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(Citation)
Funkcialaj Ekvacioj,39(3):389-397

(Issue Date)
1996-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100499911

(URL)
https://hdl.handle.net/20.500.14094/0100499911

Funkcialaj Ekvacioj, 39 (1996) 389–397

A Pohozaev-Type Inequality for a Quasilinear Haraux-Weissler Equation

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§ 1. Introduction

Since their first application in the work of Pohozaev [9], Pohozaev-identities have become a common tool to determine, whether solutions of certain classes of equations can possess zeroes or not. In this paper we want to present an application to a quasilinear porous medium equation with a source term

$$(PME) \qquad u_t - \Delta(|u|^{m-1}u) = |u|^{p-1}u \quad \text{in } \boldsymbol{R}^N \times (0, T),$$

where the parameters are taken as $m > 0$ and $p > 1$. In studying this equation, the so-called selfsimilar solutions

$$u(x, t) = t^{-\alpha}U(r), \qquad r = |x|t^{-\beta},$$

are of interest, as they usually describe the large time behaviour of solutions to the Cauchy problem with general initial data (see for instance [8]). Inserting this structure assumption, we arrive at the following initial value problem, which was first considered by Haraux & Weissler [5] for the semilinear case $m = 1$:

$$(P) \qquad \begin{aligned} &V'' + \frac{N-1}{r}V' + \beta rU' + \alpha U + |U|^{p-1}U = 0 \quad \text{in } \boldsymbol{R}^+, \\ &V = |U|^{m-1}U, \qquad \alpha, \beta > 0, \qquad \alpha(m-1) + 2\beta = 1, \\ &V'(0) = 0, \qquad U(0) = a > 0. \end{aligned}$$

In fact, if the equation originates from (PME), the parameters α, β have to fulfill the additional condition $\alpha p = p + 1$ and are thus given by

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{p-m}{2(p-1)}.$$

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[1] The research was done while the author was visiting Hokkaido University, Sapporo, Japan, supported by a postdoctoral fellowship of the Japan Society for the Promotion of Science, no. 93186.

Here, as also done in [1] and [5], we want to consider them as parameters only subject to the condition stated in (P).

To our knowledge the structure of solutions of equation (P) above was studied only in the semilinear case $m = 1$: In the works of Haraux & Weissler [5] and Atkinson & Peletier [1], among other results it was proved, that if p is supercritical, i.e.

$$N > 2 \qquad \text{and} \qquad p \geq \frac{N+2}{N-2},$$

then solutions of (P) with $\alpha \leq \max\{2, N/2\}$ cannot possess zeroes. In sharp contrast to this result solutions of (P) with subcritical p may have any number of zeroes, provided the initial value is large enough (see [12]).

Another aspect of problem (P) is that it can be regarded as the equation satisfied by radial solutions of a quasilinear elliptic equation with a gradient term

$$\Delta(|u|^{m-1}u) + \beta x \cdot \nabla u + \alpha u + |u|^{p-1}u = 0 \qquad \text{in } \boldsymbol{R}^N.$$

This equation was studied by Hirose [6] in the particular case $m = 1$ and $\alpha = 0$ (thus $\beta = 1/2$). Among other results he shows the nonexistence of zeroes for solutions of (P) in this setting for all supercritical p.

A similar equation was studied by Chipot & Weissler [2] and Serrin & Zou [11]. They study the influence of another gradient term on the solutions of

$$\Delta u + |u|^{p-1}u - |\nabla u|^q = 0 \qquad \text{in } \boldsymbol{R}^N.$$

They found that again for supercritical p the solutions cannot possess zeroes, provided q is small enough. The same equation without gradient term

$$\Delta u + |u|^{p-1}u = 0 \qquad \text{in } \boldsymbol{R}^N,$$

was also studied by various authors. Again the behaviour of solutions strongly depends on the relation of p to its critical value. For an overview on results on the latter equation we would like to refer to [7] and the literature cited therein.

A very general approach to Pohozaev-identities for quasilinear elliptic equations was made in Kawano, Ni & Yotsutani [7]. However, their equation—

$$\operatorname{div}(A(|\nabla u|)|\nabla u|) + f(|x|, u) = 0 \qquad \text{in } \boldsymbol{R}^N$$

does not include gradient terms of a form we consider here. Moreover their results are applicable to the generalized Laplace equation

$$A(v) = v^{m-2}$$

only if $1 < m < N$, thus not allowing arbitrary degeneracies in the leading term.

In order to state our results, let us introduce

$$k := \frac{\alpha}{\beta}.$$

Due to the condition on these two parameters given in (P), there is a bijection between k and (α, β), so that the parameter values are easily recovered from k. Please observe that in the singular case $m < 1$ the value of k can only range from 0 to $\frac{2}{1-m}$, whereas it can be any positive number in case that $m \geq 1$.

Theorem 1.1. *Let $N > 2$, $m \geq 0$ and*

$$\frac{p}{m} \geq \frac{N+2}{N-2}.$$

Then the solution U of (P) has no finite zero, regardless of the initial value $U(0) > 0$, provided

$$k < \frac{N}{m+1}.$$

Theorem 1.2. *Let $N > 2$, $m \in \left(0, \frac{N-2}{N}\right)$ and define $p_0(m, N)$ by*

$$p_0(m, N) := \max\left\{m\frac{(N+2)m - (N-4)}{N(1-m) - 2}, 1\right\}.$$

Then, if $p \geq p_0$ (m, N), the solution U of (P) has no finite zeroes, regardless of the value of $k \in \left[0, \frac{2}{1-m}\right)$ and the initial value $U(0) > 0$.

Corollary 1.3. *If $N > 2$ and $\frac{p}{m} \geq \frac{N+2}{N-2}$, then a selfsimilar solution of (PME) satisfying (P) with some $a \in \boldsymbol{R}^+$ has to be positive.*

The corollary is easily proved by observing that such a solution must satisfy (P) and that

$$k = \frac{2}{p-m} \leq \frac{N}{m+p} < \frac{N}{m+1},$$

due to the conditions imposed on p, m and N. Thus theorem 1.1 can be applied, which proves the corollary.

Before turning to the proof of the other results, we would like to give some comments.

Remark 1.4. (i) Theorem 1.2 cannot be extended to some $m > \frac{N-2}{N}$. In this range of m values for k larger than the space dimension N are admissable, and, as shown in [3], every solution of (P) with $k > N$ possesses a zero, regardless of p. $\left(\text{Note that } p_0(m, N) \to \infty \text{ if } m \to \frac{N-2}{N}.\right)$
Let us briefly indicate the idea of the proof: The equation (P) can be rewritten as

$$r^{N-1}V'(r) + \beta r^N U(r) = -\int_0^r t^{N-1}(\alpha - \beta N + |U(t)|^{p-1})U(t)dt.$$

Assuming a nonnegative solution U and taking $k > N$ into account, the right hand side of this equation is strictly negative for all r away from zero. But the terms on the left hand side can be estimated via the known asymptotic behaviour of solutions to (P), i.e.

$$0 \leq \lim_{r \to \infty} r^k U(r) < \infty$$

(see [4]). From this one readily derives that the terms on the left hand side converge to zero, a contradiction. Thus U must possess a sign change.

(ii) Theorem 1.1 cannot be extended to $\frac{p}{m} < \frac{N+2}{N-2}$. In this case it is shown in [3] by a blow-up technique similar to the one introduced in [12] for the semilinear case, that for sufficiently large initial value the solution U of (P) can possess an arbitrary large number of zeroes, regardless of the value of k.

(iii) Concerning the bound on k in theorem 1.1, we improve the result of [1] for the semilinear case $m = 1$ and $N \geq 4$. For general $m > 0$ this result may not be sharpened, as $\frac{N}{m+1} \to N$ if $m \to 0$, and—as mentioned already—any solution of (P) with $k > N$ possesses a zero. However, k ranges only up to $\frac{2}{1-m}$, it is not clear whether this result is optimal or not.

§ 2. Proof of the main results

In this chapter we will derive a Pohozaev-type inequality involving two additional parameters. By carefully choosing these parameters, we are able to prove theorems 1.1 and 1.2.

Lemma 2.1. *Let $m > 0$, $q > 2$, $r_0 > 0$ and let V be a solution of (P).*

Moreover let V be positive in $(0, r_0)$ and $V(r_0) = 0$. Then

$$\int_0^{r_0} r^q |U'| V \le \frac{m+1}{mq} \int_0^{r_0} r^{q+1} U'V'.$$

Proof. We just apply the Cauchy-Schwartz inequality and integration by parts:

$$\left(\int_0^{r_0} r^q |U'| V\right)^2 \le \frac{1}{m^2}\left(\int_0^{r_0} r^{q-1}|U|^{m+1}\right)\left(\int_0^{r_0} r^{q+1}|U|^{1-m}|V'|^2\right)$$

$$\le \frac{m+1}{mq}\left(\int_0^{r_0} r^q |U'| V\right)\left(\int_0^{r_0} r^{q+1}\frac{1}{m}|U|^{1-m}V'V'\right),$$

which is equivalent to the statement of the lemma, since $U' = \dfrac{1}{m}|U|^{1-m}V'$.

q.e.d.

After this preparation we can now turn to the Pohozaev-type inequality.

Proposition 2.2. *Let $m > 0$, $p > 1$, $r_0 > 0$ and let V be a solution of (P). Moreover let V be positive in $(0, r_0)$ and $V(r_0) = 0$. Then*

$$\frac{1}{2} r_0^q |V'(r_0)|^2 \le \left\{q\left(\frac{1}{2} + \frac{m}{m+p}\right) + 1 - N\right\} \int_0^{r_0} r^{q-1}|V'|^2$$

$$\frac{p-1}{m+p}\left\{\alpha\frac{m+1}{q} - \beta\right\}\int_0^{r_0} r^{q+1} U'V', \qquad \textit{if } q \ge N.$$

Proof. Extending an idea introduced in [10] we define

$$G(r) := \frac{1}{2}|V'(r)|^2 + \frac{\alpha m}{m+1}|V(r)|^{1+(1/m)} + \frac{m}{m+p}|V(r)|^{1+(p/m)} + \delta\frac{V(r)V'(r)}{r}.$$

Differentiating $r^q G(r)$, where q is an additional parameter to be chosen later, we calculate

$$\begin{aligned}
\{r^q G(r)\}' &= q r^{q-1} G(r) + r^q G'(r) \\
&= \frac{q}{2} r^{q-1}|V'|^2 + \frac{\alpha m q}{m+1} r^{q-1}|U|^{m+1} + \frac{mq}{m+p} r^{q-1}|U|^{p+m} + q\delta r^{q-2} VV' \\
&\quad + r^q V'(V'' + \alpha U + |U|^{p-1}U) + \delta r^{q-1}|V'|^2 + \delta r^{q-1} VV'' - \delta r^{q-2} VV' \\
&= \left(\frac{q}{2} + \delta\right) r^{q-1}|V'|^2 + \frac{\alpha m q}{m+1} r^{q-1}|U|^{m+1} + \frac{mq}{m+p} r^{q-1}|U|^{p+m} \\
&\quad + q\delta r^{q-2} VV' - (N-1) r^{q-1}|V'|^2 - \beta r^{q-1} U'V' \\
&\quad - N\delta r^{q-2} VV' - \beta\delta r^q VU' - \alpha\delta r^{q-1}|U|^{m+1} - \delta r^{q-1}|U|^{p+m}
\end{aligned}$$

$$= \left(\frac{q}{2} - N + 1 + \delta\right) r^{q-1}|V'|^2 + (q-N)\delta r^{q-2} VV'$$

$$- \beta r^{q+1} U'V' - \delta\beta r^q VU'$$

$$+ \alpha\left(\frac{mq}{m+1} - \delta\right) r^{q-1}|U|^{m+1} + \left(\frac{mq}{m+p} - \delta\right) r^{q-1}|U|^{p+m}.$$

Integrating from 0 to r_0, we see that in order to estimate the right hand side from above without integrals containing $|U|^p$, we have to choose δ greater or equal $\frac{mq}{m+p}$. Thus we set

$$\delta = \frac{mq}{m+p}$$

and therefore arrive at

$$\begin{aligned}\frac{1}{2} r_0^q |V'(r_0)|^2 = &\left\{ q\left(\frac{1}{2} + \frac{m}{m+p}\right) + 1 - N \right\} \int_0^{r_0} r^{q-1}|V'|^2 \\ &+ \frac{mq}{m+p}(N-q) \int_0^{r_0} r^{q-2}|V'|V \\ &- \beta \int_0^{r_0} r^{q+1} U'V' + \beta \frac{mq}{m+p} \int_0^{r_0} r^q V|U'| \\ &+ \alpha \frac{mq}{m+p} \frac{p-1}{m+1} \int_0^{r_0} r^{q-1}|U|^{m+1},\end{aligned}$$

using that $V' \leq 0$ on $(0, r_0)$. We now integrate by parts in the last integral and use Lemma 1.1 to estimate the last two integrals in terms of $\int r^{q+1} U'V'$; this gives

$$\begin{aligned}\frac{1}{2} r_0^q |V'(r_0)|^2 \leq &\left\{ q\left(\frac{1}{2} + \frac{m}{m+p}\right) + 1 - N \right\} \int_0^{r_0} r^{q-1}|V'|^2 \\ &+ \frac{mq}{m+p}(N-q) \int_0^{r_0} r^{q-2}|V'|V \\ &+ \frac{p-1}{m+p} \left\{ \alpha \frac{m+1}{q} - \beta \right\} \int_0^{r_0} r^{q+1} U'V'.\end{aligned}$$

If we choose $q \geq N$, then we can just omit the second integral on the right hand side and the proposition is proved. q.e.d

We will now apply the Pohozaev inequality to determine parameter ranges, where solutions of (P) cannot possess a finite zero and thereby prove our results.

Proof of Theorem 1.1. *and* 1.2. First let us clarify the effect of q on the coefficients on the right hand side of the Pohozaev-inequality. As can be easily seen, they are nonpositive, if

$$N \le q < 2(N-1) \tag{1}$$

$$k \le \frac{q}{m+1} \tag{2}$$

$$\frac{p}{m} \ge 1 + \frac{4(q-(N-1))}{2(N-1)-q}. \tag{3}$$

So if we can choose a q satisfying (1), such that these conditions are fulfilled by p, m, N and k, and at least one of the inequalities (2) and (3) is strict, the assumption that U has a finite zero r_0 immediately leads to a contradiction, since Proposition 2.2 then implies

$$\frac{1}{2} r_0^q |V'(r_0)|^2 < 0.$$

It remains to show, that this is the case for the parameter ranges described in theorem 1.1 and 1.2.

If we choose $q = N$ in Proposition 2.2, we immediately get theorem 1.1 by evaluating the conditions (2) and (3). Concerning theorem 1.2, we observe that $k \in \left(0, \frac{2}{1-m}\right)$. So in order to cover all possible values of k in (2), we should set

$$q = \frac{2}{1-m}(m+1).$$

If $m \in \left[\frac{N-2}{N+2}, \frac{N-2}{N}\right)$, this q lies in the interval $[N, 2(N-1))$ and we apply Proposition 2.2 again.

This time the evaluation of (2) yields

$$\begin{aligned}\frac{p}{m} &\ge 1 + \frac{4(m+1) + 2(m-1)(N-1)}{(N-1)(1-m)-(m+1)} \\ &= \frac{N(1-m) - 2 + 4m + 4 - 2N(1-m) + 2 - 2m}{N(1-m)-2} \\ &= \frac{(N+2)m - (N-4)}{N(1-m)-2}.\end{aligned}$$

If $m \in \left(0, \dfrac{N-2}{N+2}\right)$, it suffices to take $q = N$, as in this range of m we have

$$k < \frac{2}{1-m} < \frac{N}{m+1};$$

so this case is already covered in theorem 1.1 and we are done.

Remark 2.4. One could ask the question whether the results can be sharpened by taking q less than N. In this case we have to estimate the integral

$$\frac{mq}{m+p}(N-q)\int_0^{r_0} r^{q-2}|V'|V$$

at the end of the proof of proposition 2.2, since now the coefficient is positive. This can be done analogously to lemma 2.1 by using the Cauchy-Schwartz inequality and integration by parts, which gives

$$\int_0^{r_0} r^{q-2}|V'|V \leq \frac{2}{q-2}\int_0^{r_0} r^{q-1}|V'|^2.$$

Proceeding as above we then obtain that the coefficients in the integrals on the right hand side of the Pohozaev-inequality are nonpositive, if

$$k \leq \frac{q}{m+1} \quad \text{and} \quad \frac{p}{m} \geq \frac{q+2}{q-2}, \quad \text{if } 2 < q \leq N.$$

Thus this range of possible q does not give sharper results.

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(Ricevita la 18-an de novembro, 1994)
(Reviziita la 4-an de aprilo, 1995)