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Elaydi, S.

Zhang, S.

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Stability and Periodicity of Difference Equations with Finite Delay

By

S. ELAYDI and S. ZHANG*

(Trinity University and the University of Rhode Island, USA)

The study of stability of difference equations has been carried out by many authors in [1, 2, 3, 5]. In [1–3] the authors dealt with equations of infinite delay, while in [5], the investigation was concerned with finite delay equations. The main purpose of this paper is to develop a general theory of stability for finite delay difference equations. Specifically, a Liapunov and Razumikhin type theory for finite delay systems will be established in section 1. In section 2, we develop a variation of constants formula for nonhomogeneous system. Then the existence and uniqueness of a periodic solution of a nonhomogeneous linear difference system are established provided that the homogeneous system is uniformly asymptotically stable. This result extends the work of the first author in [3].

1. Stability theory

Consider the difference systems with finite delay of the form:

$$x(n+1) = F(n, x(n), x(n-\tau_1(n)),\dots,x(n-\tau_m(n))),\ n\in \boldsymbol{Z}^+ \tag{1.1}$$

where $\boldsymbol{Z}^+$ denotes the set of non negative integers, $x\in \boldsymbol{R}^k$, $\tau_i\colon \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$, $o \le \tau_i(n) \le r$, $i = 1, 2,\dots,m$, for some positive integer r.

We assume

$$F(n, 0, 0,\dots,0) \equiv 0 \qquad \text{for } n\in \boldsymbol{Z}^+$$

so that (1.1) always has the zero solution $x(n) = 0$ for $n\in \boldsymbol{Z}^+$.

Obviously, for any given $n_0\in \boldsymbol{Z}^+$ and a given initial function $\varphi\colon \{-r, -r+1,\dots,-1, 0\} \to \boldsymbol{R}^k$, there is a unique solution of (1.1), denoted by $x(n, n_0, \varphi)$, such that it satisfies (1.1) for all integers $n \ge n_0$ and

$$x(n_0 + j, n_0, \varphi) = \varphi(j) \qquad \text{for } j = -r, -r+1,\dots,-1, 0.$$

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* On leave from Anhui University, Hefei, Anhui, 230039, People's Rupublic of China.

Let

$$\|\varphi\| = \sup\{|\varphi(j)| : j \in \{-r, -r+1, \ldots, -1, 0\}\}$$

and for any $H > 0$,

$$S_H = \{\varphi : \|\varphi\| < H\}.$$

Definition 1. The zero solution of (1.1) is uniformly stable if for each $\varepsilon > 0$, and any $n_0 \in \boldsymbol{Z}^+$, there exists a $\delta > 0$ independent of n_0 such that if $\|\varphi\| < \delta$, then

$$|x(n, n_0, \varphi)| < \varepsilon \qquad \text{for all } n \geq n_0.$$

Definition 2. The zero solution of (1.1) is uniformly asymptotically stable if it is uniformly stable and there is an $\eta > 0$ such that for each $\gamma > 0$, there exists an integer $N(\gamma) > 0$ independent of n_0 such that if $\|\varphi\| < \eta$ then

$$|x(n, n_0, \varphi)| < \gamma \qquad \text{for all } n \geq n_0 + N(\gamma).$$

Definition 3. A strictly increasing continuous function $W: [0, \infty) \to [0, \infty)$, with $W(0) = 0$, $W(s) > 0$ if $s > 0$, is called a wedge.

The following lemma is needed in the proof of the main results in this section.

Lemma 4. *Let $\{x(n)\}$ be a bounded sequence. Suppose that for a wedge W_1 and $r \in \boldsymbol{Z}^+$, $\sum_{j=n-r}^{n} W_1(|x(j)|) \geq \alpha$ for some $\alpha > 0$. If W_2 is another wedge, then there exists $\beta > 0$ such that*

$$\sum_{j=n-r}^{n} W_2(|x(j)|) \geq \beta.$$

Proof. Suppose

$$\sum_{j=n-r}^{n} W_1(|x(j)|) \geq \alpha.$$

Then there must exist at least one $j^* \in [n-r, n]$ such that

$$W_1(|x(r^*)|) \geq \frac{\alpha}{r+1},$$

or

$$|x(r^*)| \geq W_1^{-1}\left(\frac{\alpha}{r+1}\right).$$

Therefore,

$$\sum_{j=n-r}^{n} W_2(|x(j)|) \geq W_2(|x(j^*)|) \geq W_2\left(W_1^{-1}\left(\frac{\alpha}{r+1}\right)\right) = \beta.$$

Obviously, $\beta > 0$ is independent of n, this completes the proof.

We are now ready to state the main results of this section. The first result is a Liapunov type theorem modeled after [4].

Theorem 5. *Suppose there exists a (Liapunov) functional* $V: \boldsymbol{Z}^+ \times S_H \to [0, \infty)$ *satisfying the following two conditions*:

$$\text{(i)} \quad W_1(|\varphi(0)|) \leq V(n, \varphi) \leq W_2(|\varphi(0)|) + W_3[\sum_{j=-r}^{0} W_4(|\varphi(j)|)],$$

and

$$\text{(ii)} \quad \Delta V_{(1.1)}(n, x_n(\cdot)) \leq -W_5(|x(n)|)$$

for some wedge functions W_i, $1 \leq i \leq 5$, *where*

$$\Delta V_{(1.1)}(n, x_n(\cdot)) = V(n+1, x_{n+1}(\cdot)) - V(n, x_n(\cdot)).$$

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. First we prove uniform stability. So let $\varepsilon > 0$ be given. For any $n_0 \in \boldsymbol{Z}^+$ and $\varphi \in S_\delta$, with $\delta < \varepsilon$, let $x(n)$ denote $x(n, n_0, \varphi)$. Then

$$W_1(|x(n)|) \leq V(n, x(\cdot)) \leq V(n_0, \varphi) \leq W_2(\delta) + W_3[(r+1)W_4(\delta)]. \tag{1.2}$$

For sufficiently small δ, $W_2(\delta) + W_3[(r+1)W_4(\delta)] < W_1(\varepsilon)$. Hence from (1.2) $|x(n)| < \varepsilon$, for $n \geq n_0$. Consequently, the zero solution of (1.1) is uniformly stable. Thus there exists $\eta > 0$ such that $\varphi \in S_\eta$ implies $|x(n, n_0, \varphi)| < H$. Let $\gamma > 0$ be given. Then there exists $\alpha > 0$ such that

$$W_2(\alpha) + W_3(\alpha) < W_1(\gamma). \tag{1.3}$$

Let $\varphi \in S_\eta$. We need to find N such that for $n \geq n_0 + N$, $|x(n, n_0, \varphi)| < \gamma$. We first show that there exists $n^* > n_0$ with

$$|x(n^*)| < \alpha \quad \text{and} \quad \sum_{j=n^*-r}^{n^*} W_4(|x(j)|) < \alpha.$$

In fact, since $\Delta V_{(1.1)}(n, x(\cdot)) \leq -W_5(|x(n)|)$, there exists a sufficiently large $N^* \geq r$ such that $|x(n, n_0, \varphi)| < \alpha$ for some n in every discrete interval of length N^*. Thus there exists a sequence $\{n_i\}$, $n_i \in [n_0 + (i-1)N^*, n_0 + iN^*]$, $i = 2, 3, \ldots$, such that $|x(n_i, n_0, \varphi)| < \alpha$.

If

$$\sum_{j=n_i-r}^{n_i} W_4(|x(j)|) \geq \alpha \tag{1.4}$$

then by Lemma 4, $\sum_{j=n_i-r}^{n_i} W_5(|x(j)|) \geq \beta$ for some $\beta > 0$. Note that

$$V(n_0, \varphi) \leq W_2(H) + W_3[(r+1)W_4(\eta)] = \mu.$$

Hence for $n \geq n_{2h}$, for some positive integer h to be specified later,

$$\begin{aligned} V(n+1, x_{n+1}(\cdot)) &\leq V(n_0, \varphi) - \sum_{j=n_0}^{n} W_5(|x(j)|) \\ &\leq \mu - \sum_{i=1}^{h} \sum_{j=n_{2i}-r}^{n_{2i}} W_5(|x(j)|) \leq \mu - h\beta \\ &< 0 \text{ if } h > \frac{\mu}{\beta}. \end{aligned}$$

Thus if $h > \frac{\mu}{\beta}$, then there exists $n^* \leq n_0 + 2hN^*$ for which (1.4) fails, that is,

$$\sum_{j=n^*-r}^{n^*} W_4(|x(j)|) < \alpha.$$

Thus

$$W_1(|x(n)|) \leq V(n, x(\cdot)) \leq V(n^*, x(\cdot)) < W_2(\alpha) + W_3(\alpha) < W_1(\gamma)$$

for $n \geq n^*$. If we let $h = \left[\frac{\mu}{\beta}\right] + 1$, $N = 2hN^*$, then $|x(n)| < \gamma$ for $n \geq n_0 + N$. The proof of the theorem is now complete.

The next important result belongs to a class of theorems commonly known as of Liapunov-Razumikhin type. Here we use (Liapunov) functions in contrast to the use of (Liapunov) functionals in the preceding theorem.

Theorem 6. *Suppose there exists a function* $V: Z \times B_H \to [0, \infty)$, *where* $B_H = \{x \in \boldsymbol{R}^k : |x| < H\}$, *such that*:

(i) $W_1(|x|) \leq V(n, x) \leq W_2(|x|)$,

(ii) $\Delta V_{(1.1)}(n, x_n(n)) \leq -W_3(|x(n)|)$ *if*

$$P[V(n+1), x(n+1)] > V(s, x(s)) \qquad \text{for } n - r \leq s \leq n,$$

where W_i, $1 \leq i \leq 3$, *are wedges, and* $P: [0, \infty) \to [0, \infty)$ *is a continuous function*

with $P(u) > u$ *if* $u > 0$. *Then the zero solution of* (1.1) *is uniformly asymptotically stable. Moreover, if* $H = \infty$ *and* $W_1(u) \to \infty$ *as* $u \to \infty$, *then the zero solution of* (1.1) *is a global attractor* [4].

Proof. First we establish uniform stability. For a given $\varepsilon > 0$ ($\varepsilon < H$) choose $\delta > 0$ such that $\delta < \varepsilon$ and $W_2(\delta) < W_1(\varepsilon)$. Let

$$n_0 \in \boldsymbol{Z}^+, \ \|\varphi\| < \delta, \quad \text{and} \quad x(n) = x(n, n_0, \varphi).$$

Claim that

$$V(n, x(n)) < W_1(\varepsilon), \qquad \text{for } n \geq n_0.$$

From our construction above, we have

$$V(n, x(n)) \leq W_2(\delta) < W_1(\varepsilon), \qquad \text{for } n_0 - r \leq n \leq n_0.$$

Suppose now there exists $n_1 \geq n_0$ such that

$$V(n, x(n)) < W_1(\varepsilon), \qquad \text{for } n_0 \leq n \leq n_1,$$

and

$$V(n_1 + 1, x(n_1 + 1)) \geq W_1(\varepsilon).$$

Then

$$\begin{aligned} P(V(n_1 + 1, x(n_1 + 1))) &> V(n_1 + 1, x(n_1 + 1)) \\ &\geq W_1(\varepsilon) \\ &> V(s, x(s)), \qquad \text{for } n_1 - r \leq s \leq n_1. \end{aligned}$$

Hence by (ii) we conclude that

$$V(n_1 + 1, x(n_1 + 1)) \leq V(n_1, x(n_1)) < W_1(\varepsilon),$$

which is a contradiction. This proves our claim that $V(n, x(n_1)) < W_1(\varepsilon)$ for $n \geq n_0$. Using (i), one may now conclude uniform stability.

To prove uniform asymptotic stability, we let $\varepsilon = H$ and $\delta = \eta$ be the corresponding constant from uniform stability. Let $n_0 \in \boldsymbol{Z}^+$, $\varphi \in S_\eta$. Then from assumption (i) we have

$$V(n, x(n)) \leq W_2(H), \qquad \text{for } n \geq n_0.$$

For any $\gamma > 0$, with $W_1(\gamma) < W_2(H)$, let

$$d = \inf \{(P(u) - u) \mid W_1(\gamma) \leq u \leq W_2(H)\}. \tag{1.5}$$

Then clearly $d > 0$. Choose $h \in \boldsymbol{Z}^+$ such that

$$W_1(\gamma) + (h-1)d < W_2(H) \leq W_1(\gamma) + hd. \tag{1.6}$$

We claim that there exists $N_1 \geq n_0 + r$ such that

$$V(N_1, x(n_1)) \leq W_1(\gamma) + (h-1)d.$$

To prove the claim we assume the contrary, that is,

$$V(n, x(n)) > W_1(\gamma) + (h-1)d, \qquad \text{for all } n \geq n_0. \tag{1.7}$$

Hence

$$\begin{aligned} P(V(n+1, x(n+1))) &\geq V(n+1, x(n+1)) + d && \text{(from 1.5)} \\ &> W_1(\gamma) + hd && \text{(from 1.7)} \\ &\geq W_2(H) && \text{(from 1.6)} \\ &\geq V(s, x(s)), \qquad \text{for } n - r \leq s \leq n,\ n \geq n_0 + r. \end{aligned}$$

Thus by assumption (ii) it follows that

$$V(n+1, x(n+1)) \leq W_2(H) - \sum_{j=n_0+r}^{n} W_3(|x(j)|). \tag{1.8}$$

Using assumption (i) and (1.7) we obtain

$$W_2(|x(n)|) \geq V(n, x(n)) > W_1(\gamma).$$

Thus

$$|x(n)| > W_2^{-1}(W_1(\gamma)) = b > 0.$$

Now for $n \geq n_0 + r + [W_2(H)/W_3(b)]$, (1.8) yields

$$V(n+1, x(n+1)) \leq W_2(H) - W_3(b)([W_2(H)/W_3(b)] + 1) < 0,$$

which leads to a contradiction.
Therefore there must be some $N_1 \in [n_0 + r, n_0 + r + [W_2(H)/W_3(b)]]$ such that

$$V(N_1, x(N_1)) \leq W_1(\gamma) + (h-1)d.$$

We next show that

$$V(n, x(n)) \leq W_1(\gamma) + (h-1)d, \qquad \text{for all } n \geq N_1. \tag{1.9}$$

If (1.9) is not true then there exists $\sigma \geq N_1$ such that

$$V(n, x(n)) \leq W_1(\gamma) + (h-1)d, \qquad \text{for } N_1 \leq n \leq \sigma,$$

and

$$V(\sigma + 1, x(\sigma + 1)) > W_1(\gamma) + (h-1)d. \tag{1.10}$$

This would imply that

$$\begin{aligned} P(V(\sigma+1, x(\sigma+1))) &\geq V(\sigma+1, x(\sigma+1)) + d \\ &> W_1(\gamma) + hd \\ &\geq W_2(H) \geq V(s, x(s)), \qquad \text{for } \sigma - h \leq s \leq \sigma, \end{aligned}$$

and thus by assumption (ii) we have

$$V(\sigma+1, x(\sigma+1)) \leq V(\sigma, x(\sigma)) \leq W_1(\gamma) + (h-1)d.$$

This contradicts (1.10) and the proof of (1.9) is now complete.

Similarly, one may show that there exist positive integers $N_2, N_3, \ldots, N_h$, with $N_i \in [n_0 + (i-1)(r + [W_2(H)/W_3(b)]),\ n_0 + i(r + [W_2(H)/W_3(b)])]$ such that

$$V(n, x(n)) \leq W_1(\gamma) + (h-i)d, \qquad \text{for } n \geq n_i,\ 2 \leq i \leq h.$$

Thus

$$V(n, x(n)) \leq W_1(\gamma) \qquad \text{for } n \geq N_h.$$

This implies by assumption (i) that

$$|x(n)| < \gamma \qquad \text{for all } n \geq N_h.$$

Clearly, if we let $N = N(\gamma) = h(r + [W_2(H)/W_3(b)])$ then $\|\varphi\| < \eta$ implies $|x(n)| < \gamma$ for all $n \geq n_0 + N$. This proves uniform asymptotic stability.

From the above argument it is clear that if $\delta > 0$, $\varepsilon > 0$ are such that $W_2(\delta) < W_1(\varepsilon)$, then $\|\varphi\| < \delta$ implies $|x(n, n_0, \varphi)| < \varepsilon$ and $|x(n, n_0, \varphi)| \to 0$ as $n \to \infty$. Now if $H = \infty$ and $W_1(u) \to \infty$, then one can fix $\eta > 0$ arbitrarily large and determine H with $W_2(\eta) < W_1(H)$. This proves that the zero solution of (1.1) is a global attractor.

As an application of the above theorem let us consider the following difference system:

$$x(n+1) = A(n)x(n) + \sum_{j=1}^{m} B_j(n)x(n - \tau_j(n)) \tag{1.11}$$

where $n \in \boldsymbol{Z}^+$, $x(n) \in \boldsymbol{R}^k$, $A(n)$, $B(n)$ are $k \times k$ matrices and $\tau_j : \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ with $0 \leq \tau_j(n) \leq r$ for some positive integer r, $j = 1, 2, \ldots, m$. We assume that there are some positive numbers A and B such that

$$A = \sup_{n \in \boldsymbol{Z}^+} |A(n)|, \qquad B = \sup_{n \in Z^+} \Big(\sum_{j=1}^{m} |B_j(n)| \Big)$$

where $|\ |$ is any given norm on $\boldsymbol{R}^{k \times k}$.

We will now show that the zero solution of (1.11) is uniformly asymptotically stable if

$$A + B < 1. \tag{1.12}$$

Let $V(n, x) = |x|$. Then condition (i) of Theorem 6 is satisfied. From condition (1.12) one may find a number $\mu \in (0, 1)$ such that

$$B < \mu(1 - A). \tag{1.13}$$

Let $P(u) = \frac{1}{\mu} u$ for $u \geq 0$. Then $P(u) > u$ for $u > 0$. Moreover

$$\frac{1}{\mu} |x(n + 1)| > |x(s)|$$

for $n - r \leq s \leq n$ if and only if $P(V(n + 1), x(n + 1)) > V(s, x(s))$ for $n - r \leq s \leq n$.

This would imply by (1.11)

$$|x(n + 1)| \leq A\,|x(n)| + \left(\frac{B}{\mu}\right)|x(n + 1)|.$$

Thus

$$|x(n + 1)| \leq \frac{A}{1 - \left(\frac{B}{\mu}\right)} |x(n)|.$$

Hence

$$\begin{aligned} \Delta V_{(1.5)}(n, x(n)) &\leq - \left[1 - \frac{A}{1 - \left(\frac{B}{\mu}\right)} \right] |x(n)| \\ &= - \left[\frac{\mu(1 - A) - B}{\mu - B} \right] |x(n)| \\ &= - W_3(|x|), \text{ using (1.13).} \end{aligned}$$

By Theorem 6, this yields uniform asymptotic stability.

A special case of (1.11) is the system:

$$x(n + 1) = A(n)x(n) + \sum_{j=n-r}^{n} B(n, j)\, y(j)$$

which will be discussed in the next section. Here A is as above and

$$B = \sup_{n\in\mathbf{Z}^+} \Big(\sum_{j=n-r}^{n} B(n, j) \Big).$$

2. Periodic solutions of equations of non-convolution type

In this section we consider the difference system of finite delay:

$$y(n+1) = A(n)y(n) + \sum_{j=n-r}^{n} A(n, j)y(j) + h(n), \tag{2.0}$$

where $A(n)$, $A(n, j)$ are $k \times k$ matrices, and $y(n)$, $h(n) \in \boldsymbol{R}^k$, and $r \in \boldsymbol{Z}^+$ is the delay. If we let $B(n, n) = A(n) + A(n, n) - I$ in (2.1) we obtain the equation:

$$\Delta y(n) = \sum_{j=n-r}^{n} B(n, j)y(j) + h(n). \tag{2.1}$$

We also consider the associated homogeneous equation:

$$\Delta x(n) = \sum_{j=n-r}^{n} B(n, j)x(j). \tag{2.2}$$

We first develop a variation of constants formula for system (2.1).

Theorem 7. *The unique solution* $y(n, n_0, \varphi)$ *of* (2.1) *is given by*

$$y(n, n_0, \varphi) = x(n, n_0, \varphi) + \sum_{s=n_0}^{n-1} U(n, s+1)h(s), \qquad n \geq n_0, \tag{2.3}$$

where $x(n, n_0, \varphi)$ *is the corresponding solution of* (2.2), *and*

$$U(n, s) = \begin{cases} \displaystyle\sum_{u=s}^{n-1} \sum_{j=u-r}^{u} B(u, j)U(j, s) + I, & n \geq s \\ 0, & s - r \leq n < s \end{cases} \tag{2.4}$$

with I *being the* $k \times k$ *identity matrix.*

Proof. It is easy to see that by (2.4) the $k \times k$ matrix $U(n, s)$ is uniquely well defined. (Note that as a convention, $U(n, n) = I$.)

To verify that $y(n, n_0, \varphi)$ given by (2.3) satisfies (2.1), we write, for convenience, $y(n) = y(n, n_0, \varphi)$ and $x(n) = x(n, n_0, \varphi)$.

Then by (2.3) we have

$$\begin{aligned} \Delta y(n) &= \Delta x(n) + \sum_{s=n_0}^{n} U(n+1, s+1)h(s) - \sum_{s=n_0}^{n-1} U(n, s+1)h(s) \\ &= \sum_{j=n-r}^{n} B(n, j)x(j) + \sum_{s=n_0}^{n-1} \Delta U(n, s+1)h(s) + U(n+1, n+1)h(n) \end{aligned}$$

$$= \sum_{j=n-r}^{n} B(n, j)x(j) + \sum_{s=n_0}^{n-1} \sum_{j=n-r}^{n} B(n, j)U(j, s+1)h(s) + h(n)$$

$$= \sum_{j=n-r}^{n} B(n, j)(x(j) + \sum_{s=n_0}^{n-1} U(j, s+1)h(s)) + h(n)$$

$$= \sum_{j=n-r}^{n} B(n, j)(x(j) + \sum_{s=n_0}^{j-1} U(j, s+1)h(s)) + h(n)$$

$$= \sum_{j=n-r}^{n} B(n, j)y(j) + h(n), \qquad n \geq n_0,$$

as desired.

Moreover, for $-r \leq i \leq 0$ we have

$$y(n_0 + i, n_0, \varphi) = x(n_0 + i, n_0, n_0, \varphi) = \varphi(i).$$

This completes the proof.

As an immediate implication, we have the following:

Corollary 8. *If the zero solution of* (2.2) *is uniformly asymptotically stable, then solutions of* (2.1) *are all bounded in the positive direction for bounded* $h(n)$.

Proof. Obviously, by (2.4) we have

$$\Delta U(n, s) = \sum_{j=n-r}^{n} B(n, j)U(j, s).$$

This means that each column of $U(n, s)$ is a solution of (2.2). Hence $|U(n, s)| \leq Kv^{n-s}$, for some $K > 0$, $v \in (0, 1)$, and $n \geq s$. The conclusion of the Corollary now follows immediately from formula (2.3).

We now turn our atention to the question of the existence of periodic solutions of system (2.1). From now on it is assumed that

$$B(n + N, j + N) = B(n, j), \qquad h(n + N) = h(n) \tag{2.5}$$

for some positive integer N.

Before we establish the existence of N-periodic solutions, we introduce the following lemma.

Lemma 9. *Under the assumption* (2.5)

$$U(n + N, s + N) = U(n, s).$$

Proof. By (2.4) we have for $n \geq s$ that

$$U(n+N, s+N) = \sum_{u=s+N}^{n+N-1} \sum_{j=u-r}^{u} B(u, j)U(j, s+N) + I$$

$$= \sum_{v=s}^{n-1} \sum_{j=v+N-r}^{v+N} B(v+n, j)U(j, s+N) + I$$

$$= \sum_{v=s}^{n-1} \sum_{i=v-r}^{v} B(v+N, i+N)U(i+N, s+N) + I$$

$$= \sum_{u=s}^{n-1} \sum_{j=u-r}^{u} B(u, j)U(j+N, s+N) + I.$$

On the other hand,

$$U(n+N, s+N) = 0 \qquad \text{for } s - r \leq n < s.$$

Hence, by uniqueness we conclude that

$$U(n+N, s+N) = U(n, s).$$

Theorem 10. *Suppose that the zero solution of equation* (2.2) *is uniformly asymptotically stable. Then the equation* (2.1) *has the unique N-periodic solution*:

$$y^*(n) = \sum_{s=-\infty}^{n-1} U(n, s+1)h(s) \tag{2.6}$$

where $U(n, s)$ *is defined by* (2.4).

Proof. By Corollary 8, we may pick a solution of (2.1), say $\tilde{y}(n) = \tilde{y}(n, 0, \varphi)$, such that $|\tilde{y}(n)| \leq M$ for some constant $M > 0$ and for all $n \in \boldsymbol{Z}^+$. Now define a sequence of solutions of (2.1) on the set $\{-\ell N, -\ell N + 1, \ldots\}$ by

$$y_\ell(n) = \tilde{y}(n + \ell N) \qquad \text{for } n \geq -\ell N \text{ and } \ell = 1, 2, \ldots$$

Clearly, for each n with $-N \leq n \leq N$, $\{y_\ell(n)\}$ is a bounded sequence since $\tilde{y}(n)$ is bounded for all $n \in \boldsymbol{Z}^+$. Hence, we can extract a convergent subsequence $\{y_{\ell_{i1}}(n)\}$ which converges to, say $y^*(n) \in \boldsymbol{R}^k$, $-N \leq n \leq N$.

Next, for each n: $-2N \leq n \leq 2N$. $\{y_{\ell_{i1}}(n)\}$ is a bounded sequence so that there is a convergent subsequence $\{y_{\ell_{i2}}(n)\}$ which converges to

$$y^*(n) \in \boldsymbol{R}^k, \quad -2N \leq n \leq 2N.$$

By induction, we may show that for each $j \geq 1$, $\{y_{\ell_{ij}}(n)\}$ converges to $y^*(n) \in \boldsymbol{R}^k$ for $-jN \leq n \leq jN$, and $\{y_{\ell_{i(j+1)}}(n)\}$ converges to $y^*(n) \in \boldsymbol{R}^k$ for $-(j+1)N \leq n \leq (j+1)N$, where $\{\ell_{i(j+1)})\}$ is a subsequence of $\{\ell_{ij}\}$.

Now we claim that the above-defined sequence $\{y^*(n)\}_{-\infty}^{\infty}$ is a bounded solution of (2.1).

In fact, since $\tilde{y}(n) = \tilde{y}(n, 0, \varphi)$ is a solution of (2.1), we have

$$\Delta\tilde{y}(n + \ell_{ij}N) = \sum_{j=n+\ell_{ij}N-r}^{n+\ell_{ij}N} B(n + \ell_{ij}N, j)\tilde{y}(j) + h(n + \ell_{ij}N)$$

$$= \sum_{j=n-r}^{n} B(n, j)\tilde{y}(j + \ell_{ij}N) + h(n),$$

i.e.,

$$\Delta y_{\ell_{ij}+1}(n) = \sum_{j=n-r}^{n} B(n, j)y_{\ell_{ij}}(j) + h(n). \tag{2.7}$$

For any $L > 0$, on the set $\{-L, -L+1, -L+2, \ldots\}$ we may take the limit $\ell_{ij} \to \infty$ in (2.7) to obtain:

$$\Delta y^*(n) = \sum_{j=n-r}^{n} B(n, j)y^*(j) + h(n).$$

This equality holds for any L, and hence on $\boldsymbol{Z}$. Thus, $y^*(n)$ is a solution of (2.1) on $\boldsymbol{Z}$ and obviously:

$$|y^*(n)| \leq M \qquad \text{for all } n \in \boldsymbol{Z}.$$

We next claim that (2.1) has at most one (bounded) solution on $\boldsymbol{Z}$ and thus $y^*(n)$ is the unique N-periodic solution of (2.1) since $y^*(n + N)$ is obviously also a solution of (2.1) on $\boldsymbol{Z}$. Suppose that there are two distinct (bounded) solutions of (2.1) on $\boldsymbol{Z}$, say y and z, $y \not\equiv z$. Then there must be some $n_1 \in \boldsymbol{Z}$ and $d > 0$ such that

$$|y(n_1) - z(n_1)| = d.$$

Note that $y(n) - z(n)$ is a solution of (2.2), so that by the uniform asymptotic stability, for $\varepsilon = \dfrac{d}{2}$, there is an $N_0 \in \boldsymbol{Z}^+$ such that for any $N_0 \in \boldsymbol{Z}$ and $n \geq n_0 + N_0$ we have:

$$|y(n) - z(n)| < \varepsilon = \frac{d}{2}.$$

If we pick n_0 with $n_0 + N_0 < n_1$, we are led to a contradiction. Hence, (2.1) has at most one (bounded) solution on $\boldsymbol{Z}$. Finally, we claim that the unique N-periodic solution $y^*(n)$ of (2.1) is given by (2.6).

Indeed, by (2.3) we have:

$$\tilde{y}(n, 0, \varphi) = \tilde{x}(n, 0, \varphi) + \sum_{s=0}^{n-1} U(n, s+1)h(s), \qquad n \geq 0,$$

where $\tilde{x}(n, 0, \varphi)$ is the corresponding solution of (2.2) to $\tilde{y}(n, 0, \varphi)$. Thus

$$\tilde{y}(n + \ell_{ij}N, 0, \varphi) = \tilde{x}(n + \ell_{ij}N, 0, \varphi) + \sum_{s=0}^{n+\ell_{ij}N-1} U(n + \ell_{ij}N, s+1)h(s)$$

$$= \tilde{x}(n + \ell_{ij}N, 0, \varphi) + \sum_{s=-\ell_{ij}N}^{n-1} U(n, s+1)h(s),$$

i.e.,

$$y_{\ell_{ij}}(n) = \tilde{x}(n + \ell_{ij}N, 0, \varphi) + \sum_{s=-\ell_{ij}N}^{n-1} U(n, s+1)h(s).$$

As above, by taking the limit $\ell_{ij} \to \infty$ we arrive at

$$y^*(n) = \sum_{s=-\infty}^{n-1} U(n, s+1)h(s)$$

This completes the proof of the theorem.

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nuna adreso:
S. Elaydi
Department of Mathematics
Trinity University
San Antonio
Texas 78212–7200
USA
S. Zhang
Department of Mathematics
The University of Rhode Island
Kingston, Rhode Island 02881–0816
USA

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