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On the Equivalence of Two Notions of Weak Solutions, Viscosity Solutions and Distribution Solutions

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Introduction

We shall be mainly concerned with the linear, degenerate elliptic, partial differential equation

$$(1.1) \quad \mathcal{L}u = f \quad \text{in } \Omega,$$

where Ω is an open subset of $\mathbf{R}^N$ and $\mathcal{L}$ is the operator defined by

$$\mathcal{L}u(x) = - \sum_{i,j=1}^N a_{ij}(x)u_{x_i x_j}(x) + \sum_{i=1}^N b_i(x)u_{x_i}(x) + c(x)u(x).$$

Throughout this paper we assume that the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ and $f(x)$ are real and that the matrices $a(x) \equiv (a_{ij}(x))$ are symmetric and nonnegative definite and

$$a_{ij} \in C^{1,1}(\Omega), \quad b_i \in C^{0,1}(\Omega), \quad c, f \in C(\Omega) \quad \forall i, j = 1, \dots, N.$$

It is known that under these assumptions the square root $\sigma \equiv a^{1/2}$ of a is in $C^{0,1}(\Omega)$. E.g., see [10] for a proof of this fact.

We consider weak solutions of (1.1) in the class of continuous functions. Subsolutions in the distribution sense are defined as follows. A function $u \in C(\Omega)$ is a distribution subsolution of (1.1) if

$$(1.2) \quad \int_{\Omega} (u \mathcal{L}^* \varphi - f \varphi) dx \leq 0$$

for any $\varphi \in \mathcal{D}_+(\Omega) \equiv \{\varphi \in C_0^\infty(\Omega) \mid \varphi \geq 0\}$, where $\mathcal{L}^*$ is the formal adjoint operator of $\mathcal{L}$, i.e.,

$$\mathcal{L}^* \varphi = - \sum_{i,j=1}^N (a_{ij} \varphi)_{x_i x_j} - \sum_{i=1}^N (b_i \varphi)_{x_i} + c \varphi \quad \forall \varphi \in C^2(\Omega).$$

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Likewise, a distribution supersolution is defined to be a continuous function u which satisfies (1.2) with $\geq$ replacing $\leq$. We shall indicate that u is a distribution subsolution (respectively, a distribution supersolution) by writing

$$\mathcal{L}u \leq f \text{ in } \mathcal{D}'(\Omega) \quad (\text{respectively, } \mathcal{L}u \geq f \text{ in } \mathcal{D}'(\Omega)).$$

A distribution solution of (1.1) is a function which is both a distribution subsolution and a distribution supersolution of (1.1). Equivalently, $u \in C(\Omega)$ is a distribution solution of (1.1) if

$$\int_{\Omega} (u \mathcal{L}^* \varphi - f \varphi) dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

This is indicated by writing $\mathcal{L}u = f$ in $\mathcal{D}'(\Omega)$.

For our exposition it is convenient to consider the general second-order, degenerate elliptic partial differential equation

$$(1.3) \quad F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega.$$

Here $F: \Omega \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N \rightarrow \mathbf{R}$ is a continuous function, where $\mathbf{S}^N$ denotes the set of real $N \times N$ symmetric matrices, and Du and D^2u denote the gradient $(u_{x_1}, \dots, u_{x_N})$ and the Hessian matrix $(u_{x_i x_j})$. The precise meaning of “degenerate ellipticity” is this. The function F or equation (1.3) is degenerate elliptic if $F(x, r, p, X) \leq F(x, r, p, Y)$ provided $X \geq Y$, i.e., $X - Y$ is nonnegative definite.

A function $u \in C(\Omega)$ is a viscosity subsolution of (1.3) if $F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$ whenever $\varphi \in C^2(\Omega)$, $x \in \Omega$ and $(u - \varphi)(x) = \sup_{\Omega} (u - \varphi)$. Similarly, $u \in C(\Omega)$ is a viscosity supersolution of (1.3) if $F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0$ whenever $\varphi \in C^2(\Omega)$, $x \in \Omega$ and $(u - \varphi)(x) = \inf_{\Omega} (u - \varphi)$. $u \in C(\Omega)$ is a viscosity solution of (1.3) if it is both a viscosity subsolution and a viscosity supersolution of (1.3). When convenient, we shall indicate that u is a viscosity subsolution (respectively, a viscosity supersolution, or a viscosity solution) of (1.3) by writing

$F(x, u, Du, D^2u) \leq 0$ (respectively, ≥ 0 , or $= 0$) in Ω in the viscosity sense.

We set

$$F_{\mathcal{L}}(x, r, p, X) = -\operatorname{tr} a(x)X + \langle b(x), p \rangle + c(x)r - f(x).$$

Now (1.1) reads $F_{\mathcal{L}}(x, u, Du, D^2u) = 0$ in Ω . Since $a(x) \geq 0$, it is seen that $F_{\mathcal{L}}$ is degenerate elliptic. Subsolutions, supersolutions and solutions of (1.1) in the viscosity sense are defined with $F_{\mathcal{L}}$.

The definitions of distribution solutions and viscosity solutions are based on the integration by parts and on the maximum principle, respectively. The maximum principle here means that if $v \in C^2(\Omega)$ attains its maximum at $x \in \Omega$, then $Dv(x) = 0$ and $D^2v(x) \leq 0$.

The question we address here is if these two notions of weak solutions of (1.1) are equivalent. An affirmative answer has been given in [8] by P.-L. Lions. The arguments there are largely based on probabilistic techniques to deduce the answer. We will give here another approach based on purely PDE and viscosity solutions methods to obtain a similar conclusion.

Theorem 1 *If $u \in C(\Omega)$ is a viscosity subsolution of (1.1), then it is a distribution subsolution of (1.1).*

Theorem 2 *Assume that $\sigma \in C^1(\Omega)$. If $u \in C(\Omega)$ is a distribution subsolution of (1.1), then it is also a viscosity subsolution of (1.1).*

Our results are slightly better in the sense that the regularity requirements on a is less than those in [8]. In deed, it is assumed in [8] that σ is in $C^{1,1}(\Omega)$.

The paper is organized as follows. In Section 1 we explain an observation concerning the sup-convolution of viscosity solutions. Section 2 is devoted to the proof of Theorem 1. In Section 3 we collect solvability and regularity results (Theorems 4 and 5) of solutions of (1.1) which are needed in the proof of Theorem 2. Theorem 2 is proved in Section 4. Theorems 4 and 5 are proved in Section 5.

§1 Approximation of viscosity solutions

It is well known that the sup-convolutions and inf-convolutions yield good approximations of viscosity subsolutions and supersolutions, respectively. We give here an additional remark concerning these approximations.

Throughout this section, for simplicity of presentation we assume that Ω is bounded and only consider those solutions u which are bounded, uniformly continuous, i.e., $u \in BUC(\Omega)$. For a function $u \in BUC(\Omega)$ and $\varepsilon > 0$ the sup-convolution is defined by

$$u^\varepsilon(x) = \sup_{y \in \Omega} \left(u(y) - \frac{1}{2\varepsilon} |x - y|^2 \right).$$

We shall write $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \Omega^c) > \varepsilon\}$.

To formulate the result, we introduce some conditions on F .

(A1) For each $R > 0$ there is a function $\omega_{1R} \in C([0, \infty))$ satisfying $\omega_{1R}(0) = 0$ such that if $-R \leq r \leq s \leq R$, then $F(x, r, p, X) \leq F(x, s, p, X) + \omega_{1R}(s - r)$.

(A2) For each $R > 0$ there is a function $\omega_{2R} \in C([0, \infty))$ satisfying $\omega_{2R}(0) = 0$ such that if $|r| \leq R$ and if $\alpha > 1$ and $X, Y \in \mathbf{S}^N$ satisfy

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y, r, \alpha(x - y), -Y) \leq F(x, r, \alpha(x - y), X) + \omega_{2R}(\alpha|x - y|^2 + 1/\alpha).$$

We note that if F satisfies (A2), then F is degenerate elliptic. Note also that $F_{\mathcal{F}}$ satisfies (A1) and (A2) provided σ and b are Lipschitz continuous and c and f are uniformly continuous on Ω . See for these [2].

Theorem 3 *Let (A1) and (A2) hold. Let $u \in BUC(\Omega)$ be a viscosity subsolution of (1.3). Then for each $\varepsilon > 0$ there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$,*

$$F(x, u^\delta, Du^\delta, D^2u^\delta) \leq \varepsilon \quad \text{in } \Omega_\varepsilon \text{ in the viscosity sense.}$$

Remark The constant δ_0 can be chosen so that it depends on u only through $\sup_\Omega |u|$ and its modulus of continuity.

Proof. We choose $M > 0$ so that $M \geq 2 \sup_\Omega |u|$, and a nondecreasing function $\omega \in C([0, \infty))$ satisfying $\omega(0) = 0$ so that

$$\sup \{u(x) - u(y) \mid x, y \in \Omega, |x - y| \leq r\} \leq \omega(r) \quad \forall r \geq 0,$$

and $\max \{\omega_{1M}, \omega_{2M}\} \leq \omega$, where ω_{1M} and ω_{2M} are from (A1) and (A2) with $R = M$, respectively.

Let $\delta > 0$. It is obvious that $u \leq u^\delta$ on Ω . Therefore, it is easily seen that if $\gamma = (2\delta M)^{1/2}$ and $x \in \Omega_\gamma$, then $B(x, \gamma) \subset \Omega$ and

$$u^\delta(x) = \max \left\{ u(y) - \frac{1}{2\delta} |x - y|^2 \mid y \in B(x, \gamma) \right\}.$$

For each $x \in \Omega_\gamma$ we fix $y(x, \delta) \in B(x, \gamma)$ so that

$$u^\delta(x) = u(y(x, \delta)) - \frac{1}{2\delta} |x - y(x, \delta)|^2.$$

We observe that from the inequality $u \leq u^\delta$ on Ω that

$$\frac{1}{2\delta} |x - y(x, \delta)|^2 \leq u(y(x, \delta)) - u(x) \leq \omega(|x - y(x, \delta)|) \leq \omega(\gamma).$$

We recall that if $x \in \Omega_\gamma$ and $(p, X) \in J^{2,+}u^\delta(x)$, then $y(x, \delta) = x + \delta p$. See [2] for this, the definitions of semijets $J^{2,\pm}u$, $\bar{J}^{2,\pm}u$ and relevant facts. Now, fix $x \in \Omega_\gamma$ and $(p, X) \in J^{2,+}u^\delta(x)$. We set

$$v(z) = \langle p, z - x \rangle + \frac{1}{2} \langle X(z - x), z - x \rangle \quad \forall z \in \mathbf{R}^N,$$

and $w(y, z) = u(y) - v(z)$ for $y \in \Omega$, $z \in \mathbf{R}^N$. We observe that

$$\begin{aligned} w(y, z) - \frac{1}{2\delta} |y - z|^2 &\leq u^\delta(z) - v(z) \leq u^\delta(x) + o(|z - x|^2) \\ &= w(y(x, \delta), x) - \frac{1}{2\delta} |y(x, \delta) - x|^2 + o(|z - x|^2) \quad \text{as } z \longrightarrow x, \end{aligned}$$

i.e.,

$$\begin{aligned} &\left(\frac{1}{\delta} (y(x, \delta) - x), \frac{1}{\delta} (x - y(x, \delta)), \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right) \\ &= \left(p, -p, \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right) \in J^{2,+} w(y(x, \delta), x). \end{aligned}$$

By the maximum principle for semicontinuous functions (see [2]), we see that there are $Y, Z \in \mathbf{S}^N$ such that

$$\begin{aligned} -\frac{3}{\delta} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \leq \frac{3}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \\ (p, Y) &\in \bar{J}^{2,+} u(y(x, \delta)), \quad (p, -Z) \in \bar{J}^{2,-} v(x). \end{aligned}$$

The last inclusion implies that $-Z \leq D^2 v(x) = X$. Since u is a viscosity subsolution of (1.3), we have

$$F(y(x, \delta), u(y(x, \delta)), p, Y) \leq 0.$$

To proceed, we assume that $\delta < 1$. Assumption (A2) now yields

$$\begin{aligned} &F\left(x, u(y(x, \delta)), \frac{1}{\delta} (y(x, \delta) - x), -Z\right) \\ &\leq F(y(x, \delta), u(y(x, \delta)), \frac{1}{\delta} (y(x, \delta) - x), Y) + \omega\left(\frac{1}{\delta} |y(x, \delta) - x|^2 + \delta\right). \end{aligned}$$

Consequently,

$$\begin{aligned} 0 &\geq F(x, u(y(x, \delta)), p, -Z) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u(y(x, \delta)), p, X) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u^\delta(x) + (1/2\delta) |y(x, \delta) - x|^2, p, X) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u^\delta(x), p, X) - \omega(\omega(\gamma)) - \omega(2\omega(\gamma) + \gamma). \end{aligned}$$

Thus

$$F(x, u^\delta(x), p, X) \leq 2\omega(2\omega(\gamma) + \delta) \quad \text{in } \Omega_\gamma$$

in the viscosity sense. Noting that $\gamma \equiv (2\gamma M)^{1/2} \rightarrow 0$ and $2\omega(2\omega(\gamma) + \delta) \rightarrow 0$ as $\delta \downarrow 0$, we finish the proof. ■

§2 Proof of Theorem 1

Theorem 3 and the following lemma will be key observations in our proof of Theorem 1. We denote by $\mathcal{M}(\Omega)$ and by $\mathcal{D}'(\Omega)$ the spaces of Radon measures on Ω and of distributions on Ω , respectively. Recall that we may identify $\mathcal{M}(\Omega)$ with the dual space $C_0(\Omega)'$ of $C_0(\Omega)$.

Lemma 1 (A. D. Aleksandrov) *Let $u \in C(\mathbf{R}^N)$ be semiconvex. Then there are matrices $U = (u_{ij})_{1 \leq i, j \leq N}$ with $u_{ij} \in L^1_{loc}(\mathbf{R}^N)$ and $V = (v_{ij})_{1 \leq i, j \leq N}$ with $v_{ij} \in \mathcal{M}(\mathbf{R}^N)$ such that*

$$D^2u = U + V \text{ in } \mathcal{D}'(\Omega), \quad V \geq 0 \text{ in } \mathcal{M}(\mathbf{R}^N),$$

$$(Du(x), D^2u(x)) \in J^2u(x) \text{ a.e. in } \mathbf{R}^N,$$

where $J^2u(x) = J^{2,+}u(x) \cap J^{2,-}u(x)$. Moreover, the measures v_{ij} are singular with respect to the Lebesgue measure.

For a proof of this lemma we refer the reader to [5].

Proof of Theorem 1. Because of the local property of the assertion, we may assume that Ω is bounded and that $a \in C^{1,1}(\bar{\Omega})$, $b \in C^{0,1}(\bar{\Omega})$, $c, f \in C(\bar{\Omega})$, $\sigma \in C^{0,1}(\bar{\Omega})$ and $u \in C(\bar{\Omega})$. This guarantees that $F_{\mathcal{F}}$ satisfies (A1) and (A2).

Now, fix $\varphi \in \mathcal{D}_+(\Omega)$. Choose $\varepsilon > 0$ so that $\text{supp } \varphi \subset \Omega_\varepsilon$. By virtue of Theorem 3, there is $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then

$$(2.1) \quad F_{\mathcal{F}}(x, u^\delta, Du^\delta, D^2u^\delta) \leq \varepsilon \text{ in } \Omega_\varepsilon \text{ in the viscosity sense.}$$

Fix $\delta \in (0, \delta_0)$. By Lemma 1 we find $U_\delta = (u_{ij}^\delta)$ with $u_{ij}^\delta \in L^1_{loc}(\mathbf{R}^N)$ and $V_\delta = (v_{ij}^\delta)$ with $v_{ij}^\delta \in \mathcal{M}(\mathbf{R}^N)$ such that

$$D^2u^\delta = U_\delta + V_\delta \text{ in } \mathcal{D}'(\mathbf{R}^N), \quad V_\delta \geq 0 \text{ in } \mathcal{M}(\mathbf{R}^N),$$

$$(Du^\delta(x), U_\delta(x)) \in J^2u^\delta(x) \text{ a.e.}$$

The last inclusion and (2.1) yield

$$F_{\mathcal{F}}(x, u^\delta(x), Du^\delta(x), U_\delta(x)) \leq \varepsilon \quad \text{a.e. in } \Omega_\varepsilon,$$

and multiplying this by φ and integrating over Ω yield

$$(2.2) \quad \int_{\Omega} (F_{\mathcal{F}}(x, u^\delta(x), Du^\delta(x), U_\delta(x)) - \varepsilon) \varphi(x) dx \leq 0.$$

Now we observe that

$$\sum_{i,j=1}^N \int_{\Omega} a_{ij} \varphi dv_{ij}^{\delta}(x) = \sum_{i,j,k=1}^N \int_{\Omega} (\sigma_{ik} \varphi^{1/2})(\sigma_{jk} \varphi^{1/2}) dv_{ij}^{\delta}(x) \geq 0,$$

and that if we identify $\mathcal{M}(\mathbf{R}^N)$ with $C_0(\mathbf{R}^N)' \subset \mathcal{D}'(\mathbf{R}^N)$, then

$$\begin{aligned} \sum_{i,j=1}^N \left\{ \int_{\Omega} a_{ij} \varphi dv_{ij}^{\delta}(x) + \int_{\Omega} a_{ij} \varphi u_{ij}^{\delta} dx \right\} \\ = \sum_{i,j=1}^N \langle u_{ij}^{\delta} + v_{ij}^{\delta}, a_{ij} \varphi \rangle = \sum_{i,j=1}^N \langle u_{x_i x_j}^{\delta}, a_{ij} \varphi \rangle \\ = \sum_{i,j=1}^N \langle u^{\delta}, (a_{ij} \varphi)_{x_i x_j} \rangle = \sum_{i,j=1}^N \int_{\Omega} u^{\delta} (a_{ij} \varphi)_{x_i x_j} dx. \end{aligned}$$

Here $\langle g, \psi \rangle$ denotes the duality pairing between $g \in \mathcal{D}'(\mathbf{R}^N)$ and $\psi \in C_0^{\infty}(\mathbf{R}^N)$ and we may assume by approximation that the a_{ij} are C^{∞} . Combining these, we have

$$\sum_{i,j=1}^N \int_{\Omega} a_{ij} \varphi u_{ij}^{\delta} dx \leq \sum_{i,j=1}^N \int_{\Omega} u^{\delta} (a_{ij} \varphi)_{x_i x_j} dx.$$

Therefore, from (2.2) we obtain

$$\int_{\Omega} (u^{\delta} \mathcal{L}^* \varphi - f \varphi - \varepsilon \varphi) dx \leq 0.$$

Noting that $u^{\delta}(x) \rightarrow u(x)$ uniformly in Ω as $\varepsilon \downarrow 0$ and passing to the limit as $\varepsilon \downarrow 0$, we conclude that

$$\int_{\Omega} (u \mathcal{L}^* \varphi - f \varphi) dx \leq 0.$$

This completes the proof. ■

§3 Solvability of (1.1)

In this section we treat the case when $\Omega = \mathbf{R}^N$, and consider the solvability of (1.1). The results here are more or less known.

Concerning the regularity of a we do not assume that $a \in W^{2,\infty}(\mathbf{R}^N)$ except in the assertion (ii) of Theorem 5, and instead we only assume that $\sigma \in W^{1,\infty}(\mathbf{R}^N)$.

We define

$$c_0 = \inf_{\mathbf{R}^N} c, \quad \lambda_0 = \sup_{x \neq y} \left\{ \frac{\operatorname{tr}(\sigma(x) - \sigma(y))^2 - \langle b(x) - b(y), x - y \rangle}{|x - y|^2} \right\}.$$

We note that λ_0 may be negative.

Theorem 4 *Assume that $c_0 > 0$ and $c, f \in BUC(\mathbf{R}^N)$. Then there is a unique viscosity solution $u \in BUC(\mathbf{R}^N)$ of (1.1) and moreover,*

$$(3.1) \quad \|u\|_{L^\infty} \leq \frac{1}{c_0} \|f\|_{L^\infty}.$$

Theorem 5 *Assume that $c_0 \geq 0$, and let $u \in BUC(\mathbf{R}^N)$ be a viscosity solution of (1.1). Then: (i) if $c_0 > \lambda_0$ and $c, f \in W^{1,\infty}(\mathbf{R}^N)$, then $u \in W^{1,\infty}(\mathbf{R}^N)$ and*

$$(3.2) \quad \|Du\|_{L^\infty} \leq \frac{1}{c_0 - \lambda_0} (\|Df\|_{L^\infty} + \|Dc\|_{L^\infty} \|u\|_{L^\infty}).$$

(ii) if $c_0 > \lambda_1 \equiv \max\{\lambda_0, 2\lambda_0\}$ and $\sigma, b, c, f \in W^{2,\infty}(\mathbf{R}^N)$, then $u \in W^{2,\infty}(\mathbf{R}^N)$ and

$$(3.3) \quad \|D^2u\|_{L^\infty} \leq C(\|D^2\sigma\|_{L^\infty} + 1),$$

where

$$C = M(\lambda_1, 1/(c_0 - \lambda_1), \|D\sigma\|_{L^\infty}, \|D^2b\|_{L^\infty}, \|Df\|_{W^{1,\infty}}, \|c\|_{W^{2,\infty}}, \|u\|_{W^{1,\infty}})$$

for some continuous function M on $\mathbf{R}^7$.

Theorems 4 and 5 have been proved in [6], [7], [8], [3] and [4]. See also [9]. The condition that $c_0 > \lambda_1$ in the assertion (ii) of Theorem 5 is slightly sharper than that used in [9]. Theorem 4 and the assertion (i) of Theorem 5 are valid for Hamilton-Jacobi-Bellman-Isaacs equations under similar assumptions. Half of the assertion (ii) of Theorem 5, the estimate on solutions u

$$\langle D^2u\xi, \xi \rangle \leq C(\|D^2\sigma\|_{L^\infty} + 1) \quad \forall \xi \in \mathbf{R}^N \text{ with } |\xi| \leq 1$$

(in the viscosity sense or equivalently in the distribution sense) is valid for Hamilton-Jacobi-Bellman equation under similar assumptions. This assertion requires convexity of equations. Indeed, [6], [7] and [8] treat Hamilton-Jacobi-Bellman equations and techniques there are largely based on stochastic optimal control theory, and [3] treat Hamilton-Jacobi-Bellman-Isaacs equations.

The proof of these theorems will be postponed until Section 5.

§4 Proof of Theorem 2

We may assume that $c = 0$; otherwise we regard the original $f - cu$ as f in (1.1). Let $u \in C(\Omega)$ satisfy

$$\mathcal{L}u \leq f \quad \text{in } \mathcal{D}'(\Omega).$$

Suppose that u does not satisfy

$$\mathcal{L}u \leq f \quad \text{in } \Omega \text{ in the viscosity sense.}$$

We shall show that this yields a contradiction.

By this supposition we find $z \in \Omega$, $r > 0$ and $\varphi \in C^2(\Omega)$ such that

$$\begin{cases} \mathcal{L}\varphi(x) \geq f(x) + 2r & \forall x \in B(z, r), \\ u(z) = \varphi(z), \\ u(x) \leq \varphi(x) - |x - z|^4 & \forall x \in B(z, r). \end{cases}$$

Of course, we assume here that $B(z, r) \subset \Omega$. Set $U = B(z, r)^\circ$. By continuity, there is $\delta > 0$ such that for any $\varepsilon \in [0, \delta]$, if we define $\varphi_\varepsilon \in C^2(U)$ by $\varphi_\varepsilon(x) = \varphi(x) - \varepsilon$, then $\mathcal{L}\varphi_\varepsilon(x) \geq f(x) + r$ for $\forall x \in U$.

We assume that $\delta^{1/4} < r$, so that $B(z, \delta^{1/4}) \subset U$. Let $0 < \varepsilon \leq \delta$, and we set $w_\varepsilon(x) = u(x) - \varphi_\varepsilon(x)$ for $x \in \bar{U}$. Then $w_\varepsilon \in C(\bar{U})$, $\max_{\bar{U}} w_\varepsilon = \varepsilon$, $w_\varepsilon(x) \leq 0$ for $\forall x \in \bar{U} \setminus B(z, \varepsilon^{1/4})$ and $\mathcal{L}w_\varepsilon \leq -r$ in $\mathcal{D}'(U)$.

Fix $\zeta \in C_0^\infty(U)$ so that $0 \leq \zeta \leq 1$ in U and $\zeta(x) = 1$ for $\forall x \in B(x, \varepsilon^{1/4})$. Define the operator $\mathcal{L}_\zeta$ by

$$\mathcal{L}_\zeta \psi = \zeta^2 \mathcal{L} \psi = -\text{tr}(\zeta^2 a D^2 \psi) + \langle \zeta^2 b, D \psi \rangle.$$

Then,

$$\mathcal{L}_\zeta w_\varepsilon \leq -r \zeta^2 \quad \text{in } \mathcal{D}'(U).$$

Let $\lambda > 0$ be a constant to be fixed later on. We let $\varepsilon = \{\delta, r/\lambda\}$, so that $\lambda w_\varepsilon \leq r \zeta^2$ in U and moreover,

$$\lambda w_\varepsilon + \mathcal{L}_\zeta w_\varepsilon \leq 0 \quad \text{in } \mathcal{D}'(U).$$

Thus

$$(4.1) \quad \langle w_\varepsilon, \lambda v + \mathcal{L}_\zeta^* v \rangle \leq 0 \quad \forall v \in W^{2,\infty}(U) \text{ with } v \geq 0.$$

We put

$$\tilde{\sigma}_{ij} = \zeta \sigma_{ij}, \quad \tilde{b}_i = [\zeta^2 b_i + \sum_{j=1}^N (\zeta^2 a_{ij})_{x_j}],$$

$$\tilde{c} = - \sum_{i,j=1}^N (\zeta^2 a_{ij})_{x_i x_j} - \sum_{i=1}^N (\zeta^2 b_i)_{x_i}.$$

We extend these functions to $\mathbf{R}^N$ by assuming their values to be zero outside of U , and set $\tilde{\sigma} = (\tilde{\sigma}_{ij})_{1 \leq i,j \leq N}$, $\tilde{a} = (\tilde{\sigma})^2$ and $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_N)$. Now we may regard $\mathcal{L}_\zeta^*$ as an operator defined for functions on $\mathbf{R}^N$, i.e.,

$$\mathcal{L}_\zeta^* \psi = -\text{tr}(\tilde{a} D^2 \psi) + \langle \tilde{b}, D\psi \rangle + \tilde{c} \psi \quad \text{for } \psi \in C^2(\mathbf{R}^N).$$

Note that $\tilde{\sigma}_{ij} \in C^1(\mathbf{R}^N)$, $\tilde{b}_i \in W^{1,\infty}(\mathbf{R}^N)$ and $\tilde{c} \in L^\infty(\mathbf{R}^N)$. By using standard mollification techniques, we find C_0^∞ functions $\sigma_{ij}^\delta, b_i^\delta, c^\delta$, with $\delta \in (0, 1)$ and $1 \leq i, j \leq N$, such that

$$\begin{aligned} \|\sigma_{ij}^\delta\|_{W^{1,\infty}} &\leq \|\tilde{\sigma}_{ij}\|_{W^{1,\infty}}, \quad \|D^2 \sigma_{ij}^\delta\|_{L^\infty} \leq \frac{1}{\delta} \|D\tilde{\sigma}_{ij}\|_{L^\infty}, \\ \|b_i^\delta\|_{W^{1,\infty}} &\leq \|\tilde{b}_i\|_{W^{1,\infty}}, \quad \|c^\delta\|_{L^\infty} \leq \|\tilde{c}\|_{L^\infty}, \end{aligned}$$

and as $\delta \downarrow 0$,

$$(4.2) \quad \begin{cases} \|\sigma_{ij}^\delta - \tilde{\sigma}_{ij}\|_{L^1} = o(\delta), \\ \|b_i^\delta - \tilde{b}_i\|_{L^1} \longrightarrow 0, \quad \|c^\delta - \tilde{c}\|_{L^1} \longrightarrow 0. \end{cases}$$

We may moreover assume that the $\sigma_{ij}^\delta, b_i^\delta$ and c^δ vanish outside of a compact subset of U .

In view of Theorems 4 and 5 we set

$$\begin{aligned} \lambda_0 &= \sup \left\{ \frac{\text{tr}(\sigma^\alpha(x) - \sigma^\alpha(y))^2 - \langle b^\beta(x) - b^\beta(y), x - y \rangle}{|x - y|^2} \mid x \neq y, \alpha, \beta \in (0, 1) \right\}, \\ c_0 &= \inf \{c^\gamma(x) \mid x \in \mathbf{R}^N, 0 < \gamma < 1\}, \end{aligned}$$

and fix $\lambda > 0$ so that $\lambda > c_0 + 2 \max\{\lambda_0, 0\}$. Fix $\psi \in C_0^\infty(\mathbf{R}^N)$ so that $\text{supp } \psi \subset U$. Theorems 4 and 5 guarantee that for each $\alpha, \beta, \gamma \in (0, 1)$ there is a unique viscosity solution $v = v^{\alpha\beta\gamma} \in BUC(\mathbf{R}^N)$ of

$$\lambda v + \mathcal{L}^{\alpha\beta\gamma} v = \psi \quad \text{in } \mathbf{R}^N,$$

where

$$\mathcal{L}^{\alpha\beta\gamma} v(x) = -\text{tr} a^\alpha(x) D^2 v(x) + \langle b^\beta(x), Dv(x) \rangle + c^\gamma(x) v(x).$$

Moreover, for any $\alpha, \beta, \gamma \in (0, 1)$ we have $v^{\alpha\beta\gamma} \in W^{2,\infty}(\mathbf{R}^N)$, and

$$(4.3) \quad \begin{cases} \|D^2 v^{\alpha\beta\gamma}\|_0 \leq \frac{1}{\alpha} C_1(\beta, \gamma), \\ \|Dv^{\alpha\beta\gamma}\|_0 \leq C_2(\gamma), \\ \|v^{\alpha\beta\gamma}\|_0 \leq C_3, \end{cases}$$

where $C_1(\beta, \gamma)$, $C_2(\gamma)$ and C_3 are constants independent, respectively, of α , of α and β and of α, β and γ . Since the a^α , b^β and c^γ vanish outside of a compact subset of U , so does the $v^{\alpha\beta\gamma}$, i.e., $v^{\alpha\beta\gamma} \in C_0(U)$. Also, by the maximum principle, $v^{\alpha\beta\gamma} \geq 0$ on $\mathbf{R}^N$ for all $\alpha, \beta, \gamma \in (0, 1)$. Therefore, going back to (4.1), we obtain

$$\begin{aligned} \langle w_\varepsilon, \psi \rangle &= \langle w_\varepsilon, \lambda v^{\alpha\beta\gamma} + \mathcal{L}^{\alpha\beta\gamma} v^{\alpha\beta\gamma} \rangle \\ &= \langle w_\varepsilon, \lambda v^{\alpha\beta\gamma} + \mathcal{L}_\zeta^* v^{\alpha\beta\gamma} \rangle + \langle w_\varepsilon, \mathcal{L}^{\alpha\beta\gamma} v^{\alpha\beta\gamma} - \mathcal{L}_\zeta^* v^{\alpha\beta\gamma} \rangle \\ &\leq \|w_\varepsilon\|_0 \{ \|D^2 v^{\alpha\beta\gamma}\|_0 (\|\sigma^\alpha\|_0 + \|\tilde{\sigma}\|_0) \|\sigma^\alpha - \tilde{\sigma}\|_{L^1} \\ &\quad + \|D v^{\alpha\beta\gamma}\|_0 \|\tilde{b} - b^\beta\|_{L^1} + \|v^{\alpha\beta\gamma}\|_0 \|\tilde{c} - c^\gamma\|_{L^1} \}. \end{aligned}$$

In view of (4.2) and (4.3), sending $\alpha \downarrow 0$, $\beta \downarrow 0$ and $\gamma \downarrow 0$ in this order, we see that $\langle w_\varepsilon, \psi \rangle \leq 0$ and hence $w_\varepsilon \leq 0$ on U . This is a contradiction, which completes the proof. ■

§5 Proof of Theorems 4 and 5

In the spirit of being free from probabilistic techniques, it may be important to prove Theorems 4 and 5 without using results based on probabilistic techniques.

It is well known (see, e.g., [8] and [3]) that Theorem 4 is valid. However we give a proof for the reader's convenience.

In what follows we use the notation: For a function $u = (u_{ij}): \mathbf{R}^N \rightarrow \mathbf{R}^{m \times n}$ we write

$$\begin{aligned} \|u\|_0 &= \|(\sum_{i=1}^m \sum_{j=1}^n |u_{ij}|^2)^{1/2}\|_{L^\infty}, \quad \|u\|_1 = \|(\sum_{k=1}^N \sum_{i=1}^m \sum_{j=1}^n |u_{ijx_k}|^2)^{1/2}\|_{L^\infty}, \\ \|u\|_2 &= \|(\sum_{k,l=1}^N \sum_{i=1}^m \sum_{j=1}^n |u_{ijx_k x_l}|^2)^{1/2}\|_{L^\infty}. \end{aligned}$$

In particular, we have

$$\|u\|_{W^{1,\infty}} = \|u\|_0 + \|u\|_1 \quad \text{and} \quad \|u\|_{W^{2,\infty}} = \|u\|_0 + \|u\|_1 + \|u\|_2.$$

Proof of Theorem 4. Since $c_0 > 0$, the constants $\|f\|_0/c_0$ and $-\|f\|_0/c_0$ are a supersolution and a subsolution of (1.1), respectively. By the Perron method, we find a viscosity solution u of (1.1) with

$$-\frac{1}{c_0} \|f\|_0 \leq u \leq \frac{1}{c_0} \|f\|_0 \quad \text{on } \mathbf{R}^N.$$

The fact that $u \in UC(\mathbf{R}^N)$ follows from the comparison result for viscosity solutions (see for instance [2] and [3]). ■

Proof of Theorem 5. Assume that $c_0 > \lambda_0$. Let $u \in BUC(\mathbf{R}^N)$ be a viscosity solution of (1.1). Let $\varepsilon > 0$, $\delta > 0$ and

$$(5.1) \quad L > \frac{1}{c_0 - \lambda_0} (\|c\|_1 \|u\|_0 + \|f\|_1),$$

and set

$$\Phi(x, y) = u(x) - u(y) - L|x - y| - \delta|x|^2 - \varepsilon \quad \text{for } x, y \in \mathbf{R}^N.$$

We will show that $\Phi \leq 0$ on $\mathbf{R}^N$ for all $\varepsilon, \delta > 0$. To this end, suppose that $\sup_{\mathbf{R}^{2N}} \Phi > 0$ for some $\varepsilon > 0$ and $\delta = \delta_0 > 0$. This will lead a contradiction. Fix $\varepsilon > 0$ and $\delta_0 > 0$ so that $\sup_{\mathbf{R}^{2N}} \Phi > 0$ with this $\varepsilon > 0$ and $\delta = \delta_0$, and $0 < \delta \leq \delta_0$. Note that $\sup_{\mathbf{R}^{2N}} \Phi > 0$. Let $(\hat{x}, \hat{y}) \in \mathbf{R}^N \times \mathbf{R}^N$ be a maximum point of Φ . Writing

$$\psi(x) = |x| \quad \text{and} \quad \varphi(x, y) = L|x - y| \quad \text{for } x, y \in \mathbf{R}^N,$$

and noting that

$$D\psi(x) = \frac{x}{|x|}, \quad D^2\psi(x) = \frac{I}{|x|} - \frac{x \otimes x}{|x|^3} \leq \frac{I}{|x|},$$

and

$$D^2\varphi(x, y) \leq L \begin{pmatrix} D^2\psi(x - y) & -D^2\psi(x - y) \\ -D^2\psi(x - y) & D^2\psi(x - y) \end{pmatrix} \leq \frac{L}{|x - y|} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

we see by the maximum principle (see [2]) that for each $\theta > 1$ there are $X, Y \in \mathbf{S}^N$ such that

$$(5.2) \quad \begin{cases} (\hat{p}, X) \in \bar{J}^{2,+} u(\hat{x}) - 2\delta(\hat{x}, I), & (\hat{p}, -Y) \in \bar{J}^{2,-} u(\hat{y}), \\ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{L\theta}{|\hat{x} - \hat{y}|} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{cases}$$

where $\hat{p} = L(\hat{x} - \hat{y})/|\hat{x} - \hat{y}|$. Therefore we have

$$-\operatorname{tr} a(\hat{x})X + \langle b(\hat{x}), \hat{p} \rangle + c(\hat{x})u(\hat{x}) \leq f(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle + 2\delta \operatorname{tr} a(\hat{x})$$

and

$$-\operatorname{tr} a(\hat{y})(-Y) + \langle b(\hat{y}), \hat{p} \rangle + c(\hat{y})u(\hat{y}) \geq f(\hat{y}).$$

Hence

$$\begin{aligned} & c(\hat{x})(u(\hat{x}) - u(\hat{y})) - \operatorname{tr}(a(\hat{x})X + a(\hat{y})Y) \\ & + \langle b(\hat{x}) - b(\hat{y}), \hat{p} \rangle \leq (c(\hat{y}) - c(\hat{x}))u(\hat{y}) \end{aligned}$$

$$\begin{aligned}
& + f(\hat{x}) - f(\hat{y}) + 2\delta(\operatorname{tr} a(\hat{x}) - \langle b(\hat{x}), \hat{x} \rangle) \\
& \leq (\|c\|_1 \|u\|_0 + \|f\|_1) |\hat{x} - \hat{y}| + 2\delta(\operatorname{tr} a(\hat{x}) - \langle b(\hat{x}), \hat{x} \rangle).
\end{aligned}$$

The latter of (5.2) yields

$$\begin{aligned}
\operatorname{tr}(a(\hat{x})X + a(\hat{y})Y) &= \operatorname{tr} \left\{ (\sigma(\hat{x})\sigma(\hat{y})) \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} \sigma(\hat{x}) \\ \sigma(\hat{y}) \end{pmatrix} \right\} \\
&\leq \frac{L\theta}{|\hat{x} - \hat{y}|} \operatorname{tr} \left\{ (\sigma(\hat{x})\sigma(\hat{y})) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\hat{x}) \\ \sigma(\hat{y}) \end{pmatrix} \right\} \\
&\leq \frac{L\theta}{|\hat{x} - \hat{y}|} \operatorname{tr} (\sigma(\hat{x}) - \sigma(\hat{y}))^2.
\end{aligned}$$

Thus, recalling that $\Phi(\hat{x}, \hat{y}) > 0$, we have

$$\begin{aligned}
c_0 L |\hat{x} - \hat{y}| &\leq L |\hat{x} - \hat{y}| \frac{\operatorname{tr} (\sigma(\hat{x}) - \sigma(\hat{y}))^2 - \langle b(\hat{x}) - b(\hat{y}), \hat{x} - \hat{y} \rangle}{|\hat{x} - \hat{y}|^2} \\
&+ L(\theta - 1) \frac{\operatorname{tr} (\sigma(\hat{x}) - \sigma(\hat{y}))^2}{|\hat{x} - \hat{y}|^2} + (\|c\|_1 \|u\|_0 + \|f\|_1) |\hat{x} - \hat{y}| \\
&+ 2\delta(\|a\|_0 + \|b\|_0 |\hat{x}|).
\end{aligned}$$

Since $\theta > 1$ is arbitrary, sending $\theta \downarrow 1$, we obtain

$$(c_0 - \lambda_0) L |\hat{x} - \hat{y}| \leq (\|c\|_1 \|u\|_0 + \|f\|_1) |\hat{x} - \hat{y}| + 2\delta(\|a\|_0 + \|b\|_0 |\hat{x}|).$$

Since $\Phi(\hat{x}, \hat{y}) > 0$ and $u \in BUC(\mathbf{R}^N)$, it follows that $\delta |\hat{x}|^2 \leq 2 \|u\|_0$ and also that $\gamma \leq |\hat{x} - \hat{y}| \leq \gamma^{-1}$ for some constant $\gamma > 0$ independent of $\delta > 0$. Therefore, passing to the limit as $\delta \downarrow 0$, we see that

$$(c_0 - \lambda_0) L r \leq (\|c\|_1 \|u\|_0 + \|f\|_1) r$$

for some $r \geq \gamma$, and hence

$$L \leq \frac{1}{c_0 - \lambda_0} (\|c\|_1 \|u\|_0 + \|f\|_1).$$

This contradicts our choice (5.1) of L . Thus we know that $\Phi(x, y) \leq 0$ for all $x, y \in \mathbf{R}^N$ and $\varepsilon, \delta > 0$, which implies

$$u(x) - u(y) \leq \frac{\|c\|_1 \|u\|_0 + \|f\|_1}{c_0 - \lambda_0} |x - y| \quad \forall x, y \in \mathbf{R}^N,$$

and thus proves the assertion (i).

Next we prove (ii). We begin with preliminary calculations. Let $L > 0$, and set

$$\begin{aligned}\varphi(x, y, z) &= L|x - y|^2 + (|x - y|^4 + |x + y - 2z|^2)^{1/2} \\ &\equiv L|x - y|^2 + \varphi_1(x, y, z)\end{aligned}$$

for $x, y, z \in \mathbf{R}^N$. Let $(x, y, z) \in \mathbf{R}^{3N}$ be an arbitrary point with $\varphi_1(x, y, z) \neq 0$. We then have:

(5.3)

$$D\varphi = 2L \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} + \frac{1}{\varphi_1} \left\{ 2|x - y|^2 \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} x + y - 2z \\ x + y - 2z \\ -2x - 2y + 4z \end{pmatrix} \right\}$$

and

(5.4)

$$\begin{aligned}D^2\varphi &= 2L \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1} \left\{ 2|x - y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ &\quad \left. + 4 \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right\} - \frac{1}{\varphi_1^2} D\varphi_1 \otimes D\varphi_1 \\ &\leq 2L \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1} \left\{ 2|x - y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ &\quad \left. + 4|x - y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right\} \\ &\leq \left(2L + \frac{6|x - y|^2}{\varphi_1} \right) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}.\end{aligned}$$

Here and later φ_1 denotes its value evaluated at (x, y, z) . Now, setting

$$J = \text{tr}(\sigma(x) + \sigma(y) - 2\sigma(z))^2 - \langle b(x) + b(y) - 2b(z), x + y - 2z \rangle,$$

$$\xi = \sigma(x) + \sigma(y) - 2\sigma\left(\frac{x + y}{2}\right), \quad \eta = 2\left[\sigma\left(\frac{x + y}{2}\right) - \sigma(z)\right],$$

$$\alpha = b(x) + b(y) - 2b\left(\frac{x + y}{2}\right), \quad \beta = 2\left[b\left(\frac{x + y}{2}\right) - b(z)\right],$$

and noting that for any $g \in C_b^2(\mathbf{R}^N)$,

$$g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) \leq \|D^2g\|_0 \left|\frac{x+y}{2}\right|^2 \leq \|D^2g\|_0 |x-y|^2,$$

we calculate that

$$\begin{aligned} \text{tr } \xi^2 &\leq \|\sigma\|_2^2 |x-y|^4, \quad \text{tr } \eta^2 \leq \|\sigma\|_1^2 |x+y-2z|^2, \\ |\alpha| &\leq \|b\|_2 |x-y|^2, \quad |\beta| \leq \|b\|_1 |x+y-2z|, \end{aligned}$$

and that

$$\begin{aligned} J &\leq 4\lambda_0 \left| \frac{x+y}{2} - 2z \right|^2 + \text{tr } \xi^2 + 2 \text{tr } \xi \eta - \langle \alpha, x+y-2z \rangle \\ &\leq \lambda_1 |x+y-2z|^2 + \|\sigma\|_2^2 |x-y|^4 + 2 \|\sigma\|_1 \|\sigma\|_2 |x-y|^2 |x+y-2z| \\ &\quad + \|b\|_2 |x-y|^2 |x+y-2z| \\ &\leq \left(\lambda_1 + \frac{c_0 - \lambda_1}{2} \right) |x+y-2z|^2 \\ &\quad + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2 \right] |x-y|^4 \\ &= \left(\frac{c_0 + \lambda_1}{2} \right) |x+y-2z|^2 + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2 \right] |x-y|^4. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\left(2L + \frac{6|x-y|^2}{\varphi_1} \right) \text{tr } (\sigma(x) - \sigma(y))^2 \\ &\quad - 2 \left(L + \frac{|x-y|^2}{\varphi_1} \right) \langle b(x) - b(y), x-y \rangle + \frac{J}{\varphi_1} \\ &\leq 2 \left(L + \frac{|x-y|^2}{\varphi_1} \right) \lambda_0 |x-y|^2 + 4 \|\sigma\|_1^2 \frac{|x-y|^4}{\varphi_1} + \frac{c_0 + \lambda_1}{2\varphi_1} |x+y-2z|^2 \\ &\quad + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2 \right] \frac{|x-y|^4}{\varphi_1} \\ &\leq \left[\lambda_1 L + \lambda_1^+ + 4 \|\sigma\|_1^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2 \right] |x-y|^2 \\ &\quad + \frac{c_0 + \lambda_1}{2} |x+y-2z|, \end{aligned}$$

where $\lambda_1^+ = \max \{\lambda_1, 0\}$. We now choose $L > 0$ so that

$$\frac{c_0 + \lambda_1}{2} L \geq \lambda_1 L + \lambda_1^+ + 4 \|\sigma\|_1^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2,$$

i.e.,

$$L \geq \frac{2}{c_0 - \lambda_1} \left[\lambda_1^+ + 4 \|\sigma\|_1^2 + \frac{2}{c_0 - \lambda_1} (\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2 \right].$$

Then we have

$$(5.5) \quad \begin{aligned} & \left(2L + \frac{6|x-y|^2}{\varphi_1} \right) \text{tr}(\sigma(x) - \sigma(y))^2 - 2 \left(L + \frac{|x-y|^2}{\varphi_1} \right) \langle b(x) - b(y), x - y \rangle \\ & + \frac{J}{\varphi_1} \leq \frac{c_0 + \lambda_1}{2} (L|x-y|^2 + |x+y-2z|) \leq \frac{c_0 + \lambda_1}{2} \varphi(x, y, z) \end{aligned}$$

for all $x, y, z \in \mathbf{R}^N$.

Now, we observe that

$$(5.6) \quad \begin{aligned} f(x) + f(y) - 2f(z) &= f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \\ &+ 2\left(f\left(\frac{x+y}{2}\right) - f(z)\right) \leq \|f\|_2 |x-y|^2 + \|f\|_1 |x+y-2z| \\ &\leq \|Df\|_{W^{1,\infty}} \varphi_1(x, y, z). \end{aligned}$$

Noting that for any $g \in C_b^1(\mathbf{R}^N)$,

$$\begin{aligned} |g(x) - g(z)| &\leq \left| g(x) - g\left(\frac{x+y}{2}\right) \right| + \left| g\left(\frac{x+y}{2}\right) - g(z) \right| \\ &\leq \|g\|_1 \frac{|x-y|}{2} + \left(2 \|g\|_0 \|g\|_1 \frac{|x+y-2z|}{2} \right)^{1/2} \\ &\leq \|g\|_{W^{1,\infty}} \varphi_1(x, y, z)^{1/2}, \end{aligned}$$

we see that

$$\begin{aligned} & |(c(x) - c(z))(u(x) - u(z)) + (c(y) - c(z))(u(y) - u(z))| \\ & \leq (\|c\|_0 + \|c\|_1)(\|u\|_0 + \|u\|_1) \varphi_1(x, y, z) \leq \|c\|_{W^{1,\infty}} \|u\|_{W^{1,\infty}} \varphi_1(x, y, z) \end{aligned}$$

and hence

$$(5.7) \quad \begin{aligned} & c(x)u(x) + c(y)u(y) - 2c(z)u(z) \\ & \geq c(z)(u(x) + u(y) - 2u(z)) + (c(x) + c(y) - 2c(z))u(z) \end{aligned}$$

$$\begin{aligned}
& + (c(x) - c(z))(u(x) - u(z)) + (c(y) - c(z))(u(y) - u(z)) \\
& \geq c(z)(u(x) + u(y) - 2u(z)) - \|u\|_0 \|Dc\|_{W^{1,\infty}} \varphi_1(x, y, z) \\
& \quad - \|c\|_{W^{1,\infty}} \|u\|_{W^{1,\infty}} \varphi_1(x, y, z) \\
& \geq c(z)(u(x) + u(y) - 2u(z)) - 2\|c\|_{W^{2,\infty}} \|u\|_{W^{1,\infty}} \varphi_1(x, y, z).
\end{aligned}$$

Now we are ready to go into the proof. We shall show that

$$(5.8) \quad u(x) + u(y) - 2u(z) \leq \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}) \varphi(x, y, z)$$

for all $x, y, z \in \mathbf{R}^N$. By linearity, we then have

$$|u(x) + u(y) - 2u(z)| \leq \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}) \varphi(x, y, z)$$

for all $x, y, z \in \mathbf{R}^N$, from which follows the assertion (ii) of Theorem 5.

Fix any

$$M > \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}).$$

For $\varepsilon > 0$ and $\delta > 0$ we set

$$\Phi(x, y, z) = u(x) + u(y) - 2u(z) - M\varphi(x, y, z) - \delta|x|^2 - \varepsilon \quad \text{for } x, y, z \in \mathbf{R}^N.$$

We shall show that $\Phi \leq 0$ on $\mathbf{R}^{3N}$ for all $\varepsilon, \delta > 0$. To this end, suppose that $\sup \Phi > 0$ for some $\varepsilon > 0$ and $\delta = \delta_0 > 0$. Fix such $\varepsilon > 0$ and $\delta_0 > 0$, and fix $0 < \delta \leq \delta_0$, so that $\sup \Phi > 0$.

Let $(\hat{x}, \hat{y}, \hat{z}) \in \mathbf{R}^{3N}$ be a maximum point of Φ . Set $w(x, y, z) = u(x) - \delta|x|^2 + u(y) - 2u(z)$. Observe that $\varphi_1(\hat{x}, \hat{y}, \hat{z}) \neq 0$. We have

$$M(D\varphi(\hat{x}, \hat{y}, \hat{z}), D^2\varphi(\hat{x}, \hat{y}, \hat{z})) \in J^{2,+} w(\hat{x}, \hat{y}, \hat{z}).$$

By (5.3) and (5.4), we see that if we set

$$p = 2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \begin{pmatrix} \hat{x} - \hat{y} \\ \hat{y} - \hat{x} \\ 0 \end{pmatrix} + \frac{M}{\varphi_1} \begin{pmatrix} \hat{x} + \hat{y} - 2\hat{z} \\ \hat{x} + \hat{y} - 2\hat{z} \\ -2\hat{x} - 2\hat{y} + 4\hat{z} \end{pmatrix},$$

and

$$A = 2M \left(L + 3 \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{M}{\varphi_1} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix},$$

then

$$(p, A) \in J^{2,+} w(\hat{x}, \hat{y}, \hat{z}).$$

Here and hereafter φ_1 also denotes its value at $(\hat{x}, \hat{y}, \hat{z})$. Let $\theta > 1$. By the maximum principle for semicontinuous functions, there are $X, Y, Z \in \mathcal{S}^N$ such that

$$\begin{aligned} & \left(2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) (\hat{x} - \hat{y}) + \frac{M}{\varphi_1} (\hat{x} + \hat{y} - 2\hat{z}), X \right) \in \bar{J}^{2,+} u(\hat{x}) - 2\delta(\hat{x}, I), \\ & \left(2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) (\hat{y} - \hat{x}) + \frac{M}{\varphi_1} (\hat{x} + \hat{y} - 2\hat{z}), Y \right) \in \bar{J}^{2,+} u(\hat{y}), \\ & \quad + \left(\frac{M}{\varphi_1} (-2\hat{x} - 2\hat{y} + 4\hat{z}), Z \right) \in -2\bar{J}^{2,-} u(\hat{z}), \\ (5.9) \quad & \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \leq \theta M \left\{ 2 \left(L + 3 \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ & \quad \left. + \frac{1}{\varphi_1} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right\} \end{aligned}$$

From the first three we see that

$$\begin{aligned} & -\operatorname{tr} a(\hat{x})(X + I) + \left\langle b(\hat{x}), 2\delta\hat{x} + 2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) (\hat{x} - \hat{y}) + \frac{M}{\varphi_1} (\hat{x} + \hat{y} - 2\hat{z}) \right\rangle \\ & \quad + c(\hat{x})u(\hat{x}) \leq f(\hat{x}), \\ & -\operatorname{tr} a(\hat{y})Y + \left\langle b(\hat{y}), 2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) (\hat{y} - \hat{x}) + \frac{M}{\varphi_1} (\hat{x} + \hat{y} - 2\hat{z}) \right\rangle \\ & \quad + c(\hat{y})u(\hat{y}) \leq f(\hat{y}), \\ & -\operatorname{tr} a(\hat{z}) \left(-\frac{1}{2}Z \right) + \left\langle b(\hat{z}), \frac{M}{\varphi_1} (\hat{x} + \hat{y} - 2\hat{z}) \right\rangle + c(\hat{z})u(\hat{z}) \geq f(\hat{z}). \end{aligned}$$

From these we have

$$\begin{aligned} & -\operatorname{tr} (a(\hat{x})X + a(\hat{y})Y + a(\hat{z})Z) + 2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \langle b(\hat{x}) - b(\hat{y}), \hat{x} - \hat{y} \rangle \\ & \quad + \frac{M}{\varphi_1} \langle b(\hat{x}) + b(\hat{y}) - 2b(\hat{z}), \hat{x} + \hat{y} - 2\hat{z} \rangle \\ & \leq f(\hat{x}) + f(\hat{y}) - 2f(\hat{z}) - (c(\hat{x})u(\hat{x}) + c(\hat{y})u(\hat{y}) - 2c(\hat{z})u(\hat{z})) \\ & \quad + 2\delta \operatorname{tr} a(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle. \end{aligned}$$

From (5.9) we see that

$$\begin{aligned} \operatorname{tr}(a(\hat{x})X + a(\hat{y})Y + a(\hat{z})Z) &= \operatorname{tr} \left\{ (\sigma(\hat{x})\sigma(\hat{y})\sigma(\hat{z})) \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \begin{pmatrix} \sigma(\hat{x}) \\ \sigma(\hat{y}) \\ \sigma(\hat{z}) \end{pmatrix} \right\} \\ &\leq \theta M \left[2 \left(L + 3 \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \operatorname{tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 + \frac{1}{\varphi_1} \operatorname{tr}(\sigma(\hat{x}) + \sigma(\hat{y}) - 2\sigma(\hat{z}))^2 \right]. \end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned} 0 &\leq \theta M \left[2 \left(L + 3 \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \operatorname{tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 + \frac{1}{\varphi_1} \operatorname{tr}(\sigma(\hat{x}) + \sigma(\hat{y}) - 2\sigma(\hat{z}))^2 \right] \\ &\quad - 2M \left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1} \right) \langle b(\hat{x}) - b(\hat{y}), \hat{x} - \hat{y} \rangle \\ &\quad - \frac{M}{\varphi_1} \langle b(\hat{x} + b(\hat{y}) - 2b(\hat{z}), \hat{x} + \hat{y} - 2\hat{z}) \rangle \\ &\quad + f(\hat{x}) + f(\hat{y}) - 2f(\hat{z}) - (c(\hat{x})u(\hat{x}) + c(\hat{y})u(\hat{y}) - 2c(\hat{z})u(\hat{z})) \\ &\quad + 2\delta \operatorname{tr} a(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle. \end{aligned}$$

Sending $\theta \downarrow 1$ and using (5.5), (5.6) and (5.7), we have

$$\begin{aligned} 0 &\leq M \frac{c_0 + \lambda_1}{2} \varphi(\hat{x}, \hat{y}, \hat{z}) + \|Df\|_{W^{1,\infty}} \varphi(\hat{x}, \hat{y}, \hat{z}) \\ &\quad - c(\hat{z})(u(\hat{x}) + u(\hat{y}) - 2u(\hat{z})) + 2\|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}} \varphi(\hat{x}, \hat{y}, \hat{z}) \\ &\quad + 2\delta \operatorname{tr} a(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle. \end{aligned}$$

Since $\Phi(\hat{x}, \hat{y}, \hat{z}) > 0$ and $u \in BUC(\mathbf{R}^N)$, we have

$$u(\hat{x}) + u(\hat{y}) - 2u(\hat{z}) \geq M\varphi(\hat{x}, \hat{y}, \hat{z}) \quad \text{and} \quad \gamma \leq \varphi(\hat{x}, \hat{y}, \hat{z}) \leq \gamma^{-1},$$

where γ is a positive constant independent of $\delta > 0$. Hence,

$$(5.10) \quad 0 \leq \left(-\frac{c_0 - \lambda_1}{2} M + \|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}} \right) \varphi(\hat{x}, \hat{y}, \hat{z}) + C\delta^{1/2},$$

where C is a constant independent of δ . Moreover, sending $\delta \downarrow 0$ and (5.10) yield a contradiction. This proves that $\Phi \leq 0$ on $\mathbf{R}^{3N}$ for all $\varepsilon, \delta > 0$. It is now easily concluded that (5.8) holds. ■

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