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Existence and Nonexistence of Positive Entire Solutions of Second Order Semilinear Elliptic Systems

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1. Introduction

In this paper we consider second order semilinear elliptic systems of the form

$$(1.1) \quad \begin{cases} \Delta u = p(x)v^\alpha, \\ \Delta v = q(x)u^\beta, \end{cases} \quad \text{in } \mathbf{R}^n,$$

where $\alpha, \beta > 0$, and p and q are nonnegative continuous functions in $\mathbf{R}^n$, $n \geq 1$. We are interested in necessary and/or sufficient conditions for (1.1) to have positive entire solutions. An *entire solution* of (1.1) is defined to be a vector-valued function $(u, v) \in C^2(\mathbf{R}^n) \times C^2(\mathbf{R}^n)$ which satisfies (1.1) at every point x in $\mathbf{R}^n$.

The problem of existence and nonexistence of positive entire solutions of *scalar* elliptic equations has been investigated by many authors under various situations. To illustrate some of known results let us consider the equation

$$(1.2) \quad \Delta u = p(x)u^\sigma \quad \text{in } \mathbf{R}^n,$$

where p is a nonnegative continuous function on $\mathbf{R}^n$, and $\sigma > 0$. Roughly speaking, if

$$p(x) \leq \frac{C}{|x|^{2+\varepsilon}} \quad \text{near } \infty$$

for some constants $C, \varepsilon > 0$, then (1.2) has positive entire solutions. The precise formulation for existence theorems with such features are found in [3, 5, 7]. On the other hand, if

$$\sigma > 1 \quad \text{and} \quad p(x) \geq \frac{C}{|x|^2} \quad \text{near } \infty$$

for some constant $C > 0$, then (1.2) does not admit any positive entire solutions. This fact follows from the results in [1].

Next, we turn our attention to system (1.1). Of course we have already

known several works treating existence theorems of positive entire solutions. For example, if $n \geq 3$, and

$$p(x) \leq \frac{C}{|x|^{2+\varepsilon}}, \quad q(x) \leq \frac{C}{|x|^{2+\varepsilon}} \quad \text{near } \infty$$

for some constants $C, \varepsilon > 0$, then (1.1) can be shown to have positive entire solutions (u, v) satisfying

$$\lim_{|x| \rightarrow \infty} (u(x), v(x)) = (u_\infty, v_\infty)$$

for some constants $u_\infty, v_\infty > 0$. See the papers [3, 4] for the detail. However, as far as the author knows, there are no results which contain nonexistence criteria of positive entire solutions to (1.1) except for the work by Deng [2]. Motivated by this fact, we will study mainly this problem here. When p and q behave like positive constant multiples of $|x|^\nu$, $\nu \in \mathbf{R}$, and $n \geq 3$, we can completely characterize the existence property of positive entire solutions of (1.1).

The organization of the paper is as follows. In §2 we give nonexistence criteria of positive entire solutions for (1.1) under the basic assumption $\alpha, \beta > 1$. In §3, to reinforce our nonexistence criteria, we give existence theorems of positive entire solutions for (1.1) under the basic assumption that $\alpha\beta > 1$ and p and q have spherical symmetry.

2. Nonexistence results

In this section, we consider nonexistence of positive entire solutions of (1.1). Through the section we always assume $\alpha, \beta > 1$. For a positive entire solution (u, v) of (1.1) we denote its spherical mean $(\bar{u}(r), \bar{v}(r))$, $r \geq 0$, by

$$(2.1) \quad \begin{aligned} \bar{u}(r) &= \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x) dS, \\ \bar{v}(r) &= \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} v(x) dS, \end{aligned}$$

where dS denotes the volume element in the surface integral, ω_n denotes the surface area of the unit sphere in $\mathbf{R}^n$. Moreover we introduce the function $\hat{p}(r)$ and $\hat{q}(r)$, $r \geq 0$, by

$$(2.2) \quad \begin{aligned} \hat{p}(r) &= \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{p(x)^{\alpha'/\alpha}} \right)^{-\alpha/\alpha'}, \\ \hat{q}(r) &= \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{q(x)^{\beta'/\beta}} \right)^{-\beta/\beta'}, \end{aligned}$$

respectively, where $1/\alpha + 1/\alpha' = 1, 1/\beta + 1/\beta' = 1$. The functions $\hat{p}$ and $\hat{q}$ can be regarded as spherical means of p and q , respectively, in some sense. We note that $\hat{p} = p$ and $\hat{q} = q$ when p and q are spherically symmetry.

Theorem 2.1. *Let $n \geq 3$. Suppose that $\hat{p}$ and $\hat{q}$ satisfy*

$$(2.3) \quad \hat{p}(r) \geq \frac{K_1}{r^\lambda}, \quad \hat{q}(r) \geq \frac{K_2}{r^\mu} \quad \text{for } r \geq r_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda \leq \alpha(2 - \mu) + 2 & \text{or} \\ \mu \leq \beta(2 - \lambda) + 2. \end{cases}$$

Then (1.1) does not possess any positive entire solutions.

Proof. It suffices to treat the case that $\lambda \leq 2, \mu \leq \beta(2 - \lambda) + 2$. The proof is done by contradiction. Assume the contrary that (1.1) has a positive entire solution (u, v) . Then by taking the mean value of (1.1) and using Hölder's inequality, we see that $(\bar{u}(r), \bar{v}(r))$ satisfies the differential inequalities

$$(2.4) \quad \begin{cases} (r^{n-1}\bar{u}'(r))' \geq r^{n-1}\hat{p}(r)\bar{v}^\alpha(r), & (r^{n-1}\bar{v}'(r))' \geq r^{n-1}\hat{q}(r)\bar{u}^\beta(r), & r > 0, \\ \bar{u}(0) = a > 0, \bar{u}'(0) = 0, & \bar{v}(0) = b > 0, \bar{v}'(0) = 0. \end{cases}$$

See, for example, [1, 7, 8]. Integrating (2.4) twice over $[0, r]$, we have

$$(2.5) \quad \begin{aligned} \bar{u}(r) &\geq a + \frac{1}{n-2} \int_0^r s \hat{p}(s) \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \bar{v}^\alpha(s) ds, \\ \bar{v}(r) &\geq b + \frac{1}{n-2} \int_0^r s \hat{q}(s) \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \bar{u}^\beta(s) ds. \end{aligned}$$

(I) *The case $\lambda < 2$ and $\mu < \beta(2 - \lambda) + 2$:* First we show that

$$(2.6) \quad \bar{u}(r) \geq C_1 r^{2-\lambda} \quad \text{for } r \geq r_1 > 0,$$

where $C_1 > 0$ is some constant. Let $r > r_0$. Then from (2.3) and (2.5) we have

$$\begin{aligned} \bar{u}(r) &\geq a + \frac{1}{n-2} \int_0^r s \hat{p}(s) \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \bar{v}^\alpha(s) ds \\ &\geq a + \frac{1}{n-2} \int_{r_0}^r s \hat{p}(s) \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \bar{v}^\alpha(s) ds \\ &\geq a + \frac{b^\alpha K_1}{n-2} \int_{r_0}^r s^{1-\lambda} \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] ds \end{aligned}$$

$$\begin{aligned}
&\geq a + \frac{b^\alpha K_1}{n-2} \left[1 - \left(\frac{1}{2} \right)^{n-2} \right] \int_{r_0}^{r/2} s^{1-\lambda} ds \\
&\geq C_1 r^{2-\lambda}, \quad r \geq r_1 > 2r_0.
\end{aligned}$$

Thus we obtain (2.6).

Let us fix $R > r_1$ arbitrarily. Integrating (2.4) twice over $[R, r]$, we have

$$(2.7) \quad \bar{u}(r) \geq \bar{u}(R) + \frac{1}{n-2} \int_R^r s \hat{p}(s) \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \bar{v}^\alpha(s) ds, \quad r \geq R.$$

Using the inequality

$$s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \geq \frac{n-2}{2^{n-2}} (r-s) \quad \text{for } R \leq s \leq r \leq 2R,$$

in (2.7), we obtain

$$\begin{aligned}
\bar{u}(r) &\geq \bar{u}(R) + \frac{K_1}{2^{n-2}} \int_R^r s^{-\lambda} (r-s) \bar{v}^\alpha(s) ds \\
&\geq \frac{C_2}{R^\lambda} \int_R^r (r-s) \bar{v}^\alpha(s) ds \quad \text{for } R \leq r \leq 2R,
\end{aligned}$$

where $C_2 > 0$ is a constant independent of R . Similarly

$$\bar{v}(r) \geq \frac{C_3}{R^\mu} \int_R^r (r-s) \bar{u}^\beta(s) ds \quad \text{for } R \leq r \leq 2R,$$

where $C_3 > 0$ is a constant independent of R . Let

$$(2.8) \quad f(r) = \frac{C_2}{R^\lambda} \int_R^r (r-s) \bar{v}^\alpha(s) ds, \quad R \leq r \leq 2R,$$

$$(2.9) \quad g(r) = \frac{C_3}{R^\mu} \int_R^r (r-s) \bar{u}^\beta(s) ds, \quad R \leq r \leq 2R.$$

Then

$$\begin{aligned}
f(R) &= f'(R) = g(R) = g'(R) = 0, \\
f'(r) &= \frac{C_2}{R^\lambda} \int_R^r \bar{v}^\alpha(s) ds \geq 0, \quad R \leq r \leq 2R, \\
g'(r) &= \frac{C_3}{R^\mu} \int_R^r \bar{u}^\beta(s) ds \geq 0, \quad R \leq r \leq 2R, \\
(2.10) \quad f''(r) &= \frac{C_2}{R^\lambda} \bar{v}^\alpha(r) \geq \frac{C_2}{R^\lambda} g^\alpha(r), \quad R \leq r \leq 2R,
\end{aligned}$$

and

$$(2.11) \quad g''(r) = \frac{C_3}{R^\mu} \bar{u}^\beta(r) \geq \frac{C_3}{R^\mu} f^\beta(r), \quad R \leq r \leq 2R.$$

Now, we discuss only on the interval $[R, 2R]$ for a moment. Multiplying (2.10) by g' and integrating over $[R, r]$, we obtain

$$\int_R^r f''(s)g'(s)ds \geq \frac{C_2}{R^\lambda(\alpha+1)} g^{\alpha+1}(r), \quad R \leq r \leq 2R.$$

Integrating by parts, we have

$$f'(r)g'(r) - \int_R^r f'(s)g''(s)ds \geq \frac{C_2}{R^\lambda(\alpha+1)} g^{\alpha+1}(r), \quad R \leq r \leq 2R.$$

Hence

$$f'(r)g'(r) \geq \frac{C_4}{R^\lambda} g^{\alpha+1}(r), \quad R \leq r \leq 2R,$$

where $C_4 = C_2/(\alpha+1)$. Multiplying this relation by g' and integrating by parts, we obtain

$$f(r)\{g'(r)\}^2 \geq \frac{C_5}{R^\lambda} g^{\alpha+2}(r), \quad R \leq r \leq 2R,$$

where $C_5 = C_4/(\alpha+2)$. From (2.11), we have

$$g''(r)\{g'(r)\}^{2\beta} \geq \frac{C_6}{R^{\lambda\beta+\mu}} g^{(\alpha+2)\beta}(r), \quad R \leq r \leq 2R,$$

where $C_6 = C_3 C_5^\beta$. Multiplying this inequality by g' and integrating, we obtain

$$g'(r) \geq C_7 R^{-(\lambda\beta+\mu)/(2\beta+2)} \{g(r)\}^{((\alpha+2)\beta+1)/(2\beta+2)}, \quad R \leq r \leq 2R,$$

where $C_7 = \{C_6(2\beta+2)((\alpha+2)\beta+1)^{-1}\}^{1/(2\beta+2)}$. Since $\{(\alpha+2)\beta+1\}/(2\beta+2) > 1$, we may set $\{(\alpha+2)\beta+1\}/(2\beta+2) = \delta+1, \delta > 0$. Thus

$$g'(r)\{g(r)\}^{-\delta-1} \geq C_7 R^{-(\lambda\beta+\mu)/(2\beta+2)}, \quad R \leq r \leq 2R.$$

Integrating this relation over $[3R/2, 2R]$, we have

$$(2.12) \quad \frac{1}{\delta} \left\{ g\left(\frac{3}{2}R\right) \right\}^{-\delta} \geq \frac{C_7}{2} R^{-(\lambda\beta+\mu)/(2\beta+2)+1}.$$

Since $-(\lambda\beta+\mu)/(2\beta+2)+1 > 0$, the right hand side of (2.12) tends to ∞ as $R \rightarrow \infty$. On the other hand, from (2.6) and (2.9), we have

$$\begin{aligned}
g\left(\frac{3}{2}R\right) &\geq \frac{C_3}{R^\mu} \int_R^{3R/2} \left(\frac{3}{2}R - s\right) (C_1 s^{2-\lambda})^\beta ds \\
&\geq \frac{C_3 C_1^\beta R^{(2-\lambda)\beta}}{R^\mu} \int_R^{3R/2} \left(\frac{3}{2}R - s\right) ds \\
&= C_8 R^{\beta(2-\lambda)+2-\mu},
\end{aligned}$$

where $C_8 = C_3 C_1^\beta / 8$. Hence the left hand side of (2.12) tends to 0 as $R \rightarrow \infty$. This is a contradiction.

(II) *The case $\lambda < 2$ and $\mu = \beta(2 - \lambda) + 2$.* Let $r > r_1$. Then from (2.5) and (2.6) we have

$$\begin{aligned}
(2.13) \quad \bar{v}(r) &\geq b + \frac{1}{n-2} \int_0^r s \hat{q}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^\beta(s) ds \\
&\geq b + \frac{1}{n-2} \int_{r_1}^r s \hat{q}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^\beta(s) ds \\
&\geq b + \frac{K_2 C_1^\beta}{n-2} \int_{r_1}^r s^{1-\mu+\beta(2-\lambda)} \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] ds \\
&\geq b + \frac{K_2 C_1^\beta}{n-2} \left[1 - \left(\frac{1}{2}\right)^{n-2}\right] \int_{r_1}^{r/2} s^{-1} ds \\
&\geq C_9 \log r, \quad r \geq r_2 > 2r_1
\end{aligned}$$

for some constant $C_9 > 0$. Let $R > r_2$ be an arbitrarily fixed number. As in case (I), we introduce the functions f and g by (2.8) and (2.9), respectively. Then using a similar argument as in (I), we see that

$$(2.14) \quad \frac{1}{\delta} \left\{ f\left(\frac{3}{2}R\right) \right\}^{-\delta} \geq C_{10} R^{-(\alpha\mu+\lambda)/(2\alpha+2)+1}, \quad R > r_2.$$

where $\delta = (\alpha\beta - 1)/(2\alpha + 2)$, $C_{10} > 0$ is some constant independent of R . From (2.8) and (2.13) we have

$$\begin{aligned}
f\left(\frac{3}{2}R\right) &\geq \frac{C_2}{R^\lambda} \int_R^{3R/2} \left(\frac{3}{2}R - s\right) (C_9 \log s)^\alpha ds \\
&\geq \frac{C_2 C_9^\alpha (\log R)^\alpha}{R^\lambda} \frac{R^2}{8}.
\end{aligned}$$

Thus by (2.14) we obtain

$$\begin{aligned} \frac{1}{\delta C_{10}} &\geq R^{-(\alpha\mu+\lambda)/(2\alpha+2)+1} \left\{ f\left(\frac{3}{2}R\right) \right\}^{\delta} \\ &\geq R^{-\delta(2-\lambda)} \left(\frac{C_2 C_9^{\alpha}}{8} \right)^{\delta} (\log R)^{\omega\delta} R^{\delta(2-\lambda)} \\ &= \left(\frac{C_2 C_9^{\alpha}}{8} \right)^{\delta} (\log R)^{\omega\delta}, \quad R > r_2. \end{aligned}$$

This is a contradiction.

(III) *The case $\lambda = 2$ and $\mu \leq 2$.* Let $r > r_0$. Then a similar calculation to (2.6) shows that

$$(2.15) \quad \bar{u}(r) \geq C_{11} \log r, \quad r \geq r_3 > 2r_0$$

for some constant $C_{11} > 0$. We fix $R > r_3$ arbitrarily. Then, as in case (I) we see that

$$(2.16) \quad \frac{1}{\delta} \left\{ g\left(\frac{3}{2}R\right) \right\}^{-\delta} \geq C_{12} R^{-(2\beta+\mu)/(2\beta+2)+1},$$

where g is the function given by (2.9), and $\delta > 0$ and $C_{12} > 0$ are constants independent of R . From (2.9) and (2.15) we have

$$g\left(\frac{3}{2}R\right) \geq C_{13} R^{2-\mu} (\log R)^{\beta}.$$

Since $-(2\beta+\mu)/(2\beta+2)+1 \geq 0$, the right hand side of (2.16) is positive constant or tends to ∞ as $R \rightarrow \infty$. On the other hand, the left hand side of (2.16) tends to 0 as $R \rightarrow \infty$. This is a contradiction. This completes the proof of Theorem 2.1.

Very recently, Deng [2] proved the following nonexistence result:

Theorem A. *Let $n \geq 3$. Suppose that there exist an $r_0 > 0$, a constant $m > 1$ and nonnegative nonincreasing continuous functions $\phi(r), \psi(r)$ such that*

$$\hat{p}(r) \geq \phi(r), \quad \hat{q}(r) \geq \psi(r), \quad \text{for } r \geq r_0$$

and

$$\limsup_{r \rightarrow \infty} r^4 \phi(mr) \psi(mr) \Phi^{\beta-1}(r) \Psi^{\alpha-1}(r) = \infty,$$

where

$$\Phi(r) = \int_{r_0}^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \phi(s) ds,$$

and

$$\Psi(r) = \int_{r_0}^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] \psi(s) ds \quad \text{for } r \geq r_0.$$

Then (1.1) does not possess any positive entire solutions.

When $\phi(r) = Cr^{-\lambda}$ and $\psi(r) = Cr^{-\mu}$, $C > 0$, we can easily verify that the hypotheses in Theorem A hold if and only if (λ, μ) satisfy

$$\begin{cases} \lambda < \alpha(2 - \mu) + 2 & \text{or} \\ \mu < \beta(2 - \lambda) + 2 & \text{or} \\ \lambda = \mu = 2. \end{cases}$$

On the other hand, if (λ, μ) satisfy

$$\begin{cases} \lambda < 2, & \mu = \beta(2 - \lambda) + 2 & \text{or} \\ \mu < 2, & \lambda = \alpha(2 - \mu) + 2, \end{cases}$$

then we can see that

$$\limsup_{r \rightarrow \infty} r^4 \phi(mr) \psi(mr) \Phi^{\beta-1}(r) \Psi^{\alpha-1}(r) < \infty.$$

In this case, Theorem A cannot be applied. Therefore, Theorem 2.1 improves Theorem A.

Next we present the result for the case $n = 2$.

Theorem 2.2. *Let $n = 2$. Suppose that $\hat{p}$ and $\hat{q}$ satisfy*

$$(2.17) \quad \hat{p}(r) \geq \frac{K_1}{r^2(\log r)^\lambda}, \quad \hat{q}(r) \geq \frac{K_2}{r^2(\log r)^\mu} \quad \text{for } r \geq r_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda \leq \alpha(\beta + 2 - \mu) + 1 & \text{or} \\ \mu \leq \beta(\alpha + 2 - \lambda) + 1. \end{cases}$$

Then (1.1) does not possess any positive entire solutions.

Proof. It suffices to consider the case that $\lambda \leq \alpha + 1$ and $\mu \leq \beta(\alpha + 2 - \lambda) + 1$. Assume the contrary that (1.1) has a positive entire solution (u, v) . Then we have

$$(2.18) \quad \begin{cases} (r\bar{u}'(r))' \geq r\hat{p}(r)\bar{v}^\alpha(r), & (r\bar{v}'(r))' \geq r\hat{q}(r)\bar{u}^\beta(r), & r \geq 0, \\ \bar{u}(0) = a > 0, & \bar{u}'(0) = 0, & \bar{v}(0) = b > 0, & \bar{v}'(0) = 0, \end{cases}$$

where $\bar{u}, \bar{v}, \hat{p}$ and $\hat{q}$ are defined by (2.1) and (2.2), respectively. From (2.18), $\bar{u}(r)$ and $\bar{v}(r)$ satisfy the integral inequalities

$$(2.19) \quad \begin{cases} \bar{u}(r) \geq a + \int_0^r s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds, & r \geq 0, \\ \bar{v}(r) \geq b + \int_0^r s \log\left(\frac{r}{s}\right) \hat{q}(s) \bar{u}^\beta(s) ds, & r \geq 0. \end{cases}$$

Let $r \geq e$. Then from (2.19), we have

$$(2.20) \quad \begin{aligned} \bar{u}(r) &\geq a + \int_0^1 s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds + \int_1^e s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds \\ &\quad + \int_e^r s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds \\ &\geq a + \int_0^1 s \hat{p}(s) \bar{v}^\alpha(s) ds \log r + \int_e^r s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds \\ &\geq a + b^\alpha \int_0^1 s \hat{p}(s) ds \log r + \int_e^r s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds \\ &\geq C_1 \log r + \int_e^r s \log\left(\frac{r}{s}\right) \hat{p}(s) \bar{v}^\alpha(s) ds, \quad r \geq e \end{aligned}$$

where C_1 is a positive constant. Similarly,

$$(2.21) \quad \bar{v}(r) \geq C_2 \log r + \int_e^r s \log\left(\frac{r}{s}\right) \hat{q}(s) \bar{u}^\beta(s) ds, \quad r \geq e,$$

where C_2 is a positive constant. Let $w(r) = \bar{u}(r)/\log r$ and $z(r) = \bar{v}(r)/\log r$. Then from (2.20) and (2.21), we have

$$(2.22) \quad w(r) \geq C_1 + \int_e^r s \left(1 - \frac{\log s}{\log r}\right) \hat{p}(s) (\log s)^\alpha z^\alpha(s) ds, \quad r \geq e,$$

and

$$(2.23) \quad z(r) \geq C_2 + \int_e^r s \left(1 - \frac{\log s}{\log r}\right) \hat{q}(s) (\log s)^\beta w^\beta(s) ds, \quad r \geq e.$$

Let $t = \log s$, $\eta = \log r$, $w(r) = w(e^\eta) = \tilde{w}(\eta)$ and $z(r) = z(e^\eta) = \tilde{z}(\eta)$. Then (2.22) and (2.23) become, respectively

$$\begin{aligned}
\tilde{w}(\eta) &\geq C_1 + \int_1^\eta t \left(1 - \frac{t}{\eta}\right) e^{2t} \hat{p}(e^t) t^{\alpha-1} \tilde{z}^\alpha(t) dt \\
&= C_1 + \int_1^\eta t \left(1 - \frac{t}{\eta}\right) \tilde{p}(t) \tilde{z}^\alpha(t) dt, \quad \eta \geq 1,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{z}(\eta) &\geq C_2 + \int_1^\eta t \left(1 - \frac{t}{\eta}\right) e^{2t} \hat{q}(e^t) t^{\beta-1} \tilde{w}^\beta(t) dt \\
&= C_2 + \int_1^\eta t \left(1 - \frac{t}{\eta}\right) \tilde{q}(t) \tilde{w}^\beta(t) dt, \quad \eta \geq 1,
\end{aligned}$$

where $\tilde{p}$ and $\tilde{q}$ are given by $\tilde{p}(t) = e^{2t} \hat{p}(e^t) t^{\alpha-1}$, $\tilde{q}(t) = e^{2t} \hat{q}(e^t) t^{\beta-1}$, respectively. From (2.17), we then have

$$\tilde{p}(t) \geq e^{2t} \frac{K_1}{e^{2t} (\log e^t)^\lambda} t^{\alpha-1} = \frac{K_1}{t^{\lambda-\alpha+1}}, \quad t \geq e^{r_0},$$

and

$$\tilde{q}(t) \geq \frac{K_2}{t^{\mu-\beta+1}}, \quad t \geq e^{r_0}.$$

Since $\lambda - \alpha + 1 \leq 2$, $\mu - \beta + 1 \leq \beta(2 - (\lambda - \alpha + 1)) + 2$, using a similar argument as in the proof of Theorem 2.1, we obtain a contradiction. Thus the proof is completed.

Finally, for the case $n = 1$ we define the mean $(\bar{u}, \bar{v})$ of positive entire solution (u, v) of (1.1) by

$$\bar{u}(r) = \frac{1}{2}(u(r) + u(-r)), \quad \bar{v}(r) = \frac{1}{2}(v(r) + v(-r)), \quad r \geq 0,$$

and put

$$\begin{aligned}
\hat{p}(r) &= [\tfrac{1}{2}(p^{-\alpha'/\alpha}(r) + p^{-\alpha'/\alpha}(-r))]^{-\alpha/\alpha'}, \quad r \geq 0, \\
\hat{q}(r) &= [\tfrac{1}{2}(q^{-\beta'/\beta}(r) + q^{-\beta'/\beta}(-r))]^{-\beta/\beta'}, \quad r \geq 0,
\end{aligned}$$

where $1/\alpha + 1/\alpha' = 1$ and $1/\beta + 1/\beta' = 1$. Then using a similar argument as in [1], we find that

$$(2.24) \quad \begin{cases} \bar{u}''(r) \geq \hat{p}(r) \bar{v}^\alpha(r), & \bar{v}''(r) \geq \hat{q}(r) \bar{u}^\beta(r), & r > 0, \\ \bar{u}(0) = a > 0, & \bar{u}'(0) = 0, & \bar{v}(0) = b > 0, & \bar{v}'(0) = 0. \end{cases}$$

We have the following result.

Theorem 2.3. Let $n = 1$. Suppose that $\hat{p}$ and $\hat{q}$ satisfy

$$(2.25) \quad \hat{p}(r) \geq \frac{K_1}{r^\lambda}, \quad \hat{q}(r) \geq \frac{K_2}{r^\mu} \quad \text{for } r \geq r_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda \leq \alpha(\beta + 2 - \mu) + 1 & \text{or} \\ \mu \leq \beta(\alpha + 2 - \lambda) + 1. \end{cases}$$

Then (1.1) does not possess any positive entire solutions.

Proof. It suffices to consider the case that $\lambda \leq \alpha + 1$, $\mu \leq \beta(\alpha + 2 - \lambda) + 1$. Assume the contrary that (1.1) has a positive entire solution (u, v) . From (2.24) we have

$$\begin{aligned} \bar{u}(r) &\geq a + \int_0^1 (r-s)\hat{p}(s)\bar{v}^\alpha(s)ds + \int_1^r (r-s)\hat{p}(s)\bar{v}^\alpha(s)ds \\ &\geq a + \left(b^\alpha \int_0^1 \left(1 - \frac{s}{r}\right)\hat{p}(s)ds\right)r + \int_1^r (r-s)\hat{p}(s)\bar{v}^\alpha(s)ds \\ &\geq C_1 r + \int_1^r (r-s)\hat{p}(s)\bar{v}^\alpha(s)ds, \quad r \geq 2, \end{aligned}$$

and

$$\bar{v}(r) \geq C_2 r + \int_1^r (r-s)\hat{q}(s)\bar{u}^\beta(s)ds, \quad r \geq 2,$$

where C_1, C_2 are positive constants. Setting $w(r) = \bar{u}(r)/r$ and $z(r) = \bar{v}(r)/r$, we obtain

$$w(r) \geq C_1 + \int_1^r s\left(1 - \frac{s}{r}\right)\tilde{p}(s)z^\alpha(s)ds, \quad r \geq 2,$$

and

$$z(r) \geq C_2 + \int_1^r s\left(1 - \frac{s}{r}\right)\tilde{q}(s)w^\beta(s)ds, \quad r \geq 2,$$

where $\tilde{p}(s) = \hat{p}(s)s^{\alpha-1}$, $\tilde{q}(s) = \hat{q}(s)s^{\beta-1}$. From (2.25), we have

$$\tilde{p}(s) \geq \frac{K_1}{s^{\lambda-\alpha+1}}, \quad \tilde{q}(s) \geq \frac{K_2}{s^{\mu-\beta+1}} \quad \text{for } s \geq r_0.$$

Since $\lambda - \alpha + 1 \leq 2, \mu \leq \beta(2 - (\lambda - \alpha + 1)) + 2$, from the proof of Theorem 2.1,

we see that $(w(r), z(r))$ cannot exist on $[2, \infty)$. This contradiction completes the proof.

3. Existence results

In this section, we consider existence of spherically symmetric positive entire solutions of the semilinear elliptic system

$$(3.1) \quad \begin{cases} \Delta u = p(|x|)v^\alpha, \\ \Delta v = q(|x|)u^\beta, \end{cases} \quad \text{in } \mathbf{R}^n.$$

Through the section we always assume that $p(r), q(r), r = |x|$, are nonnegative continuous functions and $\alpha > 0, \beta > 0, \alpha\beta > 1$.

Theorem 3.1. *Let $n \geq 3$. Suppose that p and q satisfy*

$$p(r) \leq \frac{K_1}{r^\lambda}, \quad q(r) \leq \frac{K_2}{r^\mu} \quad \text{for } r \geq r_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda > \alpha(2 - \mu) + 2 \\ \mu > \beta(2 - \lambda) + 2. \end{cases}$$

Then (3.1) has infinitely many spherically symmetric positive entire solutions.

Proof. Without loss of generality, we may assume that $r_0 = 1$. We note that (u, v) is spherically symmetric entire solution of (3.1) if and only if the functions $y(r) = u(x), z(r) = v(x), r = |x|$, satisfy the system of second order ordinary differential equations

$$(3.2) \quad \begin{cases} r^{1-n}(r^{n-1}y'(r))' = p(r)z^\alpha(r), & r > 0, & y'(0) = 0, \\ r^{1-n}(r^{n-1}z'(r))' = q(r)y^\beta(r), & r > 0, & z'(0) = 0, \end{cases}$$

where $' = d/dr$. Furthermore, integrating (3.2) twice, we obtain the following integral equations equivalent to (3.2):

$$(3.3) \quad \begin{cases} y(r) = a + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] p(s) z^\alpha(s) ds, & r \geq 0, \\ z(r) = b + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] q(s) y^\beta(s) ds, & r \geq 0, \end{cases}$$

where $a = y(0)$, $b = z(0)$. So it suffices to solve (3.3). Choose constants $a > 0$ and $b > 0$ so that

$$(3.4) \quad \begin{aligned} & \frac{(2b)^\alpha}{n-2} \int_0^1 sp(s)ds \leq a/2, \\ & \frac{K_1(2b)^\alpha}{(n-2)(2-\lambda+\alpha l)} \leq a/2, \\ & \frac{(2a)^\beta}{n-2} \int_0^1 sq(s)ds \leq b/2, \\ & \frac{K_2(2a)^\beta}{(n-2)(2-\mu+\beta k)} \leq b/2, \end{aligned}$$

where $k = \{\lambda - 2 - \alpha(2 - \mu)\}/(\alpha\beta - 1) > 0$, $l = \{\mu - 2 - \beta(2 - \lambda)\}/(\alpha\beta - 1) > 0$. It is possible to choose such a and b by our assumptions. Put

$$A(r) = \begin{cases} 2a & \text{for } 0 \leq r \leq 1, \\ 2ar^k & \text{for } r \geq 1, \end{cases}$$

$$B(r) = \begin{cases} 2b & \text{for } 0 \leq r \leq 1, \\ 2br^l & \text{for } r \geq 1. \end{cases}$$

Let $C[0, \infty) \times C[0, \infty)$ denote the locally convex space of all continuous vector functions (y, z) on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{(y, z) \in C[0, \infty) \times C[0, \infty) : a \leq y(r) \leq A(r), b \leq z(r) \leq B(r), r \geq 0\}.$$

Clearly, Y is a closed convex subset of $C[0, \infty) \times C[0, \infty)$. Define the mapping $\mathcal{F} : Y \rightarrow C[0, \infty) \times C[0, \infty)$ by $\mathcal{F}(y, z) = (\tilde{y}, \tilde{z})$, where

$$\begin{aligned} \tilde{y}(r) &= a + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] p(s) z^\alpha(s) ds, & r \geq 0, \\ \tilde{z}(r) &= b + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] q(s) y^\beta(s) ds, & r \geq 0. \end{aligned}$$

We show that $\mathcal{F}$ is a continuous mapping from Y into Y such that $\mathcal{F}Y$ is relatively compact.

(i) $\mathcal{F}$ maps Y into Y . Let $(y, z) \in Y$. Clearly, $\tilde{y}(r) \geq a$. For $0 \leq r \leq 1$, we have

$$\begin{aligned}
\tilde{y}(r) &= a + \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] p(s) z^\alpha(s) ds \\
&\leq a + \frac{1}{n-2} \int_0^r s p(s) z^\alpha(s) ds \\
&\leq a + \frac{(2b)^\alpha}{n-2} \int_0^1 s p(s) ds \\
&\leq a + \frac{a}{2} \leq 2a.
\end{aligned}$$

For $r \geq 1$, we have

$$\begin{aligned}
\tilde{y}(r) &\leq a + \frac{1}{n-2} \int_0^1 s p(s) z^\alpha(s) ds + \frac{1}{n-2} \int_1^r s p(s) z^\alpha(s) ds \\
&\leq a + \frac{(2b)^\alpha}{n-2} \int_0^1 s p(s) ds + \frac{K_1(2b)^\alpha}{n-2} \int_1^r s^{1-\lambda+\alpha l} ds \\
&\leq a + \frac{a}{2} + \frac{K_1(2b)^\alpha}{(n-2)(2-\lambda+\alpha l)} r^{2-\lambda+\alpha l} \\
&\leq \frac{3}{2} a r^k + \frac{a}{2} r^k = 2a r^k.
\end{aligned}$$

Thus we obtain

$$a \leq \tilde{y}(r) \leq A(r), \quad r \geq 0.$$

A similar computation shows that

$$b \leq \tilde{z}(r) \leq B(r), \quad r \geq 0.$$

Therefore, $\mathcal{F}(y, z) \in Y$.

(ii) $\mathcal{F}$ is continuous. Let (y, z) and $(y_m, z_m), m = 1, 2, \dots$, be vector functions in Y such that $(y_m, z_m) \rightarrow (y, z)$ in Y as $m \rightarrow \infty$ uniformly on each compact subinterval of $[0, \infty)$. Let $\mathcal{F}(y_m, z_m) = (\tilde{y}_m, \tilde{z}_m)$ and $\mathcal{F}(y, z) = (\tilde{y}, \tilde{z})$. Then we have

$$\begin{aligned}
|\tilde{y}_m(r) - \tilde{y}(r)| &\leq \frac{1}{n-2} \int_0^r s \left[1 - \left(\frac{s}{r} \right)^{n-2} \right] p(s) |z_m^\alpha(s) - z^\alpha(s)| ds \\
&\leq \frac{1}{n-2} \int_0^r s p(s) |z_m^\alpha(s) - z^\alpha(s)| ds.
\end{aligned}$$

Since the functions $h_m(s) = s p(s) |z_m^\alpha(s) - z^\alpha(s)|, m \in \mathbb{N}$, satisfy $h_m(s) \leq 2s p(s) \cdot B^\alpha(s), s \geq 0$, and $h_m(s) \rightarrow 0$ as $m \rightarrow \infty$ at every point $s \geq 0$, the Lebesgue

dominated convergence theorem implies that $\tilde{y}_m(r) \rightarrow \tilde{y}(r)$ as $m \rightarrow \infty$ uniformly on each compact subinterval of $[0, \infty)$. Similarly, $\tilde{z}_m(r) \rightarrow \tilde{z}(r)$ as $m \rightarrow \infty$ uniformly on each compact subinterval of $[0, \infty)$. These imply the continuity of $\mathcal{F}$.

(iii) $\mathcal{F}Y$ is relatively compact. It suffices to prove that $\mathcal{F}Y$ is uniformly bounded and equicontinuous on each subinterval of $[0, \infty)$. This is an immediate consequence of the following inequalities holding for all $y, z \in Y$ and all $r \geq 0 : a \leq \tilde{y}(r) \leq A(r), b \leq \tilde{z}(r) \leq B(r)$,

$$0 \leq \tilde{y}'(r) = \int_0^r \left(\frac{s}{r}\right)^{n-1} p(s) z^\alpha(s) ds \leq \int_0^r p(s) B^\alpha(s) ds,$$

and

$$0 \leq \tilde{z}'(r) = \int_0^r \left(\frac{s}{r}\right)^{n-1} q(s) y^\beta(s) ds \leq \int_0^r q(s) A^\beta(s) ds.$$

Thus, we are able to apply the Schauder-Tychonoff fixed point theorem, concluding that $\mathcal{F}$ has a fixed point $(y, z) \in Y$. This fixed point $(y, z) = (y(r), z(r))$ is a solution of (3.3), and hence of (3.2). The function $(u(x), v(x)) = (y(|x|), z(|x|))$ gives a solution of (3.1) in $\mathbf{R}^n$. Since infinitely many (a, b) satisfy (3.4), we can construct an infinitude of positive entire solutions of (3.1). This completes the proof.

Theorem 3.2. Let $n = 2$. Suppose that p and q satisfy

$$p(r) \leq \frac{K_1}{r^2(\log r)^\lambda}, \quad q(r) \leq \frac{K_2}{r^2(\log r)^\mu}, \quad r \geq r_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda > \alpha(\beta + 2 - \mu) + 1 \\ \mu > \beta(\alpha + 2 - \lambda) + 1. \end{cases}$$

Then (3.1) has infinitely many spherically symmetric positive entire solutions.

Proof. Without loss of generality, we may assume that $r_0 = e$. As before, it suffices to solve the following system of integral equations:

$$(3.5) \quad \begin{cases} y(r) = a + \int_0^r s \log\left(\frac{r}{s}\right) p(s) z^\alpha(s) ds, & r \geq 0, \\ z(r) = b + \int_0^r s \log\left(\frac{r}{s}\right) q(s) y^\beta(s) ds, & r \geq 0, \end{cases}$$

where $a = y(0), b = z(0)$. Choose constants $a > 0, b > 0$ so that

$$(2b)^\alpha e \int_0^e p(s) ds \leq \frac{a}{2},$$

$$\frac{(2b)^\alpha K_1}{1 - \lambda + \alpha l} \leq \frac{a}{2},$$

$$(2a)^\beta e \int_0^e q(s) ds \leq \frac{b}{2},$$

$$\frac{(2a)^\beta K_2}{1 - \mu + \beta k} \leq \frac{b}{2},$$

where $k = \{\lambda - 2 - \alpha(2 - \mu)\}/(\alpha\beta - 1)$, $l = \{\mu - 2 - \beta(2 - \lambda)\}/(\alpha\beta - 1)$. We note that $k > 1$ and $l > 1$. It is possible to choose such a and b by our assumptions. Let

$$A(r) = \begin{cases} 2a & \text{for } 0 \leq r \leq e, \\ 2a(\log r)^k & \text{for } r \geq e, \end{cases}$$

$$B(r) = \begin{cases} 2b & \text{for } 0 \leq r \leq e, \\ 2b(\log r)^l & \text{for } r \geq e. \end{cases}$$

Consider the set

$$Y = \{(y, z) \in C[0, \infty) \times C[0, \infty) : a \leq y(r) \leq A(r), b \leq z(r) \leq B(r), r \geq 0\},$$

which is a closed convex subset of $C[0, \infty) \times C[0, \infty)$. Define the mapping $\mathcal{F} : Y \rightarrow C[0, \infty) \times C[0, \infty)$ by $\mathcal{F}(y, z) = (\tilde{y}, \tilde{z})$, where

$$\tilde{y}(r) = a + \int_0^r s \log\left(\frac{r}{s}\right) p(s) z^\alpha(s) ds, \quad r \geq 0,$$

$$\tilde{z}(r) = b + \int_0^r s \log\left(\frac{r}{s}\right) q(s) y^\beta(s) ds, \quad r \geq 0.$$

We will verify that $\mathcal{F}$ is a continuous mapping from Y into itself such that $\mathcal{F}Y$ is relatively compact.

(i) $\mathcal{F}$ maps Y into itself. It suffices to show that $(y, z) \in Y$ implies $\tilde{y}(r) \leq A(r)$ and $\tilde{z}(r) \leq B(r)$ for $r \geq 0$. Let $0 \leq r \leq e$. Then, using the inequality $0 \leq s \log(r/s) \leq r/e$ for $0 \leq s \leq r$, we have

$$\begin{aligned} \tilde{y}(r) &\leq a + \frac{r}{e} \int_0^r p(s) z^\alpha(s) ds \\ &\leq a + (2b)^\alpha \int_0^e p(s) ds \end{aligned}$$

$$\begin{aligned}
&\leq a + (2b)^\alpha e \int_0^e p(s) ds \\
&\leq a + \frac{a}{2} \leq 2a.
\end{aligned}$$

Let $r \geq e$. We then write

$$\begin{aligned}
\tilde{y}(r) &= a + \left(\int_0^1 + \int_1^e + \int_e^r \right) s \log\left(\frac{r}{s}\right) p(s) z^\alpha(s) ds \\
&\equiv a + I_1 + I_2 + I_3.
\end{aligned}$$

The inequality $0 \leq s \log(r/s) \leq \log r$ for $0 \leq s \leq 1$ implies that

$$\begin{aligned}
(3.6) \quad I_1 &\leq \int_0^1 p(s) z^\alpha(s) ds \log r \\
&\leq (2b)^\alpha \int_0^1 p(s) ds \log r \\
&\leq (2b)^\alpha e \int_0^1 p(s) ds \log r.
\end{aligned}$$

The integrals I_2 and I_3 are estimated as follows:

$$\begin{aligned}
(3.7) \quad I_2 &\leq \int_1^e s p(s) z^\alpha(s) ds \log r \\
&\leq (2b)^\alpha \int_1^e s p(s) ds \log r \\
&\leq (2b)^\alpha e \int_1^e p(s) ds \log r;
\end{aligned}$$

$$\begin{aligned}
(3.8) \quad I_3 &\leq \int_e^r s p(s) z^\alpha(s) ds \log r \\
&\leq (2b)^\alpha \int_e^r s \frac{K_1}{s^2 (\log s)^\lambda} (\log s)^{\alpha l} ds \log r \\
&= (2b)^\alpha K_1 \int_1^{\log r} t^{\alpha l - \lambda} dt \log r \\
&\leq \frac{(2b)^\alpha K_1}{1 - \lambda + \alpha l} (\log r)^{2 - \lambda + \alpha l} \\
&\leq \frac{a}{2} (\log r)^k.
\end{aligned}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
 (3.9) \quad I_1 + I_2 &\leq (2b)^\alpha e \left\{ \int_0^1 p(s) ds + \int_1^e p(s) ds \right\} \log r \\
 &= (2b)^\alpha e \int_0^e p(s) ds \log r \\
 &\leq \frac{a}{2} (\log r)^k.
 \end{aligned}$$

Thus by (3.8) and (3.9) we obtain $\tilde{y}(r) \leq 2a(\log r)^k$ for $r \geq e$. We have shown that $\tilde{y}(r) \leq A(r)$ for $r \geq 0$ as desired. A similar computation shows that $\tilde{z}(r) \leq B(r)$ for $r \geq 0$.

(ii) $\mathcal{F}$ is continuous. Let (y_m, z_m) be a sequence in Y converging to $(y, z) \in Y$ as $m \rightarrow \infty$ in the topology of $C[0, \infty) \times C[0, \infty)$. Then, we have

$$|\tilde{y}_m(r) - \tilde{y}(r)| \leq \frac{r}{e} \int_0^r p(s) |z_m^\alpha(s) - z^\alpha(s)| ds, \quad r \geq 0.$$

Since the functions $f_m(s) = p(s) |z_m^\alpha(s) - z^\alpha(s)|$, $m \in \mathbb{N}$ satisfy $f_m(s) \leq 2p(s)B^\alpha(s)$, $s \geq 0$, and $f_m(s) \rightarrow 0$ as $m \rightarrow \infty$ at every point $s \geq 0$, the Lebesgue dominated convergence theorem implies that $\tilde{y}_m(r) \rightarrow \tilde{y}(r)$ as $m \rightarrow \infty$ uniformly on each compact subinterval of $[0, \infty)$. This shows that $\tilde{y}_m \rightarrow \tilde{y}$ in Y as $m \rightarrow \infty$. Similarly, $\tilde{z}_m \rightarrow \tilde{z}$ in Y as $m \rightarrow \infty$.

(iii) $\mathcal{F}Y$ is relatively compact. This follows from the observation that if $(y, z) \in Y$, then $(\tilde{y}, \tilde{z}) = \mathcal{F}(y, z)$ satisfy

$$\begin{aligned}
 \tilde{y}'(r) &= \frac{1}{r} \int_0^r sp(s) z^\alpha(s) ds \leq \int_0^r p(s) B^\alpha(s) ds, \\
 \tilde{z}'(r) &= \frac{1}{r} \int_0^r sq(s) y^\beta(s) ds \leq \int_0^r q(s) A^\beta(s) ds,
 \end{aligned}$$

for $r \geq 0$.

Thus there exists an element $(y, z) \in Y$ such that $(y, z) = \mathcal{F}(y, z)$ by the Schauder-Tychonoff fixed point theorem. This (y, z) obviously satisfy (3.5). The function $(u(x), v(x)) = (y(|x|), z(|x|))$ is a desired solution of (3.1). The proof is completed.

Theorem 3.3. Let $n = 1$. Suppose that p and q satisfy

$$p(x) \leq \frac{K_1}{|x|^\lambda}, \quad q(x) \leq \frac{K_2}{|x|^\mu}, \quad |x| \geq x_0 > 0,$$

where K_1, K_2 are positive constants and

$$\begin{cases} \lambda > \alpha(\beta + 2 - \mu) + 1 \\ \mu > \beta(\alpha + 2 - \lambda) + 1. \end{cases}$$

Then (3.1) has infinitely many positive entire solutions.

The proof of Theorem 3.3 is the same as that of Theorem 3.1. So we leave the proof to the reader.

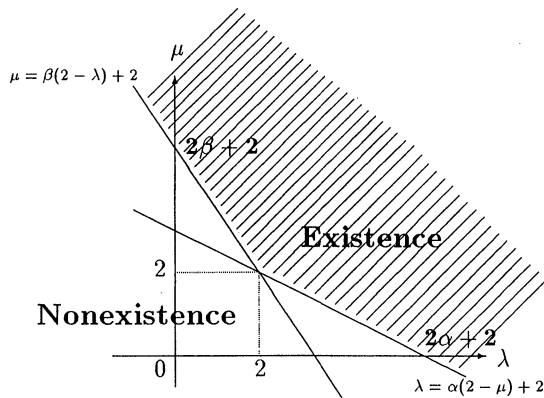
Example. Consider the system

$$(3.10) \quad \begin{cases} \Delta u = \frac{C}{(1+|x|)^\lambda} v^\alpha \\ \Delta v = \frac{C}{(1+|x|)^\mu} u^\beta \end{cases} \quad \text{in } \mathbf{R}^n,$$

where $n \geq 3, \alpha, \beta > 0, \alpha\beta > 1$ and C is a positive constant. We can completely characterize the existence of positive entire solutions of this system in terms of α, β, λ and μ . In fact, we see that

$$\begin{aligned} \frac{K_1}{|x|^\lambda} &\leq \frac{C}{(1+|x|)^\lambda} \leq \frac{K_2}{|x|^\lambda}, & |x| \geq 1, \\ \frac{K_3}{|x|^\mu} &\leq \frac{C}{(1+|x|)^\mu} \leq \frac{K_4}{|x|^\mu}, & |x| \geq 1, \end{aligned}$$

where $K_i, i = 1 \dots 4$ are some positive constants. From Theorems 2.1 and 3.1, If $\lambda \leq \alpha(2 - \mu) + 2$ or $\mu \leq \beta(2 - \lambda) + 2$, then (3.10) does not admit any positive entire solutions. On the other hand, if $\lambda > \alpha(2 - \mu) + 2$ and $\mu > \beta(2 - \lambda) + 2$, then (3.10) has infinitely many (spherically symmetric) positive entire solutions. See the figure below.



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