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Watanabe, Humihiko

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On the Defining Variety and Birational Canonical Transformations of the Fourth Painlevé Equation

By

Humihiko WATANABE

(Kyushu University, Japan)

§ 0. Introduction

In a previous paper [6], we pointed out the geometric meaning of birational canonical transformations of the Painlevé equations, and constructed those of the sixth Painlevé equation in a simple manner. In this paper, applying the method used for the sixth equation, we give a simple construction of birational canonical transformations of the fourth Painlevé equation coming from the affine Weyl group of type A_2 . In Ref. [4] Okamoto constructed such transformations by exploiting the symmetry of a second order differential equation which is satisfied by a Hamiltonian function of the fourth equation. We review the construction of the defining variety E of the fourth Painlevé equation to obtain a new symplectic patching for E which possesses the symmetry of a cyclic group of order 3 (Theorem 1). We then explicitly write out generators of the group of the birational canonical transformations of the fourth Painlevé equation by taking into consideration an affine Weyl group symmetry of the Hamiltonian structure on E (Theorem 2). Our method seems to be simpler than previous contributions to this subject.

§ 1. Defining variety

Let C^3 be the three-dimensional complex cartesian space with standard coordinate system $\mathbf{a} = (a_1, a_2, a_3)$, and let V be the hyperplane in C^3 defined by $a_1 + a_2 + a_3 = 0$. We set $B = C$ and denote its standard coordinate by t . Let (q, p) be the standard coordinate system of the two-dimensional complex cartesian space C^2 . As discussed in Section 2 of Ref. [2], a Hamiltonian H of the fourth Painlevé equation is given as a polynomial function on the product space $V \times B \times C^2$ by the following equation (see also [4]):

$$(1) \quad H = qp^2 - q^2p - 2tqp - 2(a_1 - a_2)p + 2(a_3 - a_2)q.$$

This Hamiltonian is identical to that used in Refs. [2], [4] and [5], and the parameter $\mathbf{a}$ used here is related to $\mathbf{v} = (v_1, v_2, v_3)$ used in those papers as

follows:

$$v_1 = a_2, \quad v_2 = a_1, \quad v_3 = a_3.$$

The fourth Painlevé equation is equivalent to the Hamiltonian system

$$(2) \quad \begin{cases} \frac{dq}{dt} = \frac{\partial H}{\partial p} = 2qp - q^2 - 2tq - 2(a_1 - a_2), \\ \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -p^2 + 2qp + 2tp - 2(a_3 - a_2), \end{cases}$$

with Hamiltonian H defined on $V \times B \times \mathbb{C}^2$.

In Ref. [3] Okamoto constructed a complex analytic fibration $(E, \pi, V \times B)$, E being a five-dimensional complex smooth variety and $\pi: E \rightarrow V \times B$ the projection, such that the system (2) defines a foliation of codimension 2 in E which has no singularity except that of the first class (see [3], Définition 1.4), and such that the variety E is maximal with respect to inclusion among five-dimensional complex smooth varieties possessing such a property. In this paper we call the variety E the *defining variety* of the fourth Painlevé equation. To obtain the variety E , Okamoto first considered a foliation induced by the fourth equation on a rational surface bundle over $V \times B$ whose fibres consist of the Hirzebruch surface of degree 2 and its deformations isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and fibre-wise resolved the singularities of the foliation which are not of the first class through successive blowing-up of the rational surface bundle. In Okamoto's approach, this is done until the foliation induced on the resulting variety $\bar{E}$ contains only singularities of the first class. Finally, the variety E is obtained by subtracting the vertical leaves ([3], Définition 1.5) of each fibre from $\bar{E}$. In a subsequent work, Matano, Matsumiya, and Takano [1] obtained a concrete symplectic patching for E (see also [7]).

For our purposes (see §2), we need a new symplectic patching for E . Let U, U_1, U_2, U_3 be copies of $V \times B \times \mathbb{C}^2$ with the coordinate systems $(a; t; q, p)$ for U and $(a; t; Q_i, P_i)$ for U_i ($i = 1, 2, 3$). Then our new description is as follows:

Theorem 1. *The defining variety E is obtained by patching the four copies U, U_1, U_2, U_3 of $V \times B \times \mathbb{C}^2$ by the following relations:*

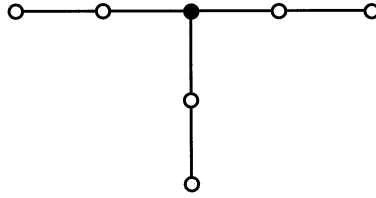
$$(3) \quad qQ_1 = 1, \quad qp + Q_1P_1 = 2(a_3 - a_2);$$

$$(4) \quad qp + Q_2P_2 = 2(a_1 - a_2), \quad pP_2 = 1;$$

$$(5) \quad qQ_3 = 1, \quad q(p - q - 2t) + Q_3P_3 = 2(a_1 - a_3 - 1).$$

Moreover, each relation fibre-wise defines a symplectic coordinate transformation between the corresponding two open sets among U, U_i ($i = 1, 2, 3$).

While Theorem 1 follows easily from Theorem 2 in Ref. [1] which comes from Okamoto's construction of the defining variety, we can directly obtain Theorem 1 by a modified construction of the variety, as explained in the following. We form the product $V \times B \times \mathbf{P}^2$, $\mathbf{P}^2$ denoting the complex projective plane, in such a manner that $V \times B \times \mathbf{P}^2$ naturally contains $U = V \times B \times \mathbf{C}^2$ as an open set, and we consider the foliation induced by (2) in $V \times B \times \mathbf{P}^2$. Here we note that the subtraction of U from $V \times B \times \mathbf{P}^2$ gives $V \times B \times L$, where L denotes the line at infinity in $\mathbf{P}^2$ with respect to the coordinate system (q, p) with self-intersection number $L^2 = 1$. Then, one can see that on every line L of every fibre $\mathbf{P}^2$ of the fibration $V \times B \times \mathbf{P}^2 \rightarrow V \times B$, the foliation in $V \times B \times \mathbf{P}^2$ has three singularities which are not of the first class. Blowing up each singularity three times, we obtain a rational surface bundle over $V \times B$ isomorphic to Okamoto's $\bar{E}$ in which the foliation induced by (2) has only singularities of the first class. Finally, from each fibre of the new fibration, subtracting seven curves each with self-intersection number -2 , we obtain the desired space E . It is easy to see that the four open sets U, U_i ($i = 1, 2, 3$) with the relations (3)–(5) cover the total space E . The configuration of the seven (-2) -curves in a given fibre is represented by the following extended Dynkin diagram of type $\bar{E}_6$:



Here the black dot represents the strict transform of the original line L .

§2. Birational canonical transformations

We can extend the Hamiltonian structure on $U = V \times B \times \mathbf{C}^2$ defined by (2) to the whole space E discussed in Section 1. To this end it is sufficient to obtain a polynomial function H_i ($i = 1, 2, 3$) on each open set U_i of E such that the three relations

$$(6) \quad dp \wedge dq - dH \wedge dt = dP_i \wedge dQ_i - dH_i \wedge dt \quad (i = 1, 2, 3),$$

hold on $U \cap U_i$, where d denotes the exterior differential with respect to the variables t, q, p, Q_i, P_i ($i = 1, 2, 3$) (therefore, $da_i = 0$). From the formulae (3)–(6), it follows easily that $H - H_1, H - H_2$, and $H - H_3 + 2Q_3^{-1}$ are rational functions only in the variables t and a_i ($i = 1, 2, 3$). Without loss of generality,

we may assume that the following hold:

(i) on $U \cap U_1$,

$$(7) \quad H = H_1 - 4t(a_3 - a_2);$$

(ii) on $U \cap U_2$,

$$(8) \quad H = H_2 - 4t(a_1 - a_2);$$

(iii) on $U \cap U_3$,

$$(9) \quad H = H_3 - \frac{2}{Q_3} + 4t(a_2 - a_3 - 1).$$

Substitution of (1) into (7)–(9) implies that

$$H_1 = Q_1^3 P_1^2 + 2(a_1 + a_2 - 2a_3)Q_1^2 P_1 + 2tQ_1 P_1 + P_1 + 4(a_3 - a_2)(a_3 - a_1)Q_1,$$

$$H_2 = -Q_2^2 P_2^3 + 2(2a_1 - a_2 - a_3)Q_2 P_2^2 + 2tQ_2 P_2 - 4(a_1 - a_2)(a_1 - a_3)P_2 - Q_2,$$

$$H_3 = Q_3^3 P_3^2 + 2(2a_3 + 2 - a_1 - a_2)Q_3^2 P_3 - 2tQ_3 P_3 - P_3$$

$$+ 4(a_1 - a_3 - 1)(a_2 - a_3 - 1)Q_3.$$

Each of these expressions is a polynomial function on its respective open set. The Hamiltonian structure on E thus obtained determines a unique foliation $\mathcal{F}$ in E . In fact, the restriction of the foliation $\mathcal{F}$ to the open set U is defined by (2), and that to each open set U_i ($i = 1, 2, 3$) is defined by the Hamiltonian system

$$\begin{cases} \frac{dQ_i}{dt} = \frac{\partial H_i}{\partial P_i}, \\ \frac{dP_i}{dt} = -\frac{\partial H_i}{\partial Q_i}. \end{cases}$$

We now explain our construction of birational canonical transformations on E which come from the affine Weyl group of type A_2 . To this end we introduce the three affine transformations s_0 , s_1 and s_2 of V by $s_0(\mathbf{a}) = (a_3 + 1, a_2, a_1 - 1)$, $s_1(\mathbf{a}) = (a_2, a_1, a_3)$ and $s_2(\mathbf{a}) = (a_1, a_3, a_2)$. These are reflections of V . Let W_a be the subgroup generated by s_0 , s_1 and s_2 in the group of all affine transformations of V . The group W_a is isomorphic to the affine Weyl group of the root system of type A_2 . Okamoto [4] proved the following theorem regarding W_a . We give an alternative proof using a simple method.

Theorem 2. *For every element $g \in W_a$, there exists a birational canonical*

transformation γ of E such that the diagram

$$(10) \quad \begin{array}{ccc} E & \xrightarrow{\gamma} & E \\ \pi \downarrow & & \downarrow \pi \\ V & \xrightarrow{g} & V \end{array}$$

is commutative. Here π denotes the natural projection $E \rightarrow V$.

Proof. Since the group W_a is generated by the three elements s_0, s_1 and s_2 , it is sufficient to construct birational transformations σ_0, σ_1 and σ_2 of E such that the diagram (10) is commutative for $\gamma = \sigma_i$ and $g = s_i$ ($i = 0, 1, 2$).

(i) Construction of σ_1 . Since the Hamiltonian H_1 is invariant under s_1 , we have the locally trivial birational canonical transformation σ_1 on U_1 :

$$\sigma_1(\mathbf{a}; t; Q_1, P_1; H_1) = (s_1(\mathbf{a}); t; Q_1, P_1; H_1).$$

Extending the domain of definition of σ_1 to U by (3) and (7), we have on U

$$\sigma_1(\mathbf{a}; t; q, p; H) = \left(s_1(\mathbf{a}); t; q, p + \frac{2(a_2 - a_1)}{q}; H + 4t(a_1 - a_2) \right).$$

In a similar way, the birational canonical transformation σ_1 on $U \cup U_1$ can be extended to the whole space E .

(ii) Construction of σ_2 . Since the Hamiltonian H_2 is invariant under s_2 , we have the locally trivial birational canonical transformation σ_2 on U_2 :

$$\sigma_2(\mathbf{a}; t; Q_2, P_2; H_2) = (s_2(\mathbf{a}); t; Q_2, P_2; H_2).$$

Extending the domain of definition of σ_2 to U by (4) and (8), we have on U

$$\sigma_2(\mathbf{a}; t; q, p; H) = \left(s_2(\mathbf{a}); t; q + \frac{2(a_2 - a_3)}{p}, p; H + 4t(a_2 - a_3) \right).$$

In a similar way, the birational canonical transformation σ_2 on $U \cup U_2$ can be extended to the whole space E .

(iii) Construction of σ_0 . Let z_0 be the affine transformation of V defined by $z_0(\mathbf{a}) = (a_3 + 2/3, a_1 - 1/3, a_2 - 1/3)$ ([5], §3; see also [4], §4). We introduce the following birational transformation ζ of $U = V \times B \times \mathbb{C}^2$:

$$\zeta(\mathbf{a}; t; q, p) = (z_0(\mathbf{a}); t; p - q - 2t, -q).$$

It is easy to see that ζ^3 is the identity. Next, we introduce a new polynomial function $\bar{H}$ on the open set $U = \zeta(U)$ such that the relation

$$dp \wedge dq - dH \wedge dt = d\bar{p} \wedge d\bar{q} - d\bar{H} \wedge dt$$

holds, where $\bar{q} = p - q - 2t$, $\bar{p} = -q$. Then it follows that $H - \bar{H} - 2\bar{p}$ is a rational function only in the variables t and a_i . Without loss of generality, we may assume that

$$H = \bar{H} + 2\bar{p} - 4t(a_1 - a_2).$$

Thus the explicit form of $\bar{H}$ is

$$\bar{H} = \bar{q}\bar{p}^2 - \bar{q}^2\bar{p} - 2t\bar{q}\bar{p} - 2(\bar{a}_1 - \bar{a}_2)\bar{p} + 2(\bar{a}_3 - \bar{a}_2)\bar{q},$$

where $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) = z_0(\mathbf{a})$. Therefore the transformation on U

$$\zeta(\mathbf{a}; t; q, p; H) = (\bar{\mathbf{a}}; t; \bar{q}, \bar{p}; \bar{H})$$

is canonical.

With respect to the new coordinate system $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ of V , the transformation s_0 is represented as $s_0(\bar{a}_1, \bar{a}_2, \bar{a}_3) = (\bar{a}_2, \bar{a}_1, \bar{a}_3)$. By the same construction as used for the transformation σ_1 , we have the birational canonical transformation σ_0 on U :

$$\sigma_0(\bar{\mathbf{a}}; t; \bar{q}, \bar{p}; \bar{H}) = \left(\bar{a}_2, \bar{a}_1, \bar{a}_3; t; \bar{q}, \bar{p} + \frac{2(\bar{a}_2 - \bar{a}_1)}{\bar{q}}; \bar{H} + 4t(\bar{a}_1 - \bar{a}_2) \right).$$

Then, using the original coordinate system $(\mathbf{a}; t; q, p)$, we have the equivalent expression

$$\begin{aligned} \sigma_0(\mathbf{a}; t; q, p; H) = & \left(s_0(\mathbf{a}); t; q - \frac{2(a_1 - a_3 - 1)}{p - q - 2t}, p - \frac{2(a_1 - a_3 - 1)}{p - q - 2t}; \right. \\ & \left. H + \frac{4(a_1 - a_3 - 1)}{p - q - 2t} + 4t(a_1 - a_2) \right). \end{aligned}$$

In a similar way, the birational canonical transformation σ_0 on U can be extended to the whole space E , although we omit the details here. Hence Theorem 2 is proved.

Moreover we have

Corollary 3. *Let $\mathbf{B}$ be the subgroup of birational canonical transformations of E generated by the three transformations $\sigma_0, \sigma_1, \sigma_2$ in the proof of Theorem 2. Then $\mathbf{B}$ is isomorphic to the group W_a , and therefore to the affine Weyl group of type A_2 .*

Proof. Obviously, we have a homomorphism $\mathbf{B} \rightarrow W_a$ by the diagram (10). By the universality of the Coxeter system $(W_a, \{s_0, s_1, s_2\})$, we have another homomorphism $W_a \rightarrow \mathbf{B}$ (cf. [8]). Obviously, these homomorphisms are mutually reciprocal. Thus the corollary follows.

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nuna adreso:

Graduate School of Mathematics

Kyushu University

Fukuoka 810-8560

Japan

E-mail: hwatanab@rcmath.rc.kyushu-u.ac.jp

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