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Kawanago, Tadashi

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Large Time Behavior of Solutions for Quasilinear Diffusion Equations with Strong Nonlinearity

By

Tadashi KAWANAGO

(Osaka University, Japan)

Dedicated to Professor Hiroki Tanabe on his 60th birthday

0. Introduction

We study the large time behavior of positive solutions of the following initial-boundary value problem:

$$(I) \quad \begin{cases} u_t = \Delta \phi(u) & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We assume that

(A1) $\phi \in C^1(\mathbf{R})$, $\phi'(r) > 0$ if $r \neq 0$ and $\phi'(0) \geq 0$,

(A2) $u_0 \in L^2(\Omega)$ and $u_0 \geq 0$ ($\neq 0$) in Ω .

If we set $k = \phi'$, then we can rewrite the equation as

$$u_t = \nabla \cdot (k(u) \nabla u).$$

When $k(0) = 0$ (i.e. the diffusion is degenerate at $u = 0$), the equation arises, for example, in gas flow through porous media. When $k(0) > 0$ (i.e. the diffusion is nondegenerate), the equation arises, for example, in heat flow through solids.

At first we state our main result on the decay of solutions. Roughly speaking, we obtain the following: If $k(r)$ is monotone for $r \geq 0$ and ϕ satisfies (A1–2) and some additional condition, then the L^2 -norm of positive solution of (I) decays at the same rate of the positive solution $y(t)$ of the ordinary equation: $y' = -\lambda_1 \phi(y)$ (see Theorem 1), where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ with Dirichlet condition. This result holds whether the equation is degenerate or nondegenerate and we can apply it to the following strongly nonlinear cases:

$$(0.1a) \quad k(r) = \frac{\theta}{(-\log r)^\rho} \quad \text{for small } r > 0,$$

$$(0.1b) \quad k(r) = 1 + \frac{\theta}{(-\log r)^\rho} \quad \text{for small } r > 0$$

and $k(r) = \theta/[\log(-\log r)]^\rho$, where $\theta, \rho \in (0, \infty)$, and $\theta \in (-\infty, \infty)$ for (0.1b). Related results were obtained by Bertsch and Peletier [2], the author [5] and so on (see the references in [2] and [5]). Also see [1] and [6] for the behavior of solutions which take values of arbitrary sign. But strongly nonlinear cases including (0.1) have not been studied in detail.

We briefly recall the related results in [2] and [5]. First let's recall the case when the diffusion is degenerate at $u = 0$. Under the condition:

$$(0.2) \quad 0 < \alpha < \phi(r)\phi''(r)/[\phi'(r)]^2 \leq 1 \quad \text{for } 0 < r < r_1$$

for some constants $\alpha \in (0, 1)$ and $r_1 > 0$

and some additional condition, Bertsch and Peletier [2] obtained the following result:

$$\frac{\phi(u(t))}{\phi(x(t))} \longrightarrow f_\mu \quad \text{in } L^\infty(\Omega) \text{ as } t \longrightarrow \infty$$

where $\mu = \mu(\phi) \in [0, 1)$ and the asymptotic profile f_μ is the unique positive solution of the elliptic problem: $-\Delta f = f^\mu$ in Ω and $f = 0$ on $\partial\Omega$, and $x(t)$ is the positive solution of $x'(t) = -\phi(x)$. The condition (0.2) is satisfied by the $\phi(r) = r^m (m > 1)$, $\phi(r) = r^m \log(1/r)$ for small r and $\phi(r) = e^{-1/r}$, but not by the case (0.1a). Indeed the condition (0.2) implies that $k(r) \leq Ar^{\alpha/(1-\alpha)}$ for $0 < r \leq r_1$, $A > 0$. (For the case (0.1a) a rough estimate for the decay of solutions was obtained in [5].) We shall show that for the case (0.1a) the solution $u(t)$ decays at the rate of $\exp\{-(\rho+1)\theta\lambda_1 t^{1/(\rho+1)}\}$ and the asymptotic profile of $u(t)$ is an eigenfunction of $-\Delta$ with Dirichlet condition (see Proposition 1). We remark that the solutions of equations considered in [2] decay at most algebraically and the decay order does not depend on Ω .

Next, let us recall the case when the diffusion is nondegenerate. In [5] we showed that the condition

$$(0.3) \quad |k(r) - 1| \leq \theta/(-\log r)^\rho \quad \text{for small } r > 0$$

for some $\theta \in (0, \infty)$ and $\rho \in (1, \infty)$

is a sufficient and an almost necessary condition for the following (0.4).

$$(0.4) \quad e^{\lambda_1 t} u(t) \longrightarrow \omega_1 \quad \text{in } H_0^1(\Omega) \text{ as } t \longrightarrow \infty,$$

where $\omega_1 (> 0 \text{ in } \Omega)$ is an eigenfunction of $-\Delta$ corresponding to λ_1 . Even when ϕ does not satisfy (0.3), our Theorem 1 gives the exact decay rate of solution under the assumption of monotonicity of k . (In particular, for the case (0.1b) see Example 2 in section 1.) Furthermore, under the same situation

we shall show that the asymptotic profile for the solution is an eigenfunction of $-\Delta$ with Dirichlet condition (see Theorem 2).

In the proofs of our results we often compute formally. But it is justified by the nonlinear semigroup theory through appropriate approximate classical solutions (see [5]).

Notation.

1. $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.
2. $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.
3. $\{\lambda_v\}_{v=1}^\infty$ ($0 < \lambda_1 < \lambda_2 < \dots$) are all distinct eigenvalues of $-\Delta$ with zero-Dirichlet condition.
4. e_1 is the eigenvector of $-\Delta$ corresponding to λ_1 which satisfies that $e_1 > 0$ in Ω and $\|e_1\|_2 = 1$.
5. $f(t) = o(g(t))$ means that $\limsup_{t \rightarrow \infty} |f(t)/g(t)| = 0$, $f(t) = O(g(t))$ that $\limsup_{t \rightarrow \infty} |f(t)/g(t)| < \infty$, and $f(t) \sim g(t)$ that $a < \liminf_{t \rightarrow \infty} |f(t)/g(t)| \leq \limsup_{t \rightarrow \infty} |f(t)/g(t)| < b$ for some $a, b > 0$.
6. $|A|$ is the measure of A for Lebesgue measurable set $A \subset \mathbb{R}^N$.
7. $C_i (i \in \mathbb{N})$ denote some positive constants.

1. Results

Before we state our results, we shall recall a definition of weak solutions of (I) by the nonlinear semigroup theory. We define operator $A: L^1(\Omega) \rightarrow L^1(\Omega)$ by

$$Au = -\Delta\phi(u) \quad \text{for } u \in D(A)$$

with $D(A) = \{u \in L^1(\Omega); \phi(u) \in W_0^{1,1}(\Omega), \Delta\phi(u) \in L^1(\Omega)\}$. The operator A is m -accretive in $L^1(\Omega)$ under the condition (A1). Therefore A generates the contraction semigroup $S_A(t)$. Hence we can define a unique weak solution of (I) by $S_A(t)u_0$ for any $u_0 \in \overline{D(A)} = L^1(\Omega)$. For the details, see [1], [5] and the references therein.

Theorem 1. *Let (A1–2) be satisfied. We assume that $k(r)$ is non-decreasing (resp. non-increasing) on $(0, r_1)$ for some $r_1 > 0$ and that*

$$(1.1) \quad \frac{1}{r} \int_0^r \sqrt{k(s)} ds \geq \sqrt{k(\alpha r)} \left(\text{resp. } \frac{\phi(r)}{r} \leq k(\alpha r) \right) \quad \text{for } 0 < r \leq r_1$$

for some $\alpha \in (0, 1)$. Also assume that $u_0 \geq ae_1$ (resp. $u_0 \leq ae_1$) in Ω for some $a > 0$. Then, the weak solution u of (I) satisfies

$$(1.2) \quad \|u(t)\|_2 \sim (u(t), e_1) \sim y(t),$$

where $y(t)$ is any fixed positive solution of the ordinary equation: $y'(t) = -\lambda_1 \phi(y)$.

Remark. If we also assume that $\phi \in C^3((0, \infty))$, then in Theorem 1 we can drop the assumption on the initial value: $u_0 \geq ae_1$ when $k(r)$ is non-decreasing on $(0, r_1)$. We shall mention the reason. Indeed, $u \in L^\infty(\Omega \times [t_0, \infty))$ for any $t_0 > 0$ ([5, Proposition 4.1]). And $u(t) \not\equiv 0$ in Ω for $t \in [0, \infty)$ since $(u(t), e_1) \geq Ce^{-\eta t}$ for some $C, \eta > 0$ by the same argument in the proof of [5, Theorem 2.3]. Moreover, the weak solution satisfies (I) also in the sense of distribution ([5, section 1]) and can be approximated by a sequence of positive classical solutions of (I) $\{u_n\}_{n=1}^\infty$ ([5, Proposition 1.3] with $\phi_n = \phi$ and $\varepsilon_n > 0$). Therefore, $u \in C(\Omega \times (t_0, \infty))$ by [4, Theorem 1] when $k(0) = 0$ and by the proof of [5, Proposition 1.1] and [7, Theorem 16] when $k(0) > 0$. Hence the set $\{x \in \Omega; u(x, t) \geq v\}$ contains an open nonempty subset of Ω for $t > t_0$ and some $v > 0$. Therefore, in view of [3, Theorem 5], $u(x, t_1) \geq Ce_1$ for some $t_1 > t_0$ and $C > 0$. Since we consider only the large time behavior of solutions, we may assume without loss of generality that $t_1 = 0$.

We state the applications of Theorem 1 to the strongly nonlinear cases.

Example 1. Let $k(r) = \theta/(-\log r)^\rho$ for $0 < r \leq r_1$, where $\theta, \rho \in (0, \infty)$ and $r_1 \in (0, 1)$ are constants. This example satisfies (1.1) for any fixed constant $\alpha \in (0, 1)$. For the initial value u_0 we assume (A2). Then the weak solution u of (I) satisfies that

$$(1.3) \quad \|u(t)\|_2 \sim \exp\{-(\rho + 1)\theta\lambda_1 t^{1/(\rho+1)}\}.$$

We shall prove (1.3) in the next section.

For the above example, we can see the asymptotic profile of solutions:

Proposition 1. *Let $k(r)$ be defined as in Example 1. And let u_0 satisfy (A2). Then the weak solution u of (I) satisfies that*

$$(1.4) \quad \frac{u(t)}{(u(t), e_1)} \longrightarrow e_1 \quad \text{in } L^2(\Omega) \quad \text{as } t \longrightarrow \infty.$$

Example 2. Let $k(r) = 1 + \theta/(-\log r)^\rho$ for $0 < r \leq r_1$, where $\theta \in (-\infty, \infty)$, $\rho \in (0, \infty)$ and $r_1 \in (0, 1)$ are constants. This example satisfies (1.1) for any fixed constant $\alpha \in (0, 1)$. For the initial value u_0 we assume (A2). When $\theta < 0$, we also assume that $u_0 \leq ae_1$ in Ω for some $a > 0$. Then the weak solution u of (I) satisfies that

$$\log \|u(t)\|_2 = -\lambda_1 t - \frac{\theta(\lambda_1 t)^{1-\rho}}{1-\rho} + o(t^{1-\rho}) \quad \text{if } \rho \in (0, 1),$$

$$\begin{aligned} \|u(t)\|_2 &\sim (t+1)^{-\theta} e^{-\lambda_1 t} && \text{if } \rho = 1, \\ \|u(t)\|_2 &\sim e^{-\lambda_1 t} && \text{if } \rho \in (1, \infty). \end{aligned}$$

We shall derive the above estimates in the next section.

For the nondegenerate case we can obtain the following result.

Theorem 2. *Let all the assumptions in Theorem 1 be satisfied. Assume also that $k(0) = 1$. And if $k(r)$ is non-increasing for $r \geq 0$, we assume that $\lim_{r \rightarrow \infty} k(r) > 0$. Then the weak solution u of (I) satisfies that*

$$(1.5) \quad \frac{u(t)}{(u(t), e_1)} \longrightarrow e_1 \quad \text{in } L^2(\Omega) \quad \text{as } t \longrightarrow \infty.$$

Remark. If we also assume that $\phi \in C^3((0, \infty))$, then in Theorem 2 we can drop the assumption on the initial value: $u_0 \geq a e_1$ when $k(r)$ is non-decreasing on $(0, r_1)$.

2. Proofs

The following proposition is very fundamental in this paper.

Proposition 2. *Let ϕ satisfy (A1). We denote by $y(t; a)$ the solution of $y' = -\lambda_1 \phi(y)$ with $y(0) = a$. Then we have*

$$y(t; a) \sim y(t; b) \quad \text{for any } a, b \in (0, \infty).$$

Proof of Proposition 2. We may assume that $a > b$. Let $t_0 > 0$ be the time such that $y(t_0; a) = b$. Set $K = \max_{r \in [0, a]} k(r) > 0$. We can easily verify that $e^{K\lambda_1 t} y(t; a)$ is increasing for $t \geq 0$. Therefore we obtain that

$$y(t; b) = y(t + t_0; a) \geq e^{-K\lambda_1 t_0} y(t; a) \quad \text{for } t \geq 0.$$

This completes the proof. $\square$

Proof of Theorem 1. First we shall consider the case when $k(r)$ is non-decreasing on $(0, r_1)$. We denote by $y(t; y_0)$ the solution of the ordinary differential equation: $y'(t) = -\lambda_1 \phi(y)$ with the initial value $y(0) = y_0 > 0$. We divide the proof into two steps.

Step 1. (Upper estimate) With the aid of integration by parts and Poincaré's inequality,

$$\frac{d}{dt} \int_{\Omega} u^2 = -2 \int k(u) |\nabla u|^2 = -2 \int |\nabla h(u)|^2 \leq -2\lambda_1 \int h(u)^2 = -2\lambda_1 \int g(u^2),$$

where we set $h(r) = \int_0^r \sqrt{k(s)} ds$ and $g(r) = h(\sqrt{r})^2$.

$$g'(r) = \sqrt{k(\sqrt{r})} \cdot \frac{h(\sqrt{r})}{\sqrt{r}} = \sqrt{k(\sqrt{r})} \cdot \frac{1}{\sqrt{r}} \int_0^{\sqrt{r}} \sqrt{k(s)} ds.$$

Hence $g(r)$ is convex for $r \geq 0$ since $g'(r)$ is nondecreasing. By Jensen's inequality,

$$\frac{d}{dt} \int u^2 \leq -2\lambda_1 |\Omega| g\left(\frac{1}{|\Omega|} \int u^2\right) = -2\lambda_1 |\Omega| h(|\Omega|^{-1/2} \|u\|_2)^2.$$

If we set $z = \alpha |\Omega|^{-1/2} \|u\|_2$, then we can obtain that

$$(2.1) \quad dz/dt \leq -\alpha^2 \lambda_1 h(z/\alpha)^2/z.$$

With the aid of (1.1),

$$(2.2) \quad h(r) \geq r \sqrt{k(\alpha r)}.$$

By (2.1) and (2.2),

$$dz/dt \leq -\lambda_1 z k(z) \leq -\lambda_1 \int_0^z k(s) ds = -\lambda_1 \phi(z).$$

Hence, we have

$$z(t) \leq y(t; z(0)).$$

Step 2. (Lower estimate) The proof is complete if we prove that

$$(2.3) \quad (u(t), e_1) \geq Cy(t) \quad \text{for } t \geq 0$$

for some $C \in (0, \infty)$. We set $\omega_1 = ce_1$. We choose $c > 0$ so small that $0 < \|\omega_1\|_\infty \leq \alpha (< 1)$. We shall show that $z(x, t) = \omega_1(x)y(t; b)$ ($b > 0$) is a subsolution of (I). By the convexity of ϕ ,

$$(2.4) \quad \begin{aligned} z_t - \Delta \phi(z) &= -\lambda_1 \omega_1 \phi(y) - k'(\omega_1 y) y^2 |\nabla \omega_1|^2 + \lambda_1 \omega_1 y k(\omega_1 y) \\ &\leq -\lambda_1 \omega_1 y \left\{ \frac{\phi(y)}{y} - k(\omega_1 y) \right\}. \end{aligned}$$

By (1.1) and Schwarz's inequality,

$$(2.5) \quad \frac{\phi(r)}{r} \geq \left(\frac{1}{r} \int_0^r \sqrt{k(s)} ds \right)^2 \geq k(\alpha r).$$

It follows from (2.4) and (2.5) that

$$(2.6) \quad z_t - \Delta \phi(z) \leq -\lambda_1 \omega_1 y \{k(\alpha y) - k(\omega_1 y)\} \leq 0.$$

Hence $z(x, t)$ is a subsolution of (I). Moreover, we can choose $b \in (0, \infty)$ so small that

$$(2.7) \quad z(x, 0) = b\omega_1(x) \leq u_0(x) \quad \text{in } \Omega.$$

It follows from (2.6), (2.7) and the comparison principle that

$$u(x, t) \geq z(x, t) = \omega_1(x)y(t).$$

This leads us to (2.3). Thus, the proof is complete for the case when $k(r)$ is non-decreasing for $r > 0$.

Next we consider the case when $k(r)$ is non-increasing on $(0, r_1)$. By the same argument as in the above Step 2 we can construct a separable supersolution of (I). Hence we obtain that

$$\|u(t)\|_\infty \leq Cy(t) \quad \text{for } t \geq 0$$

for some $C \in (0, \infty)$. This is a desired upper estimate. We shall obtain a lower estimate. It follows from the concavity of ϕ and Jensen's inequality that

$$\frac{d}{dt}(u(t), \omega_1) = -\lambda_1 \int \phi(u)\omega_1 \geq -\lambda_1 \int \phi(u\omega_1) \geq -\lambda_1 |\Omega| \phi\left(\frac{1}{|\Omega|}(u, \omega_1)\right).$$

Therefore we have

$$\frac{1}{|\Omega|}(u, \omega_1) \geq y\left(t; \frac{1}{|\Omega|}(u_0, \omega_1)\right).$$

The proof is complete. $\square$

Proof of Example 1. We will give the outline of the proof of (1.3). We set $z(t) = \exp\{-(\rho+1)\theta\lambda_1 t^{1/(1+\rho)}\}$, which is the explicit solution of $z'(t) = -\lambda_1 z k(z)$ with $z(0) = 1$. It suffices to show that $y(t) \sim z(t)$. In view of Proposition 2, we may assume that $0 < y(0) < 1$. $z(t)$ satisfies that

$$z'(t) \leq -\lambda_1 \phi(z).$$

Therefore we have $y(t) \geq z(t + t_0)$ for $t \geq 0$, where $t_0 > 0$ is a time such that $z(t_0) \leq y(0)$.

On the other hand,

$$\phi(r) = rk(r) - \rho\theta \int_0^r \frac{ds}{(-\log s)^{\rho+1}} \geq rk(r) - \frac{\rho\theta r}{(-\log r)^{\rho+1}} = rk(r) \left(1 - \frac{\rho}{-\log r}\right).$$

Hence, we have

$$(2.8) \quad y'(t) \leq -\lambda_1 yk(y) \left(1 - \frac{\rho}{-\log y}\right) \leq -\frac{1}{2}\lambda_1 yk(y) \quad \text{for } t \geq t_1,$$

where $t_1 > 0$ is a time such that $y(t_1) \leq e^{-2\rho}$. It follows that

$$(2.9) \quad y(t) \leq z(t/2) \quad \text{for } t \geq 0.$$

By (2.8) and (2.9),

$$y'(t) \leq -\lambda_1 yk(y) \{1 - C_1(t+1)^{-1/(1+\rho)}\} \quad \text{for } t \geq 0.$$

Hence we obtain that for $t \geq 0$,

$$y(t) \leq C_2 z \left(t - C_1 \int_0^t (s+1)^{-1/(1+\rho)} ds \right) \leq C_3 z(t). \quad \square$$

To prove Proposition 1 we prepare a lemma.

Lemma 1. *Let $k(r) = \theta/(-\log r)^\rho$ for $0 < r \leq r_1$, where $\theta, \rho \in (0, \infty)$ and $r_1 \in (0, 1)$ are constants. Then the weak solution u of (I) satisfies that*

$$(2.10) \quad \|u(t)\|_\infty \leq C(t+1)^\sigma \exp \{ -((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)} \} \quad \text{for } t > 0,$$

where $\sigma = N\rho/4(\rho+1)$ and $C = C(u_0) > 0$.

Proof of Lemma 1. By [5, Proposition 4.1],

$$\|u(t)\|_\infty \leq \varepsilon + \frac{C_4}{(k(\varepsilon)t)^{N/4}} \|u_0\|_2 \quad \text{for any } \varepsilon > 0 \text{ and } t > 0,$$

where $C_4 = C_4(N) > 0$. Therefore, we have

$$\|u(t)\|_\infty \leq \varepsilon + \frac{C_4}{(k(\varepsilon)t_0)^{N/4}} \|u(t-t_0)\|_2 \quad \text{for any } \varepsilon > 0 \text{ and } t \geq t_0,$$

where $t_0 \in (0, \infty)$. We substitute $\varepsilon = \|u(t-t_0)\|_2/t_0^{N/4}$ to obtain that

$$(2.11) \quad \|u(t)\|_\infty \leq \frac{1}{t_0^{N/4}} \left\{ 1 + \frac{C_4}{\theta} \left(\frac{N}{4} \log t_0 - \log \|u(t-t_0)\|_2 \right)^{N\rho/4} \right\} \|u(t-t_0)\|_2$$

for $t > t_0$. Here $t_0 > 0$ is a constant satisfying $\|u_0\|_2/t_0^{N/4} \leq r_1$. (We remark that $\|u(t)\|_2$ is non-increasing for $t \geq 0$. See the proof of Theorem 1.) We obtain (2.10) from (2.11) and (1.3) in Example 1. $\square$

Proof of Proposition 1. We set $v = u(t) - (u, e_1)e_1$ and $y(t) = \exp \{ -((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)} \}$. It is sufficient to derive that $\|v(t)\|_2 = o(y(t))$. In view of Lemma 1, we see that $\|u(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. Therefore we may assume that $\sup_{t \geq 0} \|u(t)\|_\infty$ is sufficiently small. Let $h(r)$ and $g(r)$ be the functions as defined in the proof of Theorem 1. Then, by the eigenfunction expansion of $-A$ and Jensen's inequality,

$$\begin{aligned}
 \frac{d}{dt} \int v^2 &= -2 \int |\nabla h(u)|^2 + 2\lambda_1(u, e_1)(\phi(u), e_1) \\
 &= -2 \sum_{j=1}^{\infty} \lambda_j(h(u), e_j)^2 + 2\lambda_1(u, e_1)(\phi(u), e_1) \\
 &\leq 2(\lambda_2 - \lambda_1)(h(u), e_1)^2 - 2\lambda_2 \int h(u)^2 + 2\lambda_1(u, e_1)(\phi(u), e_1) \\
 &\leq -2\lambda_2 |\Omega| g\left(\frac{1}{|\Omega|} \int u^2\right) + 2(\lambda_2 - \lambda_1)(h(u), e_1)^2 + 2\lambda_1(u, e_1)(\phi(u), e_1).
 \end{aligned}$$

By Schwarz' inequality,

$$(2.12) \quad (h(u), e_1)^2 \leq (\sqrt{u\phi(u)}, e_1)^2 = (\sqrt{ue_1}, \sqrt{\phi(u)e_1})^2 \leq (u, e_1)(\phi(u), e_1).$$

From the integration by parts,

$$\begin{aligned}
 h(r) &= \sqrt{\theta} \left\{ \frac{r}{(-\log r)^{\rho/2}} - \frac{\rho}{2} \int_0^r \frac{ds}{(-\log s)^{\rho/2+1}} \right\} \\
 &\geq \sqrt{\theta} \left\{ \frac{r}{(-\log r)^{\rho/2}} - \frac{\rho r}{2(-\log r)^{\rho/2+1}} \right\}.
 \end{aligned}$$

Therefore we obtain that

$$g(r) = h(\sqrt{r})^2 \geq 2^\rho r k(r) \left\{ 1 - \frac{2\rho}{-\log r} \right\} \quad \text{for } r \in (0, r_1).$$

It follows that

$$\begin{aligned}
 (2.13) \quad \frac{d}{dt} \int v^2 &\leq -\frac{2\theta\lambda_2}{(2^{-1} \log |\Omega| - \log \|u(t)\|_2)^\rho} \left\{ \int v^2 + (u, e_1)^2 \right\} \\
 &\quad + 2\lambda_2(u, e_1)(\phi(u), e_1) + \frac{2\rho\theta\lambda_2}{(2^{-1} \log |\Omega| - \log \|u(t)\|_2)^{\rho+1}} \int u^2.
 \end{aligned}$$

We shall estimate

$$(2.14) \quad I = -\frac{2\theta\lambda_2}{(2^{-1} \log |\Omega| - \log \|u\|_2)^\rho} (u, e_1)^2 + 2\lambda_2(u, e_1)(\phi(u), e_1).$$

Since $k(r)$ is nondecreasing, we have

$$\phi(r) = \int_0^r k(s)ds \leq rk(r) \quad \text{for } r \geq 0.$$

Therefore, we obtain that

$$(2.15) \quad I \leq 2\theta\lambda_2(u, e_1)^2 \left\{ \frac{1}{(-\log \|u\|_\infty)^\rho} - \frac{1}{(2^{-1} \log |\Omega| - \log \|u(t)\|_2)^\rho} \right\}.$$

It follows from (2.15), (1.3), (2.10) and the mean value theorem that

$$(2.16) \quad I \leq \frac{C_5 \log(t+2)}{t+1} y(t)^2.$$

In view of (2.13), (2.14) and (2.16), we have

$$\frac{d}{dt} \int v^2 dx \leq -A(t) \int v^2 dx + C_6 B(t),$$

where

$$A(t) = \frac{2\theta\lambda_2}{((\rho+1)\theta\lambda_1 t)^{\rho/(\rho+1)} + C_7},$$

$$B(t) = \frac{\log(t+2)}{t+1} \exp\{-2((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)}\}.$$

It follows that

$$(2.17) \quad \int v(t)^2 dx \leq \exp\left\{-\int_0^t A(s) ds\right\} \left[\int v(0)^2 dx + C_6 \int_0^t \exp\left\{\int_0^s A(\tau) d\tau\right\} B(s) ds \right].$$

We shall estimate the right-hand side.

$$\begin{aligned} & \int_0^t A(s) ds - 2\theta\lambda_2 \int_1^t ((\rho+1)\theta\lambda_1 s)^{-\rho/(\rho+1)} ds \\ &= \int_0^1 A(s) ds - \int_1^t \frac{2C_7\theta\lambda_2 ds}{((\rho+1)\theta\lambda_1 s)^{\rho/(\rho+1)} \{((\rho+1)\theta\lambda_1 s)^{\rho/(\rho+1)} + C_7\}}, \end{aligned}$$

which implies that

$$(2.18) \quad \int_0^t A(s) ds = \frac{2\lambda_2}{\lambda_1} ((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)} + o(t^{1/(\rho+1)}).$$

Since $2/(1+\lambda_1/\lambda_2) > 1$, we obtain that for sufficiently large $t > 0$,

$$(2.19) \quad \begin{aligned} & \frac{d}{dt} \left\{ \frac{\log(t+2)}{(t+1)^{1/(\rho+1)}} \exp(-at^{1/(\rho+1)}) \right\} \\ & \geq \frac{\log(t+2)}{(t+1)^{1/(\rho+1)}} \frac{d}{dt} \{\exp(-at^{1/(\rho+1)})\} \end{aligned}$$

$$\geq -\frac{a}{\rho+1} \cdot \frac{2}{1+\lambda_1/\lambda_2} \cdot \frac{\log(t+2)}{t+1} \exp(-at^{1/(\rho+1)}),$$

where $a > 0$ is any fixed constant. By (2.19) and the integration by parts,

$$\begin{aligned} & \int_0^t \exp \left\{ \int_0^s A(\tau) d\tau \right\} B(s) ds \\ & \geq -C_8 - \frac{(1+\lambda_1/\lambda_2)(\rho+1)^{\rho/(\rho+1)}}{2(\theta\lambda_1)^{1/(\rho+1)}} \int_0^t \exp \left\{ \int_0^s A(\tau) d\tau \right\} \frac{d}{ds} \left\{ \frac{\log(s+2)}{(s+1)^{1/(\rho+1)}} y(s)^2 \right\} ds \\ & = -C_8 + C_9 \left[-\log 2 + \exp \left\{ \int_0^t A(\tau) d\tau \right\} \cdot \frac{\log(t+2)}{(t+1)^{1/(\rho+1)}} y(t)^2 \right] \\ & \quad + \frac{1}{2} \left(1 + \frac{\lambda_2}{\lambda_1} \right) \int_0^t \frac{((\rho+1)\theta\lambda_1(s+1))^{\rho/(\rho+1)}}{((\rho+1)\theta\lambda_1 s)^{\rho/(\rho+1)} + C_2} \exp \left\{ \int_0^s A(\tau) d\tau \right\} B(s) ds. \end{aligned}$$

It follows that for $t \geq 0$

$$(2.20) \quad \int_0^t \exp \left\{ \int_0^s A(\tau) d\tau \right\} B(s) ds \leq C_{10} \left[1 + \exp \left\{ \int_0^t A(s) ds \right\} \cdot \frac{\log(t+2)}{(t+1)^{1/(\rho+1)}} y(t)^2 \right].$$

We obtain from (2.17), (2.18) and (2.20) that

$$\|v(t)\|_2 \leq \frac{C_{11} \sqrt{\log(t+2)}}{(t+1)^{1/(2\rho+2)}} y(t) \quad \text{for } t \geq 0.$$

Thus we have $\|v(t)\|_2 = o(y(t))$. $\square$

Proof of Example 2. We shall derive the decay estimates. We shall state only the case: $\rho \in (0, 1)$ and $\theta > 0$ because the argument for the other cases is similar. In view of Proposition 2, we may assume that $0 < y(0) < 1$.

$$(2.21) \quad y'(t) = -\lambda_1 y - \lambda_1 \theta \int_0^y \frac{dr}{(-\log r)^\rho} \leq -\lambda_1 y.$$

It follows that

$$(2.22) \quad y(t) \leq y(0)e^{-\lambda_1 t}.$$

By (2.21) and (2.22),

$$y' \geq -\lambda_1 y - \frac{\lambda_1 \theta y}{(-\log y)^\rho} \geq -\lambda_1 y \left\{ 1 + \frac{\theta}{(-\lambda_1 t - \log y(0))^\rho} \right\},$$

which implies that

$$(2.23) \quad y(t) \geq y(0) \exp \left\{ -\lambda_1 t - \frac{\theta(\lambda_1 t)^{1-\rho}}{1-\rho} \right\}.$$

On the other hand, by (2.22) and (2.23)

$$\begin{aligned} y' &= -\lambda_1 y - \frac{\lambda_1 \theta y}{(-\log y)^\rho} + \int_0^y \frac{\lambda_1 \theta \rho dr}{(-\log r)^{\rho+1}} \\ &\leq -\lambda_1 y \left\{ 1 + \frac{\theta}{(-\log y)^\rho} - \frac{\theta \rho}{(-\log y)^{\rho+1}} \right\} \\ &\leq -\lambda_1 y \left\{ 1 + \frac{\theta}{\{-\log y(0) + \lambda_1 t + \theta(\lambda_1 t)^{1-\rho}/(1-\rho)\}^\rho} - \frac{\theta \rho}{(-\log y(0) + \lambda_1 t)^{\rho+1}} \right\}. \end{aligned}$$

Therefore, we have

$$\log y(t) \leq -\lambda_1 t - \frac{\theta(\lambda_1 t)^{1-\rho}}{1-\rho} + o(t^{1-\rho}). \quad \square$$

Finally we shall prove Theorem 2.

Proof of Theorem 2. In view of Theorem 1, it suffices to prove that

$$(2.24) \quad \|u(t) - (u, e_1)e_1\|_2 = o(y(t)).$$

We set $v = u(t) - (u(t), e_1)e_1$. We remark that $(v, e_1) = (\nabla v, \nabla e_1) = 0$. By the integration by parts,

$$\begin{aligned} (2.25) \quad \frac{d}{dt} \int_{\Omega} v^2 &= 2 \int \nabla v \cdot k(u) \nabla u \\ &= -2 \int k(u) |\nabla v|^2 - 2(u, e_1) \int \{k(u) - 1\} \nabla v \cdot \nabla e_1. \end{aligned}$$

Since $\|u(t)\|_{\infty} \rightarrow 0$ as $t \rightarrow \infty$ (see [5, Theorem 2.2]), we may assume without loss of generality that $\sup_{t \geq 0} \|u(t)\|_{\infty}$ is sufficiently small. Thus, we may assume that $k(u(x, t)) \geq 1 - \varepsilon$ for any $(x, t) \in \Omega \times [0, \infty)$, where we set $\varepsilon \in (0, 1)$ so small that

$$(2.26) \quad \lambda_1 + \varepsilon - (1 - 2\varepsilon)\lambda_2 < 0.$$

It follows that

$$\begin{aligned} \frac{d}{dt} \int v^2 &\leq -2(1 - \varepsilon) \|\nabla v\|_2^2 + 2(u, e_1) \|k(u) - 1\|_{\infty} \|\nabla v\|_2 \|\nabla e_1\|_2 \\ &\leq -2(1 - 2\varepsilon) \|\nabla v\|_2^2 + C_{12} \|k(u) - 1\|_{\infty}^2 (u, e_1)^2 \\ &\leq -2(1 - 2\varepsilon)\lambda_2 \|v\|_2^2 + C_{12} \|k(u) - 1\|_{\infty}^2 (u, e_1)^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|v(t)\|_2^2 &\leq e^{-2(1-2\varepsilon)\lambda_2 t} \|v(0)\|_2^2 \\ &\quad + C_{12} e^{-2(1-2\varepsilon)\lambda_2 t} \int_0^t e^{2(1-2\varepsilon)\lambda_2 s} \|k(u(s)) - 1\|_\infty^2 (u(s), e_1)^2 ds. \end{aligned}$$

On the other hand, we can easily verify that $y(t) \geq C_{13} e^{-(\lambda_1 + \varepsilon)t}$ for $t \geq 0$. It follows from (2.26) that

$$(2.27) \quad \lim_{t \rightarrow \infty} e^{2(1-2\varepsilon)\lambda_2 t} y(t)^2 = \infty.$$

If $\int_0^\infty e^{2(1-2\varepsilon)\lambda_2 s} \|k(u(s)) - 1\|_\infty^2 y(s)^2 ds < \infty$, then the claim (2.24) obviously holds. If otherwise, with the aid of l'Hospital's theorem, we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\|v(t)\|_2^2}{y(t)^2} &\leq C_{14} \lim_{t \rightarrow \infty} \frac{\int_0^t e^{2(1-2\varepsilon)\lambda_2 s} \|k(u(s)) - 1\|_\infty^2 y(s)^2 ds}{e^{2(1-2\varepsilon)\lambda_2 t} y(t)^2} \\ &= C_{14} \lim_{t \rightarrow \infty} \frac{e^{2(1-2\varepsilon)\lambda_2 t} \|k(u(t)) - 1\|_\infty^2 y(t)^2}{\{e^{2(1-2\varepsilon)\lambda_2 t} y(t)^2\}'} \\ &= C_{14} \lim_{t \rightarrow \infty} \frac{\|k(u(t)) - 1\|_\infty^2}{2(1-2\varepsilon)\lambda_2 - 2\lambda_1 \phi(y)/y} \\ &= 0. \end{aligned}$$

Hence, we have (2.24). $\square$

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nuna adreso:

Tadashi Kawanago

Department of Mathematics

Faculty of Science

Osaka University

Toyonaka 560, Japan

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