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Lu, Wudu

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Nonoscillation and Oscillation for First Order Nonlinear Neutral Equations

By

Wudu LU

(The Chinese University of Hong Kong, Hong Kong)

1. Introduction

Recently oscillations of first order linear neutral equations have been discussed in many papers [1, 3–10, 15, 16]. However, there are few results for oscillations of first order nonlinear neutral equations and there are only three papers [3, 7, 19] dealing with the existence of nonoscillatory solutions of first order neutral equations with variable coefficients. [3] and [7] deal with linear neutral equations and [19] discusses nonlinear neutral equations which have nonoscillatory solutions $x(t)$ with $\liminf_{t\to\infty}|x(t)|>0$.

We first discuss the existence of nonoscillatory solutions for the first order nonlinear neutral equation

$$(1)\qquad [x(t)-\sum_{i=1}^{K}c_i(t)x(t-\gamma_1)]'+f(t,x(t-\sigma_1),\dots,x(t-\sigma_n))=0,$$

and obtain a new sufficient criterion. Next, we discuss oscillations of the nonlinear neutral equation

$$(2)\qquad [x(t)-\sum_{i=1}^{K}c_i(t)x(t-\gamma_i)]'+p(t)[\prod_{k=1}^{m}|x(t-\sigma_k)|^{\alpha_k}]\operatorname{sgn}x(t)=0,$$

and obtain a new condition for all solutions of (2) to oscillate.

Our conditions are "sharp" in the sense that when (1) and (2) are linear neutral equations with constant coefficients, the conditions become both necessary and sufficient.

We refer to [2, 11, 14, 17, 18] for oscillations of higher order neutral equations.

A solution of (1) or (2) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

2. Existence of nonoscillatory solutions

Consider the equation

(1)
$$[x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i)]' + f(t, x(t-\sigma_1),\dots,x(t-\sigma_n)) = 0, \qquad t \geq t_0 > 0,$$

where $\gamma_i > 0$, $i \in I_K = \{1, 2,\dots, K\}$, $\sigma_j \geq 0$, $j \in I_n = \{1, 2,\dots, n\}$; $c_i(t)$ $(i \in I_K)$ and f are continuous functions and satisfy the following conditions:

(i) $c_i(t) \geq 0$, $\sum_{i=1}^{K} c_i(t) \leq C$ $(0 < C < 1)$ for all sufficiently large t and there is a $c_i(t) \geq c_0 > 0$.

(ii) $f(t, y_1,\dots, y_n) \geq 0$ when $y_j \geq 0$ for all $j \in I_n$; $f(t, z_1,\dots, z_n) \geq f(t, y_1,\dots, y_n)$ when $z_j \geq y_j \geq 0$ for all $j \in I_n$.

Definition: A family of functions is equicontinuous on $[t_0, +\infty)$ if for any given $\varepsilon > 0$, the interval $[t_0, +\infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ε.

A set of functions in $C[t_0, +\infty)$ with $\|x\| = \sup_{t \geq t_0} |x(t)|$ is relatively compact if it is uniformly bounded and equicontinuous on $[t_0, +\infty)$ [12, 13].

Theorem 1. *Assume that* (i) *and* (ii) *hold,*

(3)
$$|c_i(t_2) - c_i(t_1)| \leq k_0 |t_2 - t_1|$$

where $k_0 > 0$ is the same number, and there exists a $k_1 > 0$ such that

(4)
$$\sup_{t \geq t_0} f(t, \exp(-k_1(t-\sigma_1)),\dots,\exp(-k_1(t-\sigma_n))) = M < \infty \qquad \textit{and}$$

(5)
$$\sum_{i=1}^{K} c_i(t) \exp(k_1\gamma_i) + \exp(k_1 t) \int_t^{\infty} f(s, \exp(-k_1(s-\sigma_1)),\dots, \exp(-k_1(s-\sigma_n)))\, ds \leq 1$$

for all sufficiently large t.

Then (1) *has a nonoscillatory solution which tends to zero.*

Proof. Set

$$S = \left\{ x(t) \in C[t_0, +\infty) : \begin{matrix} \exp(-k_2 t) \leq x(t) \leq \exp(-k_1 t), \\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|, t_2 \geq t_1 \geq t_0 \end{matrix} \right\}$$

where k_2 is sufficiently large such that $k_2 > k_1$ and $\sum_{i=1}^{K} c_i(t) \exp(k_2\gamma_i) \geq 1$; $L \geq \max\{k_0, k_2\}$ and $C + \dfrac{M}{L} < 1$.

We denote by C_B all bounded continuous functions in $C[t_0, +\infty)$ and define a norm $\|x\| = \sup_{t \geq t_0} |x(t)|$ in C_B. Then C_B is a Banach space and S is a bounded convex closed set in C_B.

Define a mapping as follows:

$$(6) \qquad (Px)(t) = \begin{cases} \sum_{i=1}^{K} c_i(t)x(t-\gamma_i) + \int_t^\infty f(s, x(s-\sigma_1), \dots, x(s-\sigma_n))ds, & t \geq T, \\ \exp\left(\dfrac{\ln (Px)(T)}{T} t\right), & t_0 \leq t < T, \end{cases}$$

where T is sufficiently large such that $T \geq t_0 + \max\{\gamma_1, \dots, \gamma_K, \sigma_1, \dots, \sigma_n\}$, (5) holds and

$$(7) \qquad \sum_{i=1}^{K} c_i(t_2) + \sum_{i=1}^{K} \exp(-k_1(t_1-\gamma_i)) + \frac{M}{L} \leq 1 \qquad \text{for } t_2 \geq t_1 \geq T.$$

We need to prove

a) $PS \subset S$. When $t \geq T$, we have for $x \in S$

$$\begin{aligned} (Px)(t) &\leq \sum_{i=1}^{K} c_i(t) \exp(-k_1(t-\gamma_i)) \\ &\quad + \int_t^\infty f(s, \exp(-k_1(s-\sigma_1)), \dots, \exp(-k_1(s-\sigma_n)))ds \\ &= \exp(-k_1 t)[\sum_{i=1}^{K} c_i(t) \exp(k_1\gamma_i) \\ &\quad + \exp(k_1 t) \int_t^\infty f(s, \exp(-k_1(s-\sigma_1)), \dots, \exp(-k_1(s-\sigma_n)))ds] \\ &\leq \exp(-k_1 t) \end{aligned}$$

and

$$\begin{aligned} (Px)(t) &\geq \sum_{i=1}^{K} c_i(t) \exp(-k_2(t-\gamma_i)) \\ &= \exp(-k_2 t) \sum_{i=1}^{K} c_i(t) \exp(k_2\gamma_i) \geq \exp(-k_2 t). \end{aligned}$$

Hence $\exp(-k_2 T) \leq (Px)(T) \leq \exp(-k_1 T)$. Then

$$(8) \qquad -k_2 \leq \frac{\ln (Px)(T)}{T} \leq -k_1.$$

From (6) and (8), we have $(Px)(t) \in C[t_0, \infty)$ and

$$\exp(-k_2 t) \le (Px)(t) \le \exp(-k_1 t) \qquad \text{for } t \ge t_0.$$

When $t_2 \ge t_1 \ge T$, we have

$$\begin{aligned}
&|(Px)(t_2) - (Px)(t_1)| \\
&\le \sum_{i=1}^{K} |c_i(t_2)x(t_2 - \gamma_i) - c_i(t_1)x(t_1 - \gamma_i)| + \int_{t_1}^{t_2} f(s, x(s - \sigma_1), \ldots, x(s - \sigma_n))ds \\
&\le \sum_{i=1}^{K} [c_i(t_2)|x(t_2 - \gamma_i) - x(t_1 - \gamma_i)| + |c_i(t_2) - c_i(t_1)|x(t_1 - \gamma_i)] \\
&\quad + \int_{t_1}^{t_2} f(s, \exp(-k_1(s - \sigma_1)), \ldots, \exp(-k_1(s - \sigma_n)))ds \\
&\le \{\sum_{i=1}^{K} [c_i(t_2) + \exp(-k_1(t_1 - \gamma_i))]\} L|t_2 - t_1| \\
&\quad + \sup_{s \ge T} f(s, \exp(-k_1(s - \sigma_1)), \ldots, \exp(-k_1(s - \sigma_n)))|t_2 - t_1| \\
&\le \left[\sum_{i=1}^{K} c_i(t_2) + \sum_{i=1}^{K} \exp(-k_1(t_1 - \gamma_i)) + \frac{M}{L}\right] L|t_2 - t_1| \\
&\le L|t_2 - t_1|.
\end{aligned}$$

When $t_0 \le t_1 \le t_2 \le T$, using the Mean Value Theorem we have

$$\begin{aligned}
|(Px)(t_2) - (Px)(t_1)| &= \left|\exp\left(\frac{\ln(Px)(T)}{T} t_2\right) - \exp\left(\frac{\ln(Px)(T)}{T} t_1\right)\right| \\
&\le k_2|t_2 - t_1| \\
&\le L|t_2 - t_1|.
\end{aligned}$$

Then

$$|(Px)(t_2) - (Px)(t_1)| \le L|t_2 - t_1| \qquad \text{for } t_2 \ge t_1 \ge t_0.$$

Hence $Px \in S$.

b) P is a continuous mapping. Set $x_k \in S$ and $\lim_{k \to \infty} \|x_k - x\| = 0$. Then $x \in S$. When $t \ge T$,

$$\begin{aligned}
&|(Px_k)(t) - (Px)(t)| \\
&\le \sum_{i=1}^{K} c_i(t)|x_k(t - \gamma_i) - x(t - \gamma_i)|
\end{aligned}$$

$$+ \int_t^\infty |f(s, x_k(s-\sigma_1),\dots,x_k(s-\sigma_n)) - f(s, x(s-\sigma_1),\dots,x(s-\sigma_n))|ds$$

$$\le \sum_{i=1}^{K} c_i(t)\,\|x_k - x\| + \int_T^\infty G_k(s)ds$$

$$\le \|x_k - x\| + \int_T^\infty G_k(s)ds$$

where $G_k(s) = |f(s, x_k(s-\sigma_1),\dots,x_k(s-\sigma_n)) - f(s, x(s-\sigma_1),\dots,x(s-\sigma_n))|$. Obviously, $\lim_{k\to\infty} G_k(s) = 0$ and

$$G_k(s) \le 2f(s, \exp(-k_1(s-\sigma_1)),\dots,\exp(-k_1(s-\sigma_n))).$$

From the Lebesgue Theorem, we have

$$\lim_{k\to\infty} \int_T^\infty G_k(s)ds = 0.$$

Hence

$$\lim_{k\to\infty} (\sup_{t\ge T} |(Px_k)(t) - (Px)(t)|) = 0. \tag{9}$$

Then

$$\lim_{k\to\infty} |(Px_k)(T) - (Px)(T)| = 0. \tag{10}$$

When $t_0 \le t \le T$,

$$|(Px_k)(t) - (Px)(t)| = \left|\frac{\ln (Px_k)(T)}{T} - \frac{\ln (Px)(T)}{T}\right| t \tag{11}$$

$$\le |\ln (Px_k)(T) - \ln (Px)(T)|.$$

Combining (10) and (11), we have

$$\lim_{k\to\infty} [\sup_{t_0\le t\le T} |(Px_k)(t) - (Px)(t)|] = 0. \tag{12}$$

From (9) and (12), it follows that

$$\lim_{k\to\infty} \|Px_k - Px\| = 0.$$

c) *PS* is relatively compact. Obviously, *PS* is uniformly bounded. For any $x \in S$, we have

$$|(Px)(t)| \leq \exp(-k_1 t)$$

and

$$|(Px)(t_2) - (Px)(t_1)| \leq L|t_2 - t_1| \qquad \text{for } t_2 \geq t_1 \geq t_0.$$

Then for any given $\varepsilon > 0$, there exists a sufficiently large $T' > t_0$ such that $\exp(-k_1 t) < \frac{\varepsilon}{2}$ for $t \geq T'$ and then

$$|(Px)(t_2) - (Px)(t_1)| < \varepsilon \qquad \text{for } t_2 \geq t_1 \geq T'. \tag{13}$$

Let $\delta = \varepsilon/L$. When $t_0 \leq t_1 \leq t_2 \leq T'$ and $|t_2 - t_1| \leq \delta$,

$$|(Px)(t_2) - (Px)(t_1)| < \varepsilon. \tag{14}$$

From (13) and (14), PS is equicontinuous on $[t_0, \infty)$. Hence PS is a relatively compact set. According to Schauder's fixed point theorem, P has a fixed point $x^*(t)$ in S. Obviously, $x^*(t)$ is a nonoscillatory solution of (1) which tends to zero. The proof is complete.

From Theorem 1, we have

Corollary 1. *Assume that $c_i(t)$ and $p_j(t)$ are nonnegative continuous functions and $c_i(t)$ satisfies (i) and (3). If $c_i(t) \leq c_i$, $p_j(t) \leq p_j$ and there exists a positive μ such that*

$$\sum_{i=1}^{K} c_i(t) \exp(\mu\gamma_i) + \frac{1}{\mu} \sum_{j=1}^{n} p_j \exp(\mu\sigma_j) \leq 1, \tag{15}$$

then the equation

$$[x(t) - \sum_{i=1}^{K} c_i(t) x(t-\gamma_i)]' + \sum_{j=1}^{n} p_j(t) x(t-\sigma_j) = 0, \qquad t \geq t_0 > 0, \tag{16}$$

has a nonoscillatory solution which tends to zero.

Remark 1. When $c_i(t) \equiv c_i$ and $p_j(t) \equiv p_j$, (15) is equivalent to that the characteristic equation of (16) has no real roots. Hence (15) is a necessary and sufficient condition for (16) with constant coefficients to have a nonoscillatory solution [1, 6, 15, 16].

Remark 2. All nonoscillation theorems of [3] can be derived from Corollary 1 or Theorem 1.

Corollary 2. *Consider*

$$[x(t) - \sum_{i=1}^{K} c_i(t) x(t-\gamma_i)]' + \sum_{j=1}^{n} p_j(t) [\prod_{k=1}^{m_j} |x(t-\sigma_{jk})|^{\alpha_{jk}}] \operatorname{sgn} x(t) = 0, \tag{17}$$

where $t \ge t_0 > 0$, $\gamma_i > 0$, $\sigma_{j_k} \ge 0$, $\alpha_{j_k} \ge 0$ $(i \in I_K, j \in I_{n'}\ k \in I_{m_j} = \{1, 2, \ldots, m_j\}$; $c_i(t)$ *and* $p_j(t)$ *are nonnegative continuous functions*; $c_i(t)$ *satisfies* (i) *and* (3). *If there exists a positive number* μ *such that for some sufficiently large* T,

$$\sup_{t \ge T} [p_j(t) \exp(-\mu \sum_{k=1}^{m_j} \alpha_{j_k} t)] < \infty \qquad \textit{for all } j \in I_n \tag{18}$$

and

$$\sup_{t \ge T} \left\{ \sum_{i=1}^{K} c_i(t) \exp(\mu\gamma_i) + \sum_{j=1}^{n} \exp(\mu \sum_{k=1}^{m_j} \alpha_{j_k}\sigma_{j_k}) \times \int_t^\infty p_j(s) \exp[-\mu(\sum_{k=1}^{m_j} \alpha_{j_k} s - t)] ds \right\} \le 1, \tag{19}$$

then (17) *has a nonoscillatory solution which tends to zero.*

3. Oscillation

Consider the equation

$$[x(t) - \sum_{i=1}^{K} c_i(t) x(t-\gamma_i)]' + p(t) \prod_{k=1}^{m} |x(t-\sigma_k)|^{\alpha_k} \operatorname{sgn} x(t) = 0, \qquad t \ge t_0 > 0, \tag{2}$$

where $0 < \gamma_1 \le \gamma_2 \le \cdots \le \gamma_K$, $0 \le \sigma_1 \le \cdots \le \sigma_m$, $\alpha_k > 0$ and $\sum_{k=1}^{m} \alpha_k \le 1$; $c_i(t) \ge 0$ $(i \in I_K)$ and $p(t) > 0$ are continuous.

Lemma 1. *Assume that* $\sum_{i=1}^{K} c_i(t) \le C < 1$ *and* $\int^\infty p(s) ds = \infty$. *If* $x(t)$ *is an eventually positive solution of* (2), *then* $y(t) > 0$ *eventually monotonically tends to zero, where* $y(t) = x(t) - \sum_{i=1}^{K} c_i(t) x(t-\gamma_i)$.

Proof. From (2), we have $y'(t) < 0$ eventually. Then

$$\lim_{t \to \infty} y(t) = -\infty \tag{20}$$

or

$$\lim_{t \to \infty} y(t) = a > -\infty. \tag{21}$$

If (20) holds, then $x(t)$ is unbounded and there exists a sequence $\{t_k\}$ such that $\lim_{k \to \infty} t_k = +\infty$ and $x(t_k) = \max_{s \le t_k} x(s)$. We have

$$(22)\qquad \begin{aligned} y(t_k) &= x(t_k) - \sum_{i=1}^{K} c_i(t_k)x(t_k - \gamma_i) \\ &\geq x(t_k)(1 - \sum_{i=1}^{K} c_i(t_k)) \\ &\geq 0 \end{aligned}$$

which contradicts (20). Hence (21) holds. From (22), $x(t)$ must be bounded. Set $\lim_{k\to\infty} x(t'_k) = \limsup_{t\to\infty} x(t)$. Without loss of generality, we assume that $\lim_{k\to\infty} c_i(t'_k)$ and $\lim_{k\to\infty} x(t'_k - \gamma_i)$ exist. Then

$$\begin{aligned} a &= \lim_{k\to\infty} y(t'_k) \\ &\geq \limsup_{t\to\infty} x(t)[1 - \lim_{k\to\infty} \sum_{i=1}^{K} c_i(t'_k)] \\ &\geq 0. \end{aligned}$$

If $a > 0$, then from (2) and $x(t) \geq y(t)$ we have

$$a - y(T) = -\int_T^{+\infty} p(s) \prod_{k=1}^{m} |x(s - \sigma_k)|^{\alpha_k} ds = -\infty.$$

This contradiction implies that $a = 0$. The proof is complete.

Theorem 2. *If* $\sum_{i=1}^{K} c_i(t) \leq C < 1$ *and there exists some sufficiently large* T *such that*

$$(23)\qquad \inf_{t\geq T, \mu>0} \left\{ D(t) \left[\frac{1}{\mu} p(t) \exp (\mu \sum_{k=1}^{m} \alpha_k \sigma_k) + \sum_{i=1}^{K} E_i(t) \exp (\mu\gamma_i) \right] \right\} > 1$$

where $D(t) = \prod_{k=1}^{m} [1 + \sum_{i=1}^{K} c_i(t - \sigma_k)]^{\alpha_k - 1}$ *and* $E_i(t) = \dfrac{p(t)}{p(t - \gamma_i)} \prod_{k=1}^{m} c_i(t - \sigma_k)$ $(i \in I_K)$, *then all solutions of* (2) *oscillate.*

Proof. By (23) we can prove that there exists some $d > 0$ such that $p(t) \geq d$ $(t \geq T)$. Otherwise, $\inf_{t\geq T} p(t) = 0$ and then there exists a sequence $\{t_n\}$ such that $p(t_n) = \min_{t\leq t_n} p(t)$ and $\lim_{n\to\infty} p(t_n) = 0$. Then

$$(24)\qquad \lim_{n\to\infty} \frac{1}{\mu} p(t_n) \exp (\mu \sum_{k=1}^{m} \alpha_k \sigma_k) = 0, \qquad \mu > 0,$$

and

$$\sum_{i=1}^{K} E_i(t_n)\exp(\mu\gamma_i) \le \sum_{i=1}^{K} c_i(t_n-\sigma_1)\exp(\mu\gamma_i) \tag{25}$$
$$\le \sum_{i=1}^{K} c_i(t_n-\sigma_1)\exp(\mu\gamma_K)$$
$$\le C\exp(\mu\gamma_K).$$

From (24) and (25), noting that $\left(\frac{1}{2}\right)^m \le D(t) \le 1$, when $\mu > 0$ is sufficiently small and n is sufficiently large we have

$$D(t_n)\left[\frac{1}{\mu}p(t_n)\exp(\mu\sum_{k=1}^{m}\alpha_k\sigma_k) + \sum_{i=1}^{K}E_i(t_n)\exp(\mu\gamma_i)\right] \le 1$$

which contradicts (23). If (2) has a nonoscillatory solution $x(t) > 0$, then set

$$y(t) = x(t) - \sum_{i=1}^{K} c_i(t)x(t-\gamma_i). \tag{26}$$

According to Lemma 1, there exists a T such that when $t \ge T - \gamma_K - \sigma_m$, $x(t) > 0$, $0 < y(t) \le 1$ and $y'(t) < 0$. Set

$$u(t) = -\frac{y'(t)}{y(t)}, \qquad t \ge T. \tag{27}$$

Then

$$\frac{y(t_1)}{y(t_2)} = \exp\left(\int_{t_1}^{t_2} u(s)ds\right) \qquad \text{for } t_1, t_2 \in [T, \infty). \tag{28}$$

From (27) and (2), using Jensen's inequality, when $t \ge T$ we have

(29)

$$u(t)$$
$$= \frac{p(t)}{y(t)}\prod_{k=1}^{m}[y(t-\sigma_k) + \sum_{i=1}^{K}c_i(t-\sigma_k)x(t-\sigma_k-\gamma_i)]^{\alpha_k}$$
$$\ge \frac{p(t)}{y(t)}\prod_{k=1}^{m}[1+\sum_{i=1}^{K}c_i(t-\sigma_k)]^{\alpha_k-1}\prod_{k=1}^{m}[y^{\alpha_k}(t-\sigma_k)+\sum_{i=1}^{K}c_i(t-\sigma_k)x^{\alpha_k}(t-\sigma_k-\gamma_i)]$$
$$\ge \frac{p(t)}{y(t)}\prod_{k=1}^{m}[1+\sum_{i=1}^{K}c_i(t-\sigma_k)]^{\alpha_k-1}$$
$$\times [\prod_{k=1}^{m} y^{\alpha_k}(t-\sigma_k) + \sum_{i=1}^{K}\prod_{k=1}^{m}c_i(t-\sigma_k)\prod_{k=1}^{m}x^{\alpha_k}(t-\sigma_k-\gamma_i)]$$

$$\geq D(t)\left[p(t)\prod_{k=1}^{m}\left(\frac{y(t-\sigma_k)}{y(t)}\right)^{\alpha_k}+\frac{1}{y(t)}\sum_{i=1}^{K}E_i(t)p(t-\gamma_i)\prod_{k=1}^{m}x^{\alpha_k}(t-\sigma_k-\gamma_i)\right]$$

$$=D(t)\left[p(t)\prod_{k=1}^{m}\exp\left(\alpha_k\int_{t-\sigma_k}^{t}u(s)ds\right)+\sum_{i=1}^{K}E_i(t)\frac{-y'(t-\gamma_i)}{y(t-\gamma_i)}\,\frac{y(t-\gamma_i)}{y(t)}\right]$$

$$=D(t)\left[p(t)\exp\left(\sum_{k=1}^{m}\alpha_k\int_{t-\sigma_k}^{t}u(s)ds\right)+\sum_{i=1}^{K}E_i(t)u(t-\gamma_i)\exp\left(\left(\int_{t-\gamma_i}^{t}u(s)ds\right)\right]\right..$$

Set $\lambda_0=0$,

$$\lambda_n=\inf_{t\geq T}\{D(t)[p(t)\exp(\lambda_{n-1}\sum_{k=1}^{m}\alpha_k\sigma_k)+\sum_{i=1}^{K}E_i(t)\lambda_{n-1}\exp(\lambda_{n-1}\gamma_i)]\}, \tag{30}$$

$$n=1,2,\dots.$$

By induction, it is easy to prove

$$\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots$$

When $t\geq T$, $\lambda_0<u(t)$. Using (29), (30) and induction, we easily prove that $\lambda_n\leq u(t)$ for $t\geq T+n\max\{\gamma_K,\sigma_m\}$. Set

$$\lim_{n\to\infty}\lambda_n=\lambda^*.$$

If $\lambda^*=\infty$, then $\lim_{t\to\infty}u(t)=+\infty$. Integrating (2) from $t-\frac{\sigma_1}{2}$ to t, then dividing it by $y\left(t-\frac{\sigma_1}{2}\right)$ and noting that $y(t)\leq 1$ is decreasing, we easily have

$$\frac{y(t)}{y\left(t-\frac{\sigma_1}{2}\right)}-1+\frac{1}{y\left(t-\frac{\sigma_1}{2}\right)}\int_{t-\frac{\sigma_1}{2}}^{t}p(s)\prod_{k=1}^{m}y^{\alpha_k}(s-\sigma_k)ds\leq 0,\qquad t\geq T.$$

Then $$\frac{y(t)}{y\left(t-\frac{\sigma_1}{2}\right)}-1+\frac{y(t-\sigma_1)}{y\left(t-\frac{\sigma_1}{2}\right)}\int_{t-\frac{\sigma_1}{2}}^{t}p(s)ds\leq 0,\qquad t\geq T,\quad\text{and}$$

$$\exp\left(\int_{t}^{t-\frac{\sigma_1}{2}}u(s)ds\right)-1+\frac{d\sigma_1}{2}\exp\left(\int_{t-\sigma_1}^{t-\frac{\sigma_1}{2}}u(s)ds\right)\leq 0,\qquad t\geq T. \tag{31}$$

Letting $t\to\infty$, then the first term of (31) tends to zero and the third term of (31) tends to $+\infty$. This leads to a contradiction. Hence, $0<\lambda^*<+\infty$.

Set

(32) $$\varphi_n(t) = D(t)[p(t)\exp(\lambda_{n-1}\sum_{k=1}^{m}\alpha_k\sigma_k) + \sum_{i=1}^{K}E_i(t)\lambda_{n-1}\exp(\lambda_{n-1}\gamma_i)],$$

and

(33) $$\varphi(t) = D(t)[p(t)\exp(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k) + \sum_{i=1}^{K}E_i(t)\lambda^*\exp(\lambda^*\gamma_i)].$$

For any given $\varepsilon > 0$, there exists a $t_n \geq T$ for each $\varphi_n(t)$ such that

(34) $$\varphi_n(t_n) \leq \lambda_n + \varepsilon \leq \lambda^* + \varepsilon.$$

By (34), it is easy to prove that $\{p(t_n)\}$ and $\{E_i(t_n)\}$ $(i \in I_K)$ are bounded. Without loss of generality, assume that $\lim_{n\to\infty} D(t_n)$, $\lim_{n\to\infty} p(t_n)$ and $\lim_{n\to\infty} E_i(t_n)$ $(i \in I_K)$ exist. Set

$$\varphi^* = \lim_{n\to\infty} D(t_n)[p(t_n)\exp(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k) + \sum_{i=1}^{K}E_i(t_n)\lambda^*\exp(\lambda^*\gamma_i)].$$

Then $\lim_{n\to\infty}\varphi_n(t_n) = \varphi^*$. Hence $\inf_{t\geq T}\varphi(t) \leq \varphi^* \leq \lambda^* + \varepsilon$. Letting $\varepsilon \to 0$, we have

$$\inf_{t\geq T}\varphi(t) \leq \lambda^*.$$

Then

$$\inf_{t\geq T}\left\{D(t)\left[\frac{1}{\lambda^*}p(t)\exp(\lambda^*\sum_{k=1}^{m}\alpha_k\sigma_k) + \sum_{i=1}^{K}E_i(t)\exp(\lambda^*\gamma_i)\right]\right\} \leq 1$$

which contradicts (23). The proof is complete.

Remark 3. When $m = 1$, $\alpha_1 = 1$, $c_i(t) \equiv c_i$ and $p(t) \equiv p$, (23) becomes

(35) $$\frac{1}{\mu}p\exp(\mu\sigma_1) + \sum_{i=1}^{K}c_i\exp(\mu\gamma_i) > 1 \qquad \text{for all } \mu > 0.$$

By Corollary 2 or Corollary 1, it is easy to prove that (35) is a necessary and sufficient condition for all solutions of (2) oscillate.

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nuna adreso:
Department of Mathematics
Lady Shaw Building
The Chinese University of Hong Kong
Shatin, N. T.
Hong Kong

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