t+s)))\psi_x(t+s)|_{H^0}\, ds \\
&\leq K|\phi(t) - \psi(t)|_{H_0^1} + \eta |E'(\phi(t)) - E'(\psi(t))|_\infty + \mu \int_0^{+\infty} |D(\tau)|\, d\tau\, |\phi - \psi|_{\tilde{H}_0^1} \\
&+ \eta \int_0^{+\infty} |D(\tau)|\, d\tau \sup_{\xi \in \boldsymbol{R}} |g'(\phi(\xi)) - g'(\psi(\xi))|_\infty
\end{aligned}$$

where $|\phi(t)|_\infty = \operatorname{ess\,sup} \{|\phi(t, x)|: 0 \leq x \leq \pi\}$. Using the inequality $|\phi(t) - \psi(t)|_\infty \leq L|\phi(t) - \psi(t)|_{H_0^1}$ for some $L > 0$, we conclude that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi - \psi|_{\tilde{H}_0^1} < \delta$ implies that

$$|f(t, \phi(\cdot)) - f(t, \psi(\cdot))|_{\tilde{H}_0^1} < \varepsilon.$$

Thus (B_4) of Theorem 3.1.1 holds. The assertion of Lemma 3.1.3. follows.

Since all conditions of Theorem 3.1.1 are satisfied, we conclude that (3.1.9) has a T-periodic solution for $\lambda = 1$. This completes the proof of Theorem 3.1.2.

3.2. Strongly damped wave equations

Consider the following damped wave equation

$$u_{tt} = \Delta u + 2\alpha \Delta u_t + \lambda f(t, u(\cdot), u_t(\cdot)) \quad \text{in } \Omega \subset \boldsymbol{R}^n \tag{3.2.1}$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\alpha > 0$ is a constant, Δ is the Laplacian operator, and Ω is a bounded domain with smooth boundary, $\lambda \in [0, 1]$,

$$f(t, u(\cdot), u_t(\cdot)) = f(t + T, u(\cdot), u_t(\cdot)) \quad \text{for some } T > 0.$$

We rewrite (3.2.1) as an abstract equation in a Banach space X_1 to be defined:

$$w'(t) = Cw(t) + \lambda F(t, w(\cdot)) \tag{3.2.2}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I \\ -A & -2\alpha A \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{3.2.3}$$

A is defined in (3.1.4) and $f(t, w(\cdot)) = f(t, u(\cdot), v(\cdot))$.

Now let $X_1 = H_0^1(\Omega) \times H^0(\Omega)$ with $|w|_{X_1} = |u|_{H^1} + |v|_{H^0}$ and $X_2 = D(A) \times H_0^1(\Omega)$ with norm $|w|_{X_2} = |u|_{H^2} + |v|_{H^1}$. By Massatt [26], C with $D(C) = D(A) \times D(A)$ generates a C_0-semigroup e^{Ct} on X_1. A has a complete countable orthonormal set of eigenfunctions $\phi_1, \phi_2, \cdots$ and eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$, $\lambda_m \to +\infty$ as $m \to +\infty$, and for each m, $\phi_m \in C^{+\infty}(\Omega)$, $\phi_m = 0$ on $\partial\Omega$ and $A\phi_m = \lambda_m \phi_m$. We employ the method of separation of variables to give the following explicit representation of e^{Ct} (cf. Webb [36], p. 639). Let $(\phi, \psi) \in X_1$; then $e^{Ct}(\phi, \psi) = (u(t, \phi, \psi), v(t, \phi, \psi))$,

$$u(t, \phi, \psi) = \sum_{n=1}^{\infty} [\langle \phi, \phi_n \rangle U_n(t) + \langle \psi, \phi_n \rangle V_n(t)] \phi_n,$$

$$v(t, \phi, \phi) = \sum_{n=1}^{\infty} [\langle \phi, \phi_n \rangle U_n'(t) + \langle \psi, \phi_n \rangle V_n'(t)] \phi_n,$$

$$U_n(t) = \begin{cases} e^{-\alpha\lambda_n t}\{\cos[(\lambda_n - \alpha^2\lambda_n^2)^{\frac{1}{2}} t] + \alpha\lambda_n(\lambda_n - \alpha^2\lambda_n^2)^{-\frac{1}{2}} \sin(\lambda_n - \alpha^2\lambda_n^2)^{\frac{1}{2}} t\}, \\ \qquad\qquad \text{if } \alpha^2\lambda_n < 1, \\ e^{-\alpha\lambda_n t}[1 + \alpha\lambda_n t], \qquad \text{if } \alpha^2\lambda_n = 1, \end{cases}$$

$$\left| \begin{array}{l} e^{-\alpha\lambda_n t}\{\cosh(\lambda_n^2\alpha^2-\lambda_n)^{\frac{1}{2}}t + \alpha\lambda_n(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}\sinh(\alpha^2\lambda_n^2-\lambda_n)^{\frac{1}{2}}t\}, \\ \qquad\qquad \text{if } \alpha^2\lambda_n > 1. \end{array} \right.$$

$$V_n(t) = \begin{cases} e^{-\alpha\lambda_n t}\{(\lambda_n-\alpha^2\lambda_n^2)^{-\frac{1}{2}}\sin[(\lambda_n-\alpha^2\lambda_n^2)^{\frac{1}{2}}t]\} & \text{if } \alpha^2\lambda_n < 1, \\ e^{-\alpha\lambda_n t}t & \text{if } \alpha^2\lambda_n = 1, \\ e^{-\alpha\lambda_n t}\{(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}\sinh(\lambda_n^2\alpha^2-\lambda_n)^{\frac{1}{2}}t\} & \text{if } \alpha^2\lambda_n > 1. \end{cases}$$

For $\alpha^2\lambda_n > 1$, we have

$$U_n'(t) = -e^{-\alpha\lambda_n t}\lambda_n(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}\sinh(\alpha^2\lambda_n^2-\lambda_n)^{\frac{1}{2}}t.$$

$$V_n'(t) = e^{-\alpha\lambda_n t}\{\cosh(\alpha^2\lambda_n^2-\lambda_n)^{\frac{1}{2}}t - \alpha\lambda_n(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}\sinh(\alpha^2\lambda_n^2-\lambda_n)^{\frac{1}{2}}t\}.$$

From the representation of e^{Ct}, it follows that

$$|e^{Ct}| \le C_1 e^{-b_1 t} \qquad \text{for } t \ge 0,$$

where C_1, b_1 are positive constants.

Next note that for $\alpha^2\lambda_n > 1$, we have

(3.2.4)
$$\begin{aligned} U_n(t) - 1 &= \frac{1}{2}[1 + \alpha\lambda_n(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}](e^{[-\alpha\lambda_n+(\alpha^2\lambda_n^2-\lambda_n)^{1/2}]t}-1) \\ &\quad + \frac{1}{2}[1 - \alpha\lambda_n(\alpha^2\lambda_n^2-\lambda_n)^{-\frac{1}{2}}](e^{[-\alpha\lambda_n-(\alpha^2\lambda_n^2-\lambda_n)^{1/2}]t}-1). \end{aligned}$$

Let $\mu = \alpha\lambda_n - (\alpha^2\lambda_n^2-\lambda_n)^{1/2}$, $\gamma = \alpha\lambda_n + (\alpha^2\lambda_n^2-\lambda_n)^{1/2}$. Then

$$\begin{aligned} |U_n(t)-1| &\le C_2(t\mu)^{1/2}(1-e^{-\mu t})/(t\mu)^{1/2} \\ &\quad + C_2(t\gamma)^{1/2}(1-e^{-\gamma t})/(t\gamma)^{1/2}. \end{aligned}$$

Note that the function $(1-e^{-y^2})/y$ for $y > 0$ is bounded and $\lambda_m \to +\infty$ as $m \to +\infty$. There exists $C_3 > 0$ such that

$$|U_m(t)-1| \le C_3(t\lambda_m)^{1/2} \qquad \text{for } t > 0,\ \alpha^2\lambda_m > 1.$$

The same argument shows that

$$|U_m(t)-1| \le C_4(t\lambda_m)^{1/2} \qquad \text{for } t > 0,$$

where $\alpha^2\lambda_m \le 1$ and $C_4 > 0$. Thus there exists $C_5 > 0$ such that

(3.2.5)
$$|U_m(t)-1| \le C_5(t\lambda_m)^{1/2} \qquad \text{for } t \ge 0 \text{ and } m = 1, 2, \cdots.$$

Using the same proof following (3.2.4), we can show that

$$|\lambda_m^{-1/2}(U_m(t)-1)| + |V_m(t)| \le C_0 t^{1/2}$$

and

$$|U_m'(t)| + |V_m'(t)-1| \le C_0(t\lambda_m)^{1/2} \qquad \text{for } t \ge 0,$$

where $C_0 > 0$ is a constant.

Let us consider

$$(e^{Ct}-I)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}u(t,\phi,\psi)-\phi\\ v(t,\phi,\psi)-\psi\end{pmatrix} \qquad \text{for } \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in X_2.$$

Then,

$$u(t,\phi,\psi)-\phi = \sum_{n=1}^{\infty} [\langle \phi, \phi_n\rangle (U_n(t)-1) + \langle \psi, \phi_n\rangle V_n(t)]\phi_n$$

and

$$v(t,\phi,\psi)-\psi = \sum_{n=1}^{\infty} [\langle \phi, \phi_n\rangle U_n'(t) + \langle \psi, \phi_n\rangle (V_n'(t)-1)]\phi_n.$$

Hence,

$$\begin{aligned}(3.2.6)\qquad |u_x(t,\phi,\psi)-\phi_x|_{H^0}^2 &= |A^{1/2}(u(t,\phi,\psi)-\phi)|_{H^0}^2\\ &= \sum_{n=1}^{\infty} \lambda_n[\langle \phi, \phi_n\rangle (U_n(t)-1) + \langle \psi, \phi_n\rangle V_n(t)]^2\\ &\le \sum_{n=1}^{\infty} [2\lambda_n(U_n(t)-1)^2|\langle \phi, \phi_n\rangle|^2 + 2\lambda_n V_n^2(t)|\langle \psi, \phi_n\rangle|^2]\\ &\le \sum_{n=1}^{\infty} [2C_0^2\lambda_n^2|\langle \phi, \phi_n\rangle|^2 + 2\lambda_n C_0^2|\langle \psi, \phi_n\rangle|^2]t\\ &\le 2C_0^2 t \sum_{n=1}^{\infty} (\lambda_n^2|\langle \phi, \phi_n\rangle|^2 + \lambda_n|\langle \psi, \phi\rangle|^2) = 2C_0^2 t(|A\phi|_{H^0}^2 + |A^{1/2}\psi|_{H^0}^2).\end{aligned}$$

In the same way we can show that

$$(3.2.7)\qquad |v(t,\phi,\psi)-\psi|_{H^0}^2 \le 2C_0^2 t(|A\phi|_{H^0}^2 + |A^{1/2}\psi|_{H^0}^2).$$

Thus there exists $\tilde{C} > 0$ such that

$$(3.2.8)\qquad |(e^{Ct}-I)w|_{X_1} \le \tilde{C}t^{1/2}|w|_{X_2}, \qquad \text{where } w \in X_2.$$

One also can verify that

$$(3.2.9)\qquad |e^{Ct}w|_{X_2} \le \tilde{C}(1/t^{1/2})e^{-bt}|\phi|_{H^0} \qquad \text{for } t > 0,$$

where $w = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$, and $\tilde{C}$, $b > 0$.

Theorem 3.2.1. *Suppose the following conditions hold*:

(C_1) *there exists a constant* $B > 0$ *such that if* $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ *is a T-periodic solution of* (3.2.2), *then* $|u(t)|_{H^2} + |v(t)|_{H^1} < B$ *for* $t \geq 0$;

(C_2) $f: \boldsymbol{R} \times \tilde{X}_2 \to H^0$ *takes bounded sets into bounded sets*;

(C_3) $f(\cdot, w(\cdot)): [0, T] \to H^0$ *is Hölder continuous whenever* $w \in \tilde{X}_2$ *and* $w: [0, T] \to X_1$ *is Hölder continuous*;

(C_4) *for any* $\mu > 0$, *there exists* $\beta > 0$ *such that*

$$[\phi, \psi \in \tilde{X}_2, |\phi|_{\tilde{X}_2} \leq \mu, |\psi|_{\tilde{X}_2} \leq \mu, t \in \boldsymbol{R}]$$

imply that

$$|f(t, \phi(\cdot)) - f(t, \psi(\cdot))|_{H^0} \leq \beta |\phi - \psi|_{\tilde{X}_1}.$$

Then (3.2.2) *has a T-periodic solution for* $\lambda = 1$.

Proof of Theorem 3.2.1. Since $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$, it follows that $F: \boldsymbol{R} \times \tilde{X}_2 \to X_1$ takes bounded sets into bounded sets. Thus for any $\mu > 0$, there exists an $M_\mu > 0$ such that $[\phi \in \tilde{X}_2, |\phi|_{X_2} \leq \mu, s \in \boldsymbol{R}]$ imply that $|F(s, \phi(\cdot))|_{X_1} \leq M_\mu$. By (3.2.9), we have

$$\left| \int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(\cdot)) \, ds \right|_{X_2} \leq \int_{-\infty}^{0} |e^{-Cs} F(t + s, \phi(\cdot))|_{X_2} \, ds$$

$$\leq \tilde{C} \int_{-\infty}^{0} (1/|s|^{1/2}) e^{bs} \, ds \sup_{\xi \in \boldsymbol{R}} |f(\xi, \phi(\cdot))|_{H^0} \leq \tilde{C} E_{1/2} M_\mu.$$

Thus conditions (ii) and (v) of Theorem 1.1 hold.

Next we show that $\int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(\cdot)) \, ds \in D(C)$ if $\phi \in \tilde{X}_2$ and $\phi: [0, T] \to X_1$ is Hölder continuous. By (C_3), $F(\cdot, \phi(\cdot)): [0, T] \to X_1$ is Hölder continuous for $\phi \in \tilde{X}_2$ and $\phi: [0, T] \to X_1$ is Hölder continuous. It is noted in [17] that e^{Ct} is an analytic semigroup on $D(A) \times H^0$. Thus

$$\int_{-\infty}^{t} e^{C(t-s)} F(s, \phi(\cdot)) \, ds \in D(C).$$

and (iv) of Theorem 1.1 is satisfied.

Now let $\mu > 0$. By (C_4) there exists $\beta > 0$ such that $[\phi, \psi \in \tilde{X}_2, |\phi|_{\tilde{X}_2} \leq \mu, |\psi|_{\tilde{X}_2} \leq \mu, s \in \boldsymbol{R}]$ imply that

$$|f(s, \phi(\cdot)) - f(s, \psi(\cdot))|_{H^0} \leq \beta |\phi - \psi|_{\tilde{X}_1}.$$

Let $\varepsilon > 0$ and $\delta = \varepsilon / 2\tilde{C}\beta E_{1/2}$. Suppose that $|\phi - \psi|_{\tilde{X}_1} < \delta$; then

$$\left| \int_{-\infty}^{t} e^{C(t-s)} [F(s, \phi(\cdot)) - F(t, \psi(\cdot))] \, ds \right|_{\tilde{X}_2}$$

$$\leq \int_{-\infty}^{0} |e^{-Cs} [F(t+s, \phi(\cdot)) - F(t+s, \psi(\cdot))]|_{X_2} \, ds$$

$$\leq \tilde{C} \int_{0}^{+\infty} s^{-1/2} e^{-bs} \, ds \sup_{\xi \in \boldsymbol{R}} |f(\xi, \phi(\cdot)) - f(\xi, \psi(\cdot)|_{H^0}$$

$$\leq \tilde{C} E_{1/2} \beta |\phi - \psi|_{\tilde{X}_1} < \varepsilon.$$

Finally, (C_3) and (C_4) imply that condition (iii) of Theorem 1.1 holds. The proof is complete.

Example 3.2.1. Consider the three dimensional equation

$$u_{tt} = \Delta u + 2\alpha \Delta u_t + \lambda [E(u) + K(u_t) + \int_{-\infty}^{t} D(t-s) g(u(s, x)) \, ds + p(t, x)], \quad \text{in } \Omega \subset \boldsymbol{R}^3, \tag{3.2.10}$$

$u = 0$ on $\partial\Omega$, where $\alpha > 0$ and Ω is a bounded domain with smooth boundary. Let

$$\begin{cases} uK(u) \leq \gamma u^2 + M \text{ for some } \gamma > 0 \text{ and } M > 0, \\ E'(u) + \displaystyle\int_{0}^{+\infty} D(\tau) \, d\tau g'(u) \leq k, \ k \in [0, \lambda_1) \\ \text{where } \lambda_1 \text{ is the first eigenvalue of } -\Delta \text{ on } H_0^1(\Omega), \\ E, K, g \in C^1, D \text{ is continuous. } p(t+T, x) = p(t, x) \\ \text{for some } T > 0. \end{cases} \tag{3.2.11}$$

we define $P: \boldsymbol{R} \to H^0(\Omega)$ by $P(t)(x) = p(t, x)$ for $x \in \Omega$ and assume that

$$\begin{cases} |g'(u)| \leq \mu, \ |K'(u)| \leq L \text{ for some } \mu \text{ and } L > 0. \\ \mu \displaystyle\int_{0}^{+\infty} \left| \int_{s}^{+\infty} D(\tau) \, d\tau \right| ds + \gamma < 2\alpha\lambda_1, \text{ where } \gamma, \lambda_1 \text{ are} \\ \text{given in (3.2.11). } P \in \tilde{H}^0 \text{ and } P: [0, T] \longrightarrow H^0 \\ \text{is Hölder continuous.} \end{cases} \tag{3.2.12}$$

For the sake of brevity, we only show the existence of an a priori bound on all possible T-periodic solutions of (3.2.10). Let $X_1 = H_0^1(\Omega) \times H^0(\Omega)$, $X_2 = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. We rewrite (3.2.10) as an abstract equation in X_1:

$$w'(t) = Cw(t) + \lambda F(t, w(t)) \tag{3.2.13}$$

where $w = \begin{pmatrix} u \\ v \end{pmatrix}$, $C = \begin{pmatrix} 0 & I \\ -A & -2\alpha A \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$ with A defined in (3.1.4) for $n = 3$ and

$$f(t, w(\cdot)) = E(u) + K(v) + \int_{-\infty}^{t} D(t-s)g(u(s))\,ds + P(t).$$

Lemma 3.2.1. *Suppose that* (3.2.11) *and* (3.2.12) *hold. Then there exists a constant* $B_1 > 0$ *such that if* w *is a* T*-periodic solution of* (3.2.13), *then* $|w(t)|_{X_2} < B_1$ *for* $t \geq 0$.

Proof of Lemma 3.2.1. Let $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ be a T-periodic solution of (3.2.13) and define

$$V_1(t) = \int_\Omega [|\nabla u|^2 + v^2]\,dx_1 dx_2 dx_3 - 2\lambda \int_\Omega \int_0^u \left[E(\xi) + \int_0^{+\infty} D(\tau)\,d\tau g(\xi) \right] d\xi dx_1 dx_2 dx_3 \tag{3.2.14}$$

so that

$$V_1'(t) = \int_\Omega \{2\nabla u \cdot \nabla u_t + 2u_t[\Delta u + 2\alpha \Delta u_t + \lambda f(t, w(\cdot))]\}\,dx - 2\lambda \int_\Omega u_t \left[E(u) + \int_0^{+\infty} D(\tau)\,d\tau g(u) \right] dx. \tag{3.2.15}$$

Here, $dx = dx_1 dx_2 dx_3$, subsequently, ds is the differential of surface on $\partial\Omega$.. We will use the following formula and the divergence theorem. Let $\phi: \boldsymbol{R}^3 \to \boldsymbol{R}$, $\psi: \boldsymbol{R}^3 \to \boldsymbol{R}$ be smooth, then

$$\operatorname{div}(\phi \nabla \psi) = \phi \Delta \psi + \nabla \psi \cdot \nabla \phi$$

and

$$\int_\Omega [\phi \Delta \psi + \nabla \psi \cdot \nabla \phi]\,dx = \int_{\partial\Omega} \vec{n} \cdot (\phi \nabla \psi)\,ds \tag{3.2.16}$$

where $\vec{n}$ is the outward unit normal on $\partial\Omega$. If we take $\phi = u_t$, $\psi = u$ in (3.2.16) then

$$\int_\Omega [u_t \Delta u + \nabla u \cdot \nabla u_t]\, dx = 0 \tag{3.2.17}$$

while

$$\int_\Omega u_t \Delta u_t dx = -\int_\Omega |\nabla u_t|^2 dx \tag{3.2.18}$$

if we take $\phi = \psi = u_t$ in (3.2.16). Now

$$\begin{aligned}
V_1'(t) &= -4\alpha \int_\Omega |\nabla u_t|^2 dx + 2\lambda \int_\Omega u_t f(t, w(\cdot))\, dx \\
&\quad - 2\lambda \int_\Omega u_t \left[E(u) + \int_0^{+\infty} D(\tau) d\tau g(u) \right] dx \\
&= -4\alpha \int_\Omega |\nabla u_t|^2\, dx \\
&\quad + 2\lambda \int_\Omega u_t \left[K(u_t) - \frac{d}{dt} \int_{-\infty}^{t} \int_t^{+\infty} D(\tau - s) d\tau g(u(s, x)) ds + p(t, x) \right] dx \\
&= -4\alpha \int_\Omega |\nabla u_t|^2\, dx + 2\lambda \int_\Omega u_t K(u_t)\, dx + 2\lambda \int_\Omega u_t p(t, x)\, dx \\
&\quad - 2\lambda \int_\Omega u_t \left[\int_{-\infty}^{0} \int_{-s}^{+\infty} D(\tau) d\tau g'(u(t + s, x)) u_t(t + s, x)\, ds \right] dx \\
&\leq -4\alpha \int_\Omega |\nabla u_t|^2\, dx + 2\lambda\gamma \int_\Omega u_t^2\, dx + 2M|\Omega| \\
&\quad + \mu \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \int_\Omega u_t^2\, dx \\
&\quad + \mu \int_\Omega \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| u_t^2(t + s, x) ds dx + 2\lambda \int_\Omega u_t p(t, x) dx
\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω. We will use the inequalities

$$\lambda_1 \int_\Omega |\phi|^2 dx \leq \int_\Omega |\nabla \phi|^2 dx \quad \text{and} \quad \lambda_1 \int_\Omega |\nabla \phi|^2 dx \leq \int_\Omega |\Delta \phi|^2 dx$$

where λ_1 is the first eigenvalue of $-\Delta$ in H_0^1 (cf. Hale [23], p. 136). Integrating V_1' from 0 to T and using the fact that u and V_1 are T-periodic, we have

$$
\begin{aligned}
0 &= V_1(T) - V_1(0) \\
&\le -\left[4\alpha - \frac{\mu}{\lambda_1}\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)d\tau\right|ds - 2\frac{\gamma}{\lambda_1}\right]\int_0^T\int_\Omega |\nabla u_t|^2\,dxdt \\
&\quad + \mu\int_\Omega\int_{-\infty}^0\left|\int_{-s}^{+\infty} D(\tau)d\tau\right|\int_0^T u_t^2(t+s,x)dtdsdx \\
&\quad + 2MT|\Omega| + 2\lambda\int_0^T\int_\Omega u_t p(t,x)dxdt \\
&= -2\left[2\alpha\lambda_1 - \mu\int_0^{+\infty}\left|\int_s^{+\infty} D(\tau)d\tau\right|ds - \gamma\right]\frac{1}{\lambda_1}\int_0^T\int_\Omega |\nabla u_t|^2\,dxdt \\
&\quad + 2\lambda\int_0^T\int_\Omega u_t p(t,x)dxdt + 2MT|\Omega|.
\end{aligned}
$$

This implies that there exists $M_1 > 0$ such that

$$
\int_0^T\int_\Omega |\nabla u_t|^2 dxdt \le M_1. \tag{3.2.19}
$$

Next define

$$
\begin{aligned}
V_2(t) &= \int_\Omega [u_t - 2\alpha\Delta u]^2 dx + \int_\Omega |\nabla u|^2 dx \\
&\quad - 2\lambda\int_\Omega\int_0^u\left[E(s) + \int_0^{+\infty} D(\tau)d\tau g(s)\right]dsdx.
\end{aligned} \tag{3.2.20}
$$

Then

$$
\begin{aligned}
V_2'(t) &= 2\int_\Omega (u_t - 2\alpha\Delta u)(u_{tt} - 2\alpha\Delta u_t)\,dx \\
&\quad + 2\int_\Omega \nabla u\cdot\nabla u_t dx - 2\lambda\int_\Omega u_t\left[E(u) + \int_0^{+\infty} D(\tau)d\tau g(u)\right]dx
\end{aligned} \tag{3.2.21}
$$

Using (3.2.17) we have

$$
\begin{aligned}
V_2'(t) &= -4\alpha\int_\Omega \Delta u(u_{tt} - 2\alpha\Delta u_t)dx \\
&\quad + 2\int_\Omega u_t[u_{tt} - 2\alpha\Delta u_t - \Delta u - \lambda\left[E(u) + \int_0^{+\infty} D(\tau)d\tau g(u)\right]dx \\
&= -4\alpha\int_\Omega \Delta u\left\{\Delta u + \lambda\left[E(u) + \int_0^{+\infty} D(\tau)d\tau g(u)\right] + \lambda K(u_t)\right\}dx
\end{aligned}
$$

$$+ 4\alpha\lambda \int_\Omega \Delta u \frac{d}{dt} \int_{-\infty}^{t} \int_t^{+\infty} D(\tau - s) d\tau g(u(s, x)) ds dx - 4\alpha\lambda \int_\Omega \Delta u p(t, x) dx$$

$$+ 2\lambda \int_\Omega u_t \left[K(u_t) - \frac{d}{dt} \int_{-\infty}^{t} \int_t^{+\infty} D(\tau - s) d\tau g(u(s, x)) ds + p(t, x) \right] dx$$

$$= -4\alpha \int_\Omega |\Delta u|^2 dx - 4\alpha\lambda \int_\Omega \Delta u \left[E(u) + \int_0^{+\infty} D(\tau) d\tau g(u) \right] dx$$

$$- 4\alpha\lambda \int_\Omega \Delta u K(u_t) dx$$

$$+ 4\alpha\lambda \int_\Omega \Delta u \int_{-\infty}^{0} \int_{-s}^{+\infty} D(\tau) d\tau g'(u(t + s, x)) u_t(t + s, x) ds dx$$

$$- 4\alpha\lambda \int_\Omega \Delta u p(t, x) dx + 2\lambda \int_\Omega u_t K(u_t) dx + 2\lambda \int_\Omega u_t p(t, x) dx$$

$$- 2\lambda \int_\Omega u_t \int_{-\infty}^{0} \int_{-s}^{+\infty} D(\tau) d\tau g'(u(t + s, x)) u_t(t + s, x) ds dx$$

$$= -4\alpha \int_\Omega |\Delta u|^2 dx + 4\alpha\lambda \int_\Omega \left[E'(u) + \int_0^{+\infty} D(\tau) d\tau g'(u) \right] \nabla^2 u dx$$

$$- 4\alpha\lambda \int_{\partial\Omega} \vec{n} \cdot \nabla u \left[E(u) + \int_0^{+\infty} D(\tau) d\tau g(u) \right] dS$$

$$- 4\alpha\lambda \int_{\partial\Omega} \vec{n} \cdot (K(u_t) \nabla u) dS + 4\alpha\lambda \int_\Omega K'(u_t) \nabla u \cdot \nabla u_t dx$$

$$+ 4\alpha\lambda \int_\Omega \Delta u \int_{-\infty}^{0} \int_{-s}^{+\infty} D(\tau) d\tau g'(u(t + s, x)) u_t(t + s, x) ds dx$$

$$- 4\alpha\lambda \int_\Omega \Delta u p(t, x) dx + 2\lambda \int_\Omega u_t K(u_t) dx + 2\lambda \int_\Omega u_t p(t, x) dx$$

$$- 2\lambda \int_\Omega u_t \int_{-\infty}^{0} \int_{-s}^{+\infty} D(\tau) d\tau g'(u(t + s, x)) u_t(t + s, x) ds dx.$$

Notice that

$$\int_{\partial\Omega} \vec{n} \cdot \nabla u \left[E(u) + \int_0^{+\infty} D(\tau) d\tau g(u) \right] dS$$

$$= \left[E(0) + \int_0^{+\infty} D(\tau) d\tau g(0) \right] \int_{\partial\Omega} \vec{n} \cdot \nabla u dS$$

$$= \left[E(0) + \int_0^{+\infty} D(\tau) d\tau g(0) \right] \int_\Omega \Delta u dx$$

and

$$\int_{\partial\Omega} \vec{n} \cdot (K(u_t)\nabla u) dS = K(0) \int_{\Omega} \Delta u dx.$$

Using (3.2.11) and (3.2.19), we have

$$V_2'(t) \leq -4\alpha(\lambda_1 - k)\lambda_1^{-1} \int_{\Omega} |\Delta u|^2 dx + 4\alpha\delta \int_{\Omega} |\Delta u|^2 dx + \alpha_1 \int_{\Omega} |\nabla u_t|^2 dx + \beta_1 \int_{\Omega} \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| u_t^2(t+s, x) ds dx + \gamma_1, \tag{3.2.22}$$

where $\alpha_1, \beta_1, \gamma_1$ are positive constants and $\delta > 0$ is such that

$$\lambda_1 - k - \delta\lambda_1 > 0.$$

Integrating (3.2.22) from 0 to T, we obtain

$$\int_0^T \int_{\Omega} |\Delta u|^2 dx dt \leq M_2 \qquad \text{for some } M_2 > 0. \tag{3.2.23}$$

Hence

$$T \int_{\Omega} [|\nabla u_t(t_1, x)|^2 + |\Delta u(t_1, x)|^2] dx \leq M_1 + M_2 \tag{3.2.24}$$

for some $t_1 \in [0, T]$,

$$\sup_{x \in \Omega} |u(t_1, x)| \leq C_1 \qquad \text{for some } C_1 > 0.$$

This implies that

$$\left| \int_0^{u(t_1, x)} \left[E(\xi) + \int_0^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi \right| \leq \max_{0 \leq |u| \leq C_1} \left| \int_0^{u} \left[E(\xi) + \int_0^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi \right| \stackrel{\text{def}}{=} C_2. \tag{3.2.25}$$

Let $V(t) = V_1(t) + V_2(t)$; then

$$V(t_1) = \int_{\Omega} u_t^2(t_1, x) dx + 2 \int_{\Omega} |\nabla u(t_1, x)|^2 dx + \int_{\Omega} [u_t(t_1, x) - 2\alpha \Delta u(t_1, x)]^2 dx$$

$$
\begin{aligned}
&- 4\lambda \int_{\Omega} \int_{0}^{u(t_1,x)} \left[E(\xi) + \int_{0}^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi dx \\
&\leq \int_{\Omega} u_t^2(t_1, x) dx + 2 \int_{\Omega} |\nabla u(t_1, x)|^2 dx \\
&\quad + \int_{\Omega} [u_t(t_1, x) - 2\alpha \Delta u(t_1, x)]^2 dx + 4|\Omega| C_2 \\
&\leq 3 \int_{\Omega} u_t^2(t_1, x) dx + 2(4\alpha^2 + 1/\lambda_1) \int_{\Omega} |\Delta u(t_1, x)|^2 dx + 4|\Omega| C_2 \\
&\leq [3/\lambda_1 + 2(4\alpha^2 + 1/\lambda_1)](M_1 + M_2)/T + 4|\Omega| C_2 \overset{\text{def}}{=} C_3.
\end{aligned}
$$

Next for $t_1 \leq t \leq t_1 + T$, we have

$$
\begin{aligned}
V(t) &= V(t_1) + \int_{t_1}^{t} V'(s) ds \\
&\leq C_3 + 2M|\Omega| T + 2\lambda \int_{t_1}^{T+t_1} \int_{\Omega} |u_t p(s, x)| dx ds + \gamma_1 T \\
&\quad + \alpha_1 \int_{t_1}^{T+t_1} \int_{\Omega} |u_t|^2 dx ds \\
&\quad + (\mu + \beta_1) \int_{t_1}^{T+t_1} \int_{\Omega} \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| u_t^2(v + s, x) ds dx dv \\
&\leq C_3 + 2M|\Omega| T + \int_{0}^{T} \int_{\Omega} u_t^2(s, x) dx ds + \int_{0}^{T} \int_{\Omega} p^2(s, x) dx ds + \gamma_1 T \\
&\quad + \alpha_1 \int_{0}^{T} \int_{\Omega} |u_t(s, x)|^2 dx ds \\
&\quad + (\mu + \beta_1) \int_{0}^{+\infty} \left| \int_{s}^{+\infty} D(\tau) d\tau \right| ds \int_{0}^{T} \int_{\Omega} u_t^2(v, x) dx dv \\
&\leq C_3 + \left[(\mu + \beta_1) \int_{0}^{+\infty} \left| \int_{s}^{+\infty} D(\tau) d\tau \right| ds + 1 + \alpha_1 \right] M_1/\lambda_1 \\
&\quad + \int_{0}^{T} \int_{\Omega} p^2(s, x) dx ds + 2M|\Omega| T + \gamma_1 T \overset{\text{def}}{=} C_4.
\end{aligned}
$$

Thus,

$$
\int_{\Omega} u_t^2(t, x) dx + 2 \int_{\Omega} |\nabla u(t, x)|^2 dx + \int_{\Omega} [u_t - 2\alpha \Delta u]^2 dx \tag{3.2.26}
$$

$$\leq C_4 + 4\lambda \int_\Omega \int_0^u \left[E(\xi) + \int_0^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi dx.$$

Note that

$$E'(u) + \int_0^{+\infty} D(\tau) d\tau g'(u) \leq k \qquad \text{for } k \in [0, \lambda_1)$$

implies that

$$\int_0^u \left[E(\xi) + \int_0^{+\infty} D(\tau) d\tau g(\xi) \right] d\xi \leq \frac{1}{2} k u^2 + \frac{1}{2} \delta u^2 + M^*$$

where $\delta > 0$ is as small as we like and $M^* > 0$. From (3.2.26), we have

$$\int_\Omega u_t^2(t, x) dx + 2 \int_\Omega |\nabla u(t, x)|^2 dx + \int_\Omega [u_t - 2\alpha \Delta u]^2 dx \tag{3.2.27}$$

$$\leq C_4 + 2k \int_\Omega u^2 dx + 2\delta \int_\Omega u^2 dx + 4|\Omega| M^*.$$

This yields

$$\sup_{0 \leq t \leq T} \int_\Omega [u_t^2(t, x) + |\Delta u(t, x)|^2] dx \leq C_5 \qquad \text{for some } C_5 > 0. \tag{3.2.28}$$

Next let

$$V_0(t) = \int_\Omega |\nabla u_t(t, x)|^2 dx; \tag{3.2.29}$$

then

(3.2.30)

$$\begin{aligned} V_0'(t) &= -2 \int_\Omega u_{tt} \Delta u_t dx \\ &= -2 \int_\Omega \Delta u_t [\Delta u + 2\alpha \Delta u_t + \lambda f(t, u(\cdot), u_t(\cdot))] dx \\ &= -4\alpha \int_\Omega |\Delta u_t|^2 dx - 2 \int_\Omega \Delta u_t \Delta u dx - 2\lambda \int_\Omega \Delta u_t f(t, u(\cdot), u_t(\cdot)) dx. \end{aligned}$$

Using (3.2.28) and the fact that

$$\lambda_1 \int_\Omega |\nabla u_t|^2 dx \leq \int_\Omega |\Delta u_t|^2 dx,$$

we have

$$V_0'(t) \leq -2\alpha\lambda_1 \int_\Omega |\nabla u_t|^2 dx + M_3 \tag{3.2.31}$$

for some $M_3 > 0$. Thus

$$\begin{aligned} V_0(t) &\leq e^{-2\alpha\lambda_1 t} V_0(0) + \int_0^t e^{-2\alpha\lambda_1 (t-s)} ds M_3 \\ &\leq e^{-2\alpha\lambda_1 t} V_0(0) + M_3/2\alpha\lambda_1. \end{aligned}$$

Since $V_0(t)$ is T-periodic, it follows that

$$V_0(t) = \int_\Omega |\nabla u_t|^2 dx \leq M_3/2\alpha\lambda_1. \tag{3.2.32}$$

Combining (3.2.28) and (3.2.32), there exists $B > 0$ such that

$$|w(t)|_{X_2} < B \qquad \text{for } t \geq 0,$$

whenever w is a T-periodic solution of (3.2.13). This completes the proof of Lemma 3.2.1.

§4. Functional differential equations with infinite delay

Consider the functional differential equation

$$x'(t) = f(t, x_t) \tag{4.1}$$

where $f: \boldsymbol{R} \times B \to \boldsymbol{R}^n$ is T-periodic in t and B is the space of bounded continuous functions $\phi: (-\infty, 0] \to \boldsymbol{R}^n$ with the supremum norm $\|\cdot\|$. Let $x: (-\infty, A) \to \boldsymbol{R}^n$, $-\infty < A \leq +\infty$ be continuous. For $t < A$, define $x_t: (-\infty, 0] \to \boldsymbol{R}^n$ by $x_t(s) = x(t+s)$, $s \leq 0$. Then x_t is x on $(-\infty, t]$, translated to $(-\infty, 0]$. If $A > 0$, then x_0 is merely x restricted to $(-\infty, 0]$.

Let C be an $n \times n$ matrix all of whose characteristic roots have negative real parts. We consider

$$x'(t) = Cx(t) + \lambda F(t, x_t) \tag{4.2}$$

where

$$F(t, x_t) = -Cx(t) + f(t, x_t). \tag{4.3}$$

Let $X_1 = X_2 = \boldsymbol{R}^n$. We denote by $(P_T, \|\cdot\|)$ the Banach space of continuous T-periodic functions $\phi: \boldsymbol{R} \to \boldsymbol{R}^n$ with the supremum norm.

Theorem 4.1. *Suppose the following conditions hold*:
(D_1) *there exists* $B_1 > 0$ *such that if* x *is a* T*-periodic solution of* (4.2), *then* $|x(t)| < B_1$ *for* $t \geq 0$;
(D_2) $F: \boldsymbol{R} \times P_T \to \boldsymbol{R}^n$ *takes bounded sets into bounded sets*;
(D_3) $F(\cdot, \phi_t): \boldsymbol{R} \to \boldsymbol{R}^n$ *is continuous whenever* $\phi \in (P_T, \|\cdot\|)$;
(D_4) *for any* $\mu \geq 0$, *there exists* $\beta > 0$ *such that* $[\phi, \psi \in P_T, \|\phi\| \leq \mu, \|\psi\| \leq \mu, t \in \boldsymbol{R}]$ *imply that* $|f(t, \phi_t) - f(t, \psi_t)| \leq \beta \|\phi - \psi\|$.
Then (4.1) *has a* T*-periodic solution.*

Proof. It is clear that X_2 is compactly imbedded into X_1 and C generates a C_0-semigroup on $\boldsymbol{R}^n$. There are positive constants $\tilde{C}$, b such that

$$\begin{cases} |e^{Ct}| \leq \tilde{C}e^{-bt} & \text{for } t \geq 0, \\ |(e^{Ct} - I)\phi| \leq \tilde{C}t|\phi| & \text{for } \phi \in X_2. \end{cases} \tag{4.4}$$

Since $F: \boldsymbol{R} \times \boldsymbol{P}_T \to \boldsymbol{R}^n$ takes bounded sets into bounded sets, for any $x \in P_T$ it follows that

$$\int_{-\infty}^{t} e^{C(t-s)} F(s, x_s) ds \in \boldsymbol{R}^n \in D(C)$$

and for any $\mu \geq 0$, there exists a $\tilde{\mu} > 0$ such that $[x \in P_T, \|x\| \leq \mu, t \in \boldsymbol{R}^+]$ imply that $|F(t, x_t)| \leq \tilde{\mu}$. Thus

$$\left| \int_{-\infty}^{t} e^{C(t-s)} F(s, x_s) ds \right| \leq \int_{-\infty}^{0} \tilde{C}e^{bs} |F(t+s, x_{t+s})| ds \leq \tilde{C}\tilde{\mu}/b.$$

Next let $\mu > 0$ and $[\phi, \psi \in P_T, \|\phi\| \leq \mu, \|\psi\| \leq \mu]$; then

$$\left| \int_{-\infty}^{t} e^{C(t-s)} [F(s, \phi_s) - F(s, \psi_s)] ds \right|$$

$$\leq \tilde{C} \int_{-\infty}^{0} e^{bs} |F(t+s, \phi_{t+s}) - F(t+s, \psi_{t+s})| ds.$$

Note that

$$|F(t+s, \phi_{t+s}) - F(t+s, \psi_{t+s})| \leq (|C| + \beta) \|\phi - \psi\|.$$

Hence,

$$\left| \int_{-\infty}^{t} e^{C(t-s)} [F(s, \phi_s) - F(s, \psi_s)] ds \right| \leq (|C| + \beta) \frac{\tilde{C}}{b} \|\phi - \psi\|.$$

This implies that condition (v) of Theorem 1.1 holds.

Finally all conditions of Theorem 1.1 are satisfied. Hence, (4.2) has a

T-periodic solution for $\lambda = 1$. That is, (4.1) has a T-periodic solution. The proof of Theorem 4.1 is complete.

Example 4.1. Consider a Volterra integral-differential equation

$$x'(t) = Ax(t) + \int_{-\infty}^{t} D(t-s)x(s)ds + p(t, x(t)) \tag{4.5}$$

where $x \in \boldsymbol{R}^n$, A and D are $n \times n$ matrices with $D: R \to \boldsymbol{R}^{n\times n}$ continuous, $p: \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and $p(t, x) = p(t + T, x)$ for some $T > 0$. Let $\int_t^{+\infty} D(s)ds \in L^1[0, +\infty)$ and define

$$C = A + \int_0^{+\infty} D(s)ds.$$

We assume that all characteristic roots of C have negative real parts. Thus there exists a positive definite matrix B such that

$$C^T B + BC = -I \tag{4.6}$$

where C^T is the transpose of C. Suppose that

$$\begin{cases} |p(t, x)| \le k|x| + M \text{ for some constants } k,\ M > 0, \\ \text{for any } \mu > 0, \text{ there exists } \tilde{\mu} > 0 \text{ such that} \\ [|x| \le \mu,\ |y| \le \mu,\ t \in \boldsymbol{R}] \text{ imply that} \\ |p(t, x) - p(t, y)| \le \tilde{\mu}|x - y|. \end{cases} \tag{4.7}$$

(4.8)

$$2k|B|\left(1 + \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)d\tau \right| ds\right) + 2|B||C| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)d\tau \right| ds < 1.$$

Theorem 4.2. *Suppose that* (4.6), (4.7) *and* (4.8) *hold. Then* (4.5) *has a T-periodic solution.*

Proof. For $\lambda \in [0, 1]$, consider

$$x'(t) = Cx(t) + \lambda\left[-\frac{d}{dt} \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau)d\tau x(s)ds + p(t, x(t))\right]. \tag{4.9}$$

We will verify that

$$\begin{aligned} F(t, x_t) &\overset{\text{def}}{=} -\frac{d}{dt} \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau)d\tau x(s)ds + p(t, x(t)) \\ &= -\int_0^{+\infty} D(\tau)d\tau x(t) + \int_{-\infty}^{t} D(t-s)x(s)ds + p(t, x(t)) \end{aligned} \tag{4.10}$$

satisfies conditions (D_2)-(D_4) in Theorem 4.1 and there exists a positive constant $B_1 > 0$ such that if $x(t)$ is a T-periodic solution of (4.9), then $|x(t)| < B_1$ for $t \geq 0$. Let $x(t)$ be a T-periodic solution of (4.9) and define

(4.11)
$$V(t) = \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right]^T B \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right].$$

Thus $V(t)$ is T-periodic and

(4.12)
$$\begin{aligned} V'(t) &= [Cx(t) + \lambda p(t, x)]^T B \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right] \\ &\quad + \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right]^T B[Cx(t) + \lambda p(t, x(t))] \\ &= x^T(t)[C^T B + BC]x(t) + \lambda x^T(t) C^T B \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds \\ &\quad + \lambda \left[\int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right]^T BCx(t) \\ &\quad + \lambda p^T(t, x(t)) B \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right] \\ &\quad + \lambda \left[x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right]^T Bp(t, x(t))\Bigg] \\ &\leq -|x(t)|^2 + 2|x(t)||B||C| \int_{-\infty}^{t} \left|\int_{t-s}^{+\infty} D(\tau) d\tau\right| |x(s)| ds \\ &\quad + 2|p(t, x(t))||B| \left|x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right| \\ &\leq -|x(t)|^2 + |B||C| \int_{0}^{+\infty} \left|\int_{s}^{+\infty} D(\tau) d\tau\right| ds |x(t)|^2 \\ &\quad + |B||C| \int_{-\infty}^{0} \left|\int_{-s}^{+\infty} D(\tau) d\tau\right| |x(t+s)|^2 ds \\ &\quad + 2(k|x(t)| + M)|B| \left|x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds\right| \\ &\leq -\left(1 - |B||C| \int_{0}^{+\infty} \left|\int_{s}^{+\infty} D(\tau) d\tau\right| ds\right) |x(t)|^2 \\ &\quad + |B||C| \int_{-\infty}^{0} \left|\int_{-s}^{+\infty} D(\tau) d\tau\right| |x(t+s)|^2 ds \end{aligned}$$

$$+ 2k|B||x(t)|^2 + k|B| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau)\, d\tau \right| ds |x(t)|^2$$

$$+ k|B| \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| |x(t+s)|^2 ds$$

$$+ 2|B| M \left| x(t) + \lambda \int_{-\infty}^{t} \int_{t-s}^{+\infty} D(\tau) d\tau x(s) ds \right|.$$

Integrating (4.12) from 0 to T, we have

$$\begin{aligned}
0 &= V(T) - V(0) \\
&\leq -\left(1 - 2k|B| - k|B| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \right. \\
&\quad \left. - |B||C| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \right) \int_0^T x^2(t) dt \\
&\quad + |B||C| \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| \int_0^T |x(t+s)|^2 dt ds \\
&\quad + k|B| \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| \int_0^T |x(t+s)|^2 dt ds \\
&\quad + 2|B| M \int_0^T |x(t)| dt \\
&\quad + 2|B| M \int_{-\infty}^{0} \left| \int_{-s}^{+\infty} D(\tau) d\tau \right| \int_0^T |x(t+s)| dt ds \\
&\leq -\left(1 - 2k|B| - 2k|B| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \right. \\
&\quad \left. - 2|B||C| \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \right) \int_0^T x^2(t) dt \\
&\quad + 2|B| M \left(1 + \int_0^{+\infty} \left| \int_s^{+\infty} D(\tau) d\tau \right| ds \right) \int_0^T |x(t)|\, dt.
\end{aligned}$$

By (4.8), this implies that

$$\int_0^T |x(t)|^2 dt \leq M_1 \qquad \text{for some } M_1 > 0. \tag{4.13}$$

Next from (4.9), we have

$$\int_0^T |x'(t)|\, dt \leq \left(|A| + \int_0^{+\infty} |D(\tau)| d\tau \right) \int_0^T |x(t)|\, dt + k \int_0^T |x(t)|\, dt + TM$$

Using (4.13), we get

$$\int_0^T |x'(t)|\,dt \le M_2 \qquad \text{for some } M_2 > 0. \tag{4.14}$$

By Sobolev's inequality, we have

$$\sup_{0\le t\le T} |x(t)| \le M_3 \qquad \text{for some } M_3 > 0. \tag{4.15}$$

By (4.7), conditions (D_2), (D_3) and (D_4) are satisfied. From Theorem 4.1, we conclude that (4.5) has a T-periodic solution.

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