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Local Wellposedness and Illposedness in the Critical Besov Spaces for Semilinear Wave Equations with Quadratic Forms

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1. Introduction

In this paper we study time-local wellposedness of semilinear wave equations in the following forms:

$$(1.1) \quad \begin{cases} \square u = B(Du, u), \\ u(0) = \varphi, \quad \dot{u}(0) = \psi, \end{cases}$$

$$(1.2) \quad \begin{cases} \square u = B(Du, Du), \\ u(0) = \varphi, \quad \dot{u}(0) = \psi. \end{cases}$$

Here we denote by $B(\cdot, \cdot)$ arbitrary bilinear form on any couple of Euclidean spaces. $(t, x) \in \mathbf{R}^{1+n}$, $u = u(t, x)$ is $\mathbf{R}^N$ -valued unknown function and $D = (\partial_t, \nabla_x)$. It is well-known that in the case $n = 3$, (1.1) ((1.2)) is time-locally wellposed in H^s for $s > 1$ ($s > 2$), but in general illposed in H^1 (H^2) [5, 10, 1, 8]. In this paper we show that we may regain wellposedness in the critical regularity $s = 1$ ($s = 2$) if we replace the Sobolev space with the Besov space $B_{2,1}^s$. The same phenomenon was recently observed in the case of the equation of wave map type [9, 11]. However, such a phenomenon does not always occur. In fact, we will also give an illposedness result in the Besov space $B_{2,1}^s$ for the equation of wave map type in the case $n = 1$.

The wellposedness result is obtained by the critical bilinear estimate for wave equation in Besov spaces. The bilinear estimates for wave equation in homogeneous Sobolev spaces have been investigated, e.g., in [5, 6, 1, 7, 3]. It is shown in [5, 3] that the following estimate is the critical case and does not hold for any $\sigma \in \mathbf{R}$.

$$(1.3) \quad \|uv\|_{\dot{H}^\sigma(\mathbf{R}^4)} \leq C \|\varphi\|_{\dot{H}^1(\mathbf{R}^3)} \|\psi\|_{\dot{H}^\sigma(\mathbf{R}^3)},$$

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where u and v are solutions of the linear wave equation with initial data $u(0) = \varphi$, $\dot{u}(0) = 0$, $v(0) = \psi$ and $\dot{v}(0) = 0$. In this paper we regain this estimate by replacing all the Sobolev spaces with the homogeneous Besov spaces with the third exponent 1, and then the wellposedness in the Besov spaces follows.

The illposedness result is obtained for the equation of the following form:

$$(1.4) \quad \begin{cases} \square u + f(u)(\dot{u}^2 - |\nabla u|^2) = 0, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi, \end{cases}$$

for $n = N = 1$. It is well known that the critical regularity for this equation is $s = n/2$. We will construct a solution which remains small in L^∞ but blows up at once in the Sobolev space $H^{1/2}$. Such a phenomenon is peculiar to $n = 1$ and intimately related to the following estimate:

$$(1.5) \quad \|\varphi\psi\|_{\dot{B}_{2,1}^{n/2-1}} \leq C \|\varphi\|_{\dot{B}_{2,1}^{n/2}} \|\psi\|_{\dot{B}_{2,1}^{n/2-1}},$$

which holds if and only if $n \geq 2$.

The rest of this paper is organized as follows. In Section 2, we introduce several notations and definitions. In Section 3, we prove the critical bilinear estimates in the Besov spaces. In Section 4, we prove the wellposedness result. In Section 5, we show that the bilinear estimates proved in Section 3 are optimal in the family of Besov spaces. In Section 6, we give the illposedness result.

2. Definition and Notations

In this section we introduce several definitions and notations. We denote the Fourier transform by

$$(2.1) \quad \tilde{f}(\xi) = \int f(x) e^{-ix\xi} dx,$$

and by $\mathcal{F}^{-1}$ the inverse Fourier transform (we use the same notation for the space-time Fourier transform). For any function φ , we denote by $\varphi(\omega)$ the operator:

$$(2.2) \quad f \mapsto \mathcal{F}^{-1}(\varphi(|\xi|)\tilde{f}(\xi)).$$

We denote L^2 -norm by $\|\varphi\|_2$. Define $\{\chi_k^n\}_{k \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ by their Fourier images as

$$(2.3) \quad \tilde{\chi}_k^n(\xi) = \begin{cases} 1, & \text{if } 2^{k-1} \leq |\xi| < 2^k, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can define the homogeneous Besov spaces for $\sigma \in \mathbf{R}$ and $1 \leq r \leq \infty$ as follows (cf. [2]).

$$(2.4) \quad \dot{B}_{2,r}^\sigma(\mathbf{R}^n) = \{\varphi \in \mathcal{S}'(\mathbf{R}^n) \mid \tilde{\varphi} \in L_{loc}^1(\mathbf{R}^n), \|2^{j\sigma} \|\chi_j^n * \varphi\|_2\|_{\ell^r(\mathbf{Z})} < \infty\}.$$

If $\sigma < n/2$ or $\sigma \leq n/2$ and $r = 1$, then $\dot{B}_{2,r}^\sigma(\mathbf{R}^n)$ is a Banach space embedded in $\mathcal{S}'$. (We will use the above spaces only in such cases in this paper.) For $\sigma > 0$, we may define the inhomogeneous Besov spaces as follows.

$$(2.5) \quad B_{2,r}^\sigma(\mathbf{R}^n) = \{\varphi \in L^2(\mathbf{R}^n) \mid \|\varphi\|_2 + \|2^{j\sigma} \|\chi_j^n * \varphi\|_2\|_{\ell^r(\mathbf{N})} < \infty\}.$$

The homogeneous Sobolev spaces may be defined by $\dot{H}^\sigma = \dot{B}_{2,2}^\sigma$ and the inhomogeneous Sobolev spaces $H^\sigma = B_{2,2}^\sigma$.

3. Critical bilinear estimates for wave equations

In this section we prove the critical bilinear estimates for wave equations in the Besov spaces.

Proposition 3.1. *Let $-1/2 < \sigma \leq 1$, $\varphi \in \dot{B}_{2,1}^1(\mathbf{R}^3)$ and $\psi \in \dot{B}_{2,1}^\sigma(\mathbf{R}^3)$. Let $u = e^{i\omega t}\varphi$ and $v = e^{\pm i\omega t}\psi$. Then we have the following estimate, where $C > 0$ depends only on σ .*

$$(3.1) \quad \|uv\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} \leq C \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)}.$$

Proof. For $j, k \in \mathbf{Z}$, define $C_{j,k}$ by

$$(3.2) \quad C_{j,k} = \sup_{\varphi, \psi \in L^2(\mathbf{R}^3)} \frac{\|(e^{i\omega t}\chi_j^3 * \varphi)(e^{\pm i\omega t}\chi_k^3 * \psi)\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}}{\|\varphi\|_2 \|\psi\|_2}.$$

Then, by the homogeneity, we have

$$(3.3) \quad C_{j,k} = C_{k,j} = 2^{(1+\sigma)a} C_{j-a, k-a}.$$

So if we can prove that

$$(3.4) \quad C_{0,k} \leq 2^k C \quad \text{for } k \leq 0,$$

then (3.1) follows as

$$(3.5) \quad \begin{aligned} \|uv\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq \sum_{j,k} \|(e^{i\omega t}\chi_j^3 * \varphi)(e^{\pm i\omega t}\chi_k^3 * \psi)\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} \\ &\leq \sum_{j \leq k} 2^{(1+\sigma)k} C_{0,j-k} \|\chi_j^3 * \varphi\|_2 \|\chi_k^3 * \psi\|_2 \end{aligned}$$

$$\begin{aligned}
& + \sum_{j>k} 2^{(1+\sigma)j} C_{0,k-j} \|\chi_j^3 * \varphi\|_2 \|\chi_k^3 * \psi\|_2 \\
& \leq \sum_{j,k} 2^{j+\sigma k} C \|\chi_j^3 * \varphi\|_2 \|\chi_k^3 * \psi\|_2 \\
& \leq \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)}.
\end{aligned}$$

Now we have only to prove (3.4). To estimate $C_{0,k}$, we may assume that $\text{supp } \tilde{\varphi} \subset \{1/2 \leq |\xi| < 1\}$ and $\text{supp } \tilde{\psi} \subset \{2^{k-1} \leq |\xi| < 2^k\}$. The space-time Fourier transform of uv can be written as

$$\begin{aligned}
(3.6) \quad \tilde{uv} &= (\delta(\tau - |\xi|) \tilde{\varphi}(\xi)) * (\delta(\tau \mp |\xi|) \tilde{\psi}(\xi)) \\
&= \int \delta(\tau \mp |\eta| - |\xi - \eta|) \tilde{\varphi}(\xi - \eta) \tilde{\psi}(\eta) d\eta.
\end{aligned}$$

First we consider the case $-10 \leq k \leq 0$. Then in the last member of (3.6), we have

$$(3.7) \quad |\xi| \leq |\xi - \eta| + |\eta| < 2, \quad |\tau| = ||\xi - \eta| \pm |\eta|| \leq |\xi - \eta| + |\eta| < 2.$$

So in this case, for $-1/2 < \sigma' < \sigma$ we have

$$\begin{aligned}
& \|uv\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} \leq C \|uv\|_{\dot{H}^{\sigma'}(\mathbf{R}^4)}, \\
(3.8) \quad & \|\varphi\|_{\dot{H}^{(\sigma'+1)/2}(\mathbf{R}^3)} \leq \|\varphi\|_2, \\
& \|\psi\|_{\dot{H}^{(\sigma'+1)/2}(\mathbf{R}^3)} \leq \|\psi\|_2,
\end{aligned}$$

and then (3.4) is reduced to the estimate in [6]:

$$(3.9) \quad \|uv\|_{\dot{H}^{\sigma'}(\mathbf{R}^4)} \leq C \|\varphi\|_{\dot{H}^{(\sigma'+1)/2}(\mathbf{R}^3)} \|\psi\|_{\dot{H}^{(\sigma'+1)/2}(\mathbf{R}^3)}.$$

Next we consider the case $k < -10$. Then in the last member of (3.6), we have

$$(3.10) \quad \frac{4}{9} < |\xi - \eta| - |\eta| \leq |\xi|, \quad \tau \leq |\xi - \eta| + |\eta| < \frac{10}{9}.$$

So we have $\|uv\|_{\dot{B}_{2,1}^\sigma} \leq C \|uv\|_2$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(3.11) \quad \|\tilde{uv}\|_2^2 &\leq \int I \left(\int \delta(\tau \mp |\eta| - |\xi - \eta|) |\tilde{\varphi}(\xi - \eta)|^2 |\tilde{\psi}(\eta)|^2 d\eta \right) d\tau d\xi \\
&\leq \sup_{\tau, \xi} I \|\tilde{\varphi}\|_2^2 \|\tilde{\psi}\|_2^2,
\end{aligned}$$

where we denote

$$(3.12) \quad I = \int \delta(\tau \mp |\eta| - |\xi - \eta|) \tilde{\chi}_0^3(\xi - \eta) \tilde{\chi}_k^3(\eta) d\eta.$$

So it suffices to bound I by 2^{2k} . Changing the variables as $r = |\xi - \eta|$, $\rho = |\eta|$, and $\ell = |\xi|$, we have

$$\begin{aligned}
 (3.13) \quad I &= \frac{2\pi}{\ell} \int_0^\infty \int_{|\rho-\ell|}^{\rho+\ell} \delta(\tau \mp \rho - r) \tilde{\chi}_0^1(r) \tilde{\chi}_k^1(\rho) r \rho dr d\rho \\
 &\leq \frac{2\pi}{\ell} \int_{2^{k-1}}^{2^k} \rho d\rho \\
 &\leq C 2^{2k}.
 \end{aligned}$$

Thus we obtain the desired result. $\square$

To apply the bilinear estimate to the nonlinear wave equation, we have to extend the estimate to the inhomogeneous wave equation. Then we have the following estimate.

Proposition 3.2. *Let $-1/2 < \sigma < 1/2$, $\varphi \in \dot{B}_{2,1}^1(\mathbf{R}^3)$, $F \in \dot{B}_{2,1}^1(\mathbf{R}^4)$, $\psi \in \dot{B}_{2,1}^\sigma(\mathbf{R}^3)$ and $G \in \dot{B}_{2,1}^\sigma(\mathbf{R}^4)$. Let*

$$\begin{aligned}
 (3.14) \quad K_0 &= (e^{\pm i\omega t} \varphi)(e^{\pm i\omega t} \psi), \\
 K_1 &= (e^{\pm i\omega t} \varphi) \left(\int_0^t e^{\pm i\omega(t-s)} G(s) ds \right), \\
 K_2 &= \left(\int_0^t e^{\pm i\omega(t-s)} F(s) ds \right) (e^{\pm i\omega t} \psi), \\
 K_3 &= \left(\int_0^t e^{\pm i\omega(t-s)} F(s) ds \right) \left(\int_0^t e^{\pm i\omega(t-s)} G(s) ds \right).
 \end{aligned}$$

Let $\lambda(t) \in C_0^\infty(\mathbf{R})$ satisfy

$$(3.15) \quad \lambda(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| \geq 2, \end{cases}$$

and denote $\lambda_T(t) = \lambda(t/T)$. Then we have

$$\begin{aligned}
 (3.16) \quad \|\lambda_T K_0\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq C \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)}, \\
 \|\lambda_T K_1\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq C \sqrt{T} \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}, \\
 \|\lambda_T K_2\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq C \sqrt{T} \|F\|_{\dot{B}_{2,1}^1(\mathbf{R}^4)} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)}, \\
 \|\lambda_T K_3\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq CT \|F\|_{\dot{B}_{2,1}^1(\mathbf{R}^4)} \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)},
 \end{aligned}$$

where C is a positive constant dependent only on σ and λ .

To prove the above estimates, we use several types of Besov spaces on $\mathbf{R}^4$ and their relationships. For $\sigma < 3/2$ and $r \geq 1$ we define

$$(3.17) \quad \|u\|_{\dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4)} = \|2^{\sigma j} \|\chi_j^3 * u\|_{L^2(\mathbf{R}^4)}\|_{\ell^r(\mathbf{Z})}.$$

Remark that in general $\dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4)$ differs from $\dot{B}_{2,r}^{\sigma}(\mathbf{R}^4)$ or $L_t^2(\dot{B}_{2,r}^{\sigma}(\mathbf{R}^3))$, even if $\sigma = 0$. But they have the following inclusion relation.

Lemma 3.3. *Let $1 \leq r \leq \infty$ and $\sigma < 3/2$. We have*

$$(3.18) \quad \dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4) \hookrightarrow L_t^2(\dot{B}_{2,r}^{\sigma}(\mathbf{R}^3)) \quad \text{if } r \leq 2,$$

$$L_t^2(\dot{B}_{2,r}^{\sigma}(\mathbf{R}^3)) \hookrightarrow \dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4) \quad \text{if } r \geq 2,$$

$$(3.19) \quad \dot{B}_{2,r}^{\sigma}(\mathbf{R}^4) \hookrightarrow \dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4) \quad \text{if } \sigma > 0,$$

$$\dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4) \hookrightarrow \dot{B}_{2,r}^{\sigma}(\mathbf{R}^4) \quad \text{if } \sigma < 0.$$

Proof. (3.18) immediately follows from Minkowski's inequality $\ell^r(L_t^2) \hookrightarrow L_t^2(\ell^r)$ for $r \leq 2$ and $L_t^2(\ell^r) \hookrightarrow \ell^r(L_t^2)$ for $r \geq 2$. (3.19) is obvious at least in the case $r = 2$, and by the interpolation (3.22) and (3.21) we extend to general r . $\square$

We also mention the real interpolation for these Besov spaces.

Lemma 3.4. *Let $0 < \theta < 1$, $1 \leq r, r_0, r_1 \leq \infty$, $\sigma_0 < \sigma_1 < 3/2$ and $\sigma = \sigma_0(1 - \theta) + \sigma_1\theta$. Then we have the following interpolation properties.*

$$(3.20) \quad (L_t^2(\dot{B}_{2,r_0}^{\sigma_0}(\mathbf{R}^3)), L_t^2(\dot{B}_{2,r_1}^{\sigma_1}(\mathbf{R}^3)))_{\theta,r} = \dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4),$$

$$(3.21) \quad (\dot{B}_{2,r_0}^{\sigma_0(\xi)}(\mathbf{R}^4), \dot{B}_{2,r_1}^{\sigma_1(\xi)}(\mathbf{R}^4))_{\theta,r} = \dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4),$$

$$(3.22) \quad (\dot{B}_{2,r_0}^{\sigma_0}, \dot{B}_{2,r_1}^{\sigma_1})_{\theta,r} = \dot{B}_{2,r}^{\sigma},$$

where $(\cdot, \cdot)_{\theta,r}$ is the usual real interpolation functor.

Proof. (3.22) is well-known (see, e.g., [2]). $\dot{B}_{2,r}^{\sigma(\xi)}(\mathbf{R}^4)$ can be seen as L_t^2 -valued Besov space, so we have (3.21) also. By (3.18), we have $\dot{B}_{2,1}^{\sigma(\xi)}(\mathbf{R}^4) \hookrightarrow L_t^2(\dot{B}_{2,r}^{\sigma}(\mathbf{R}^3)) \hookrightarrow \dot{B}_{2,\infty}^{\sigma(\xi)}(\mathbf{R}^4)$, so that (3.20) follows from (3.21). $\square$

We need the following multiplication estimate for $\dot{B}_{2,1}^*(\mathbf{R}^3)$.

Lemma 3.5. *Let $\sigma_0, \sigma_1 < 3/2$, $\sigma_0 + \sigma_1 > 0$. Then we have*

$$(3.23) \quad \|\varphi\psi\|_{\dot{B}_{2,1}^{\sigma_0+\sigma_1-3/2}(\mathbf{R}^3)} \leq C\|\varphi\|_{\dot{B}_{2,1}^{\sigma_0}(\mathbf{R}^3)}\|\psi\|_{\dot{B}_{2,1}^{\sigma_1}(\mathbf{R}^3)}.$$

Proof. This estimate is well-known, but we give the proof for convenience. By the support property of $\{\chi_j^3\}$, we have

$$(3.24) \quad \|\varphi\psi\|_{\dot{B}_{2,1}^{\sigma_0+\sigma_1-3/2}} \leq \sum_{\substack{i \leq j+1 \\ |j-k| \leq 1}} + \sum_{\substack{j \leq k+1 \\ |k-i| \leq 1}} + \sum_{\substack{k \leq i+1 \\ |i-j| \leq 1}} C I_{j,k}^i 2^{(\sigma_0+\sigma_1-3/2)i},$$

where $I_{j,k}^i = \|\tilde{\chi}_i^3((\tilde{\chi}_j^3\tilde{\varphi}) * (\tilde{\chi}_k^3\tilde{\psi}))\|_2 \leq C\|\chi_j^3 * \varphi\|_2\|\chi_k^3 * \psi\|_2 2^{3\min(i,j,k)/2}$, by Young's and Hölder's inequalities. Then we obtain

$$(3.25) \quad \begin{aligned} \|\varphi\psi\|_{\dot{B}_{2,1}^{\sigma_0+\sigma_1-3/2}} &\leq \sum_{\substack{i \leq j+1 \\ |j-k| \leq 1}} C 2^{(\sigma_0+\sigma_1)(i-j)} 2^{\sigma_0 j} \|\chi_j^3 * \varphi\|_2 2^{\sigma_1 k} \|\chi_k^3 * \psi\|_2 \\ &\quad + \sum_{\substack{j \leq k+1 \\ |k-i| \leq 1}} C 2^{(3/2-\sigma_0)(j-k)} 2^{\sigma_0 j} \|\chi_j^3 * \varphi\|_2 2^{\sigma_1 k} \|\chi_k^3 * \psi\|_2 \\ &\quad + \sum_{\substack{k \leq i+1 \\ |i-j| \leq 1}} C 2^{(3/2-\sigma_1)(k-i)} 2^{\sigma_0 j} \|\chi_j^3 * \varphi\|_2 2^{\sigma_1 k} \|\chi_k^3 * \psi\|_2 \\ &\leq C \|\varphi\|_{\dot{B}_{2,1}^{\sigma_0}(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,1}^{\sigma_1}(\mathbf{R}^3)}. \quad \square \end{aligned}$$

Finally, we mention time-localizable property of the space-time Besov spaces.

Lemma 3.6. *Let $H(t)$ be the Heaviside function, namely*

$$(3.26) \quad H(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Let $\lambda(t) \in C_0^\infty(\mathbf{R})$, $T > 0$ and $\lambda_T(t) = \lambda(t/T)$. Let $-1/2 < \sigma < 1/2$ and $1 \leq r \leq \infty$. Then we have

$$(3.27) \quad \begin{aligned} \|H(t)u\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^4)} &\leq C\|u\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^4)}, \\ \|\lambda_T(t)u\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^4)} &\leq C\|u\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^4)}, \end{aligned}$$

where $C > 0$ depends only on σ and λ .

Proof. This estimate is also well-known. See, e.g., [12, Theorems 2.8.7, 2.8.5] for the corresponding results in the inhomogeneous Besov spaces. C does not depend on T since $\dot{B}_{2,r}^\sigma(\mathbf{R}^4)$ is homogeneous. $\square$

Now we can prove the bilinear estimate including the inhomogeneous terms.

Proof of Proposition 3.2. By Proposition 3.1 and Lemma 3.6, we obtain the estimate for K_0 . We estimate K_2 using Proposition 3.1 and Lemma 3.6 as

$$\begin{aligned}
 (3.28) \quad \|\lambda_T K_2\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq \int_0^{2T} \|H(t-s)\lambda_T(t)(e^{\pm i\omega(t-s)}F(s))(e^{\pm i\omega t}\psi)\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} ds \\
 &\leq C \int_0^{2T} \|(e^{\pm i\omega(t-s)}F(s))(e^{\pm i\omega t}\psi)\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} ds \\
 &\leq C \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)} \int_0^{2T} \|F(s)\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} ds \\
 &\leq C\sqrt{T} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)} \|F\|_{L_t^2(\dot{B}_{2,1}^1(\mathbf{R}^3))} \\
 &\leq C\sqrt{T} \|\psi\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^3)} \|F\|_{\dot{B}_{2,1}^1(\mathbf{R}^4)}.
 \end{aligned}$$

In the last inequality, we used Lemma 3.3. Similarly, if we obtain the estimate for K_1 , we can estimate K_3 as

$$\begin{aligned}
 (3.29) \quad \|\lambda_T K_3\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq \int_0^{2T} \left\| H(t-s)\lambda_T(t)(e^{\pm i\omega(t-s)}F(s)) \left(\int_0^t e^{\pm i\omega(t-r)}G(r)dr \right) \right\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} ds \\
 &\leq C \int_0^{2T} \left\| (e^{\pm i\omega(t-s)}F(s)) \left(\int_0^t e^{\pm i\omega(t-r)}G(r)dr \right) \right\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} ds \\
 &\leq C\sqrt{T} \int_0^{2T} \|F(s)\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} ds \\
 &\leq CT \|F\|_{\dot{B}_{2,1}^1(\mathbf{R}^4)} \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}.
 \end{aligned}$$

Thus, we only have to estimate K_1 . If we use the same argument as in (3.28), we obtain

$$(3.30) \quad \|\lambda_T K_1\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} \leq C\sqrt{T} \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|G\|_{L_t^2(\dot{B}_{2,1}^\sigma(\mathbf{R}^3))}.$$

By the real interpolation (3.20) and (3.22) we have

$$(3.31) \quad \|\lambda_T K_1\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} \leq C\sqrt{T} \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|G\|_{\dot{B}_{2,1}^{\sigma(s)}(\mathbf{R}^4)}.$$

Thus, in the case $0 < \sigma < 1/2$, (3.31) and (3.19) give the desired result. In the case $\sigma \leq 0$, we need more calculation. We decompose $G = G_\xi + G_\tau$ in the Fourier space as

$$(3.32) \quad \begin{aligned} \tilde{G}_\xi(\tau, \xi) &= 0, & \text{if } |\tau| > |\xi|, \\ \tilde{G}_\tau(\tau, \xi) &= 0, & \text{if } |\tau| \leq |\xi|. \end{aligned}$$

And we decompose $K_1 = K_\xi + K_\tau$ correspondingly. Then we can estimate K_ξ by (3.31), since for K_ξ , $\dot{B}_{2,1}^\sigma(\mathbf{R}^4)$ -norm is equivalent to $\dot{B}_{2,1}^{\sigma(\xi)}(\mathbf{R}^4)$ -norm. To estimate K_τ , we rewrite K_τ by

$$(3.33) \quad \int_0^t e^{\pm i\omega(t-s)} G_\tau(s) ds = J(t) - e^{\pm i\omega t} J(0) \pm \int_0^t e^{\pm i\omega(t-s)} I(s) ds,$$

where we denote

$$(3.34) \quad \begin{aligned} J(t) &= \int \frac{1}{i\tau} \tilde{G}_\tau(\tau) e^{i\tau t} \frac{d\tau}{2\pi}, \\ I(t) &= \int \frac{\omega}{\tau} \tilde{G}_\tau(\tau) e^{i\tau t} \frac{d\tau}{2\pi}. \end{aligned}$$

First we estimate the I -part of K_τ . For $\sigma > -1$ we have,

$$(3.35) \quad \begin{aligned} \|I\|_{\dot{B}_{2,2}^{\sigma(\xi)}(\mathbf{R}^4)} &\leq C \left\| \frac{|\xi|^{1+\sigma}}{|\tau|} \tilde{G}_\tau \right\|_2 \\ &\leq C \|\tau\|^\sigma \|\tilde{G}_\tau\|_2 \\ &\leq C \|G\|_{\dot{H}^\sigma}. \end{aligned}$$

By the real interpolation (3.21) and (3.22), we obtain

$$(3.36) \quad \|I\|_{\dot{B}_{2,1}^{\sigma(\xi)}(\mathbf{R}^4)} \leq C \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}.$$

So the I -part can be estimated by (3.31). To estimate the J -part of K_τ , we consider $L_t^\infty(\dot{B}_{2,1}^{\sigma+1/2}(\mathbf{R}^3))$ -norm of J .

$$(3.37) \quad \begin{aligned} \|J(t)\|_{\dot{H}^{\sigma+1/2}(\mathbf{R}^3)} &\leq C \left\| \int \frac{|\xi|^{\sigma+1/2}}{|\tau|} |\tilde{G}_\tau(\tau, \xi)| d\tau \right\|_{L_\xi^2} \\ &\leq C \left\| |\tau|^{\sigma-1/2} \|2^{j/2} \tilde{G}_\tau(\tau, \xi)\|_{L_t^2(2^{j-1} < \tau < 2^j)} \| \ell_j^1 \|_{L_\xi^2} \right\| \\ &\leq C \|G_\tau\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}. \end{aligned}$$

Remark that in the second inequality we used the assumption $\sigma + 1/2 > 0$. Then by the real interpolation (3.22), we obtain

$$(3.38) \quad \|J\|_{L_t^\infty(\dot{B}_{2,1}^{\sigma+1/2}(\mathbf{R}^3))} \leq C \|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}.$$

Then, using Lemma 3.5, we can estimate the J -part $K_J := (e^{\pm i\omega t}\varphi)(J(t) - e^{\pm i\omega t}J(0))$ as

$$\begin{aligned}
 (3.39) \quad \|\lambda_T K_J\|_{L_t^2(\dot{B}_{2,1}^\sigma(\mathbf{R}^3))} &\leq C\sqrt{T}\|K_J\|_{L_t^\infty(\dot{B}_{2,1}^\sigma(\mathbf{R}^3))} \\
 &\leq C\sqrt{T}\|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)}\|J\|_{L_t^\infty(\dot{B}_{2,1}^{\sigma+1/2}(\mathbf{R}^3))} \\
 &\leq C\sqrt{T}\|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)}\|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}.
 \end{aligned}$$

By the real interpolation (3.20) and (3.22), we obtain for $-1/2 < \sigma < 0$,

$$\begin{aligned}
 (3.40) \quad \|\lambda_T K_J\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)} &\leq C\|\lambda_T K_J\|_{\dot{B}_{2,1}^{\sigma(\xi)}(\mathbf{R}^4)} \\
 &\leq C\sqrt{T}\|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)}\|G\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)}.
 \end{aligned}$$

Thus we obtain the desired estimate (3.16) for $0 < \sigma < 1/2$ and for $-1/2 < \sigma < 0$. By the real interpolation (3.22), we obtain (3.16) for $-1/2 < \sigma < 1/2$. $\square$

4. Proof of local wellposedness

In this section we prove local wellposedness using the bilinear estimate proved in the previous section.

Theorem 4.1. (1.1) and (1.2) are time-locally wellposed respectively for $(\varphi, \psi) \in \dot{B}_{2,1}^1(\mathbf{R}^3) \oplus \dot{B}_{2,1}^0(\mathbf{R}^3)$ and $\dot{B}_{2,1}^2(\mathbf{R}^3) \oplus \dot{B}_{2,1}^1(\mathbf{R}^3)$.

Proof. (1.2) can be transformed into (1.1) by letting $u' = Du$. So we have only to do with (1.1). By Proposition 3.2, we can solve the equation by the iteration method. In fact, let

$$\begin{aligned}
 (4.1) \quad v_0 &= 0, \\
 u_k &= \cos \omega t \varphi + \frac{\sin \omega t}{\omega} \psi + \int_0^t \frac{\sin \omega(t-s)}{\omega} v_k(s) ds, \\
 v_k &= B(Du_{k-1}, u_{k-1}).
 \end{aligned}$$

Then, by Proposition 3.2, for small $T > 0$, $\lambda_T(t)v_k$ converges to some $v \in \dot{B}_{2,1}^0(\mathbf{R}^4)$. Now we have only to show that $u_k - u_0$ converges in $L_t^\infty((-T, T); \dot{B}_{2,1}^1(\mathbf{R}^3))$. For that purpose we decompose $v_k = v_\xi + v_\tau$ as in (3.32), and correspondingly $u_k - u_0 = w_\xi + w_\tau$ (we omit k for simplicity). For w_ξ we have

$$\begin{aligned}
(4.2) \quad \|w_\xi(t)\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} &\leq C\sqrt{|t|} \left\| \frac{\sin \omega(t-s)}{\omega} v_\xi(s) \right\|_{L_\xi^2(-|t|,|t|;\dot{B}_{2,1}^1(\mathbf{R}^3))} \\
&\leq C\sqrt{|t|} \|v_\xi\|_{L_t^2(\dot{B}_{2,1}^0(\mathbf{R}^3))} \\
&\leq C\sqrt{|t|} \|v\|_{\dot{B}_{2,1}^0(\mathbf{R}^4)}.
\end{aligned}$$

Using the formula (3.33), we have

$$(4.3) \quad \int_0^t \sin \omega(t-s) v_\tau(s) ds = -\sin \omega t J(0) - i \int_0^t \cos \omega(t-s) I(s) ds,$$

where

$$\begin{aligned}
(4.4) \quad J(t) &= \int \frac{1}{i\tau} \tilde{v}_\tau(\tau) e^{i\tau t} \frac{d\tau}{2\pi}, \\
I(t) &= \int \frac{\omega}{\tau} \tilde{v}_\tau(\tau) e^{i\tau t} \frac{d\tau}{2\pi}.
\end{aligned}$$

Using (3.36), we can estimate the I -part as

$$\begin{aligned}
(4.5) \quad \left\| \int_0^t \cos \omega(t-s) I(s) ds \right\|_{\dot{B}_{2,1}^0(\mathbf{R}^3)} &\leq \sqrt{|t|} \|I\|_{L_t^2(\dot{B}_{2,1}^0(\mathbf{R}^3))} \\
&\leq C\sqrt{|t|} \|v\|_{\dot{B}_{2,1}^0(\mathbf{R}^4)}.
\end{aligned}$$

For the J -part we have for σ near 0,

$$\begin{aligned}
(4.6) \quad \|\sin \omega t J(0)\|_{\dot{H}^\sigma(\mathbf{R}^3)} &\leq C \left\| |\xi|^\sigma \int \frac{\sin |\xi| t}{i\tau} \tilde{v}_\tau(\tau, \xi) e^{i\tau t} d\tau \right\|_{L_\xi^2} \\
&\leq C \left\| |\xi|^\sigma \int \frac{\sqrt{|\xi| |t|}}{|\tau|} |\tilde{v}_\tau(\tau, \xi)| d\tau \right\|_{L_\xi^2} \\
&\leq C\sqrt{|t|} \|v_\tau\|_{\dot{B}_{2,1}^\sigma(\mathbf{R}^4)},
\end{aligned}$$

where the last inequality follows from the same argument as in (3.37). Using the real interpolation (3.21) and (3.22), we obtain

$$\begin{aligned}
(4.7) \quad \|w_\tau(t)\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} &\leq C \left\| \int_0^t \cos \omega(t-s) I(s) ds \right\|_{\dot{B}_{2,1}^0(\mathbf{R}^3)} + \|\sin \omega t J(0)\|_{\dot{B}_{2,1}^0(\mathbf{R}^3)} \\
&\leq C\sqrt{|t|} \|v\|_{\dot{B}_{2,1}^0(\mathbf{R}^4)}.
\end{aligned}$$

Thus $u_k - u_0$ converges in $L_t^\infty((-T, T); \dot{B}_{2,1}^1(\mathbf{R}^3))$ and we obtain a time-local solution in this class. Since the estimates for the difference of two solutions are just the same, we obtain the uniqueness and continuity of dependence on the data in the same manner. $\square$

5. Optimality of bilinear estimates

In this section we show that the bilinear estimates (Proposition 3.1) are optimal with respect to the choice of the exponents of the Besov spaces. First, we see that $\dot{B}_{2,1}^1(\mathbf{R}^3)$ can not be replaced by $\dot{B}_{2,r}^1$ for any $r > 1$.

Proposition 5.1. *For any $r > 1$, $\sigma_0, \sigma_1 \in \mathbf{R}$, the following estimate is false.*

$$(5.1) \quad \|uv\|_{H^{\sigma_0}(\mathbf{R}^4)} \leq C \|\varphi\|_{\dot{B}_{2,r}^1(\mathbf{R}^3)} \|\psi\|_{H^{\sigma_1}(\mathbf{R}^3)},$$

where $u = e^{i\omega t}\varphi$ and $v = e^{i\omega t}\psi$, and H^σ denotes the usual inhomogeneous Sobolev spaces.

Proof. Let $J \in \mathbf{Z}$ and $J \leq -5$. For any $\xi \in \mathbf{R}^3$, we denote $\xi = (\xi_1, \xi^1) \in \mathbf{R} \times \mathbf{R}^2$. For any $j \in \mathbf{Z}$ satisfying $2J \leq j \leq -5$, define E_j, F and $G \subset \mathbf{R}^3$ by

$$(5.2) \quad \begin{aligned} E_j &= \{\xi \in \mathbf{R}^3 \mid 2^{j-1} < \xi_1 < 2^j, |\xi^1| < 2^{J+j/2}\}, \\ F &= \{\xi \in \mathbf{R}^3 \mid 1/2 < \xi_1 < 1, |\xi^1| < 2^{J+1}\}, \\ G &= \{\xi \in \mathbf{R}^3 \mid 5/8 < \xi_1 < 7/8, |\xi^1| < 2^J\}, \end{aligned}$$

and we denote their characteristic functions by $E_j(\cdot)$, $F(\cdot)$, $G(\cdot)$. Now let $\{a_j\}_{j \in \mathbf{Z}}$ be a sequence of nonnegative numbers, $\tilde{\varphi}_j(\xi) = a_j E_j(\xi)$, $\tilde{\psi}(\xi) = F(\xi)$, $u_j = e^{i\omega t}\varphi_j$, $v = e^{i\omega t}\psi$, $\varphi = \sum_{j=2J}^{-5} \varphi_j$ and $u = \sum_{j=2J}^{-5} u_j$. Suppose $(\tau, \xi) \in \text{supp } \widetilde{u_j v}$, $\xi \in G$ and $\eta \in E_j$. Then, for $\xi - \eta$ we have

$$\begin{aligned} 1/2 &< 5/8 - 2^j < \xi_1 - \eta_1 < 7/8 + 2^j < 1, \\ |\xi^1 - \eta^1| &\leq 2^J + 2^{J+j/2} < 2^{J+1}, \end{aligned}$$

so that $\xi - \eta \in F$. For $\tau = |\xi - \eta| + |\eta|$, using the inequality

$$(5.3) \quad |\xi| \leq |\xi_1| + \frac{|\xi^1|^2}{2|\xi_1|},$$

we have

$$(5.4) \quad \begin{aligned} |\xi| &\leq \tau = |\xi - \eta| + |\eta| \leq \xi_1 - \eta_1 + \frac{|\xi^1 - \eta^1|^2}{2(\xi_1 - \eta_1)} + \eta_1 + \frac{|\eta^1|^2}{2\eta_1} \\ &\leq \xi_1 + 2^{2J+2} + 2^{2J+j-j} \\ &\leq |\xi| + 2^{2J+3}. \end{aligned}$$

So, by Hölder's inequality we have for $\xi \in G$,

$$\begin{aligned}
 (5.5) \quad \|\tilde{u}v(\tau, \xi)\|_{L_t^2} &\geq 2^{-(2J+3)/2} \|\tilde{u}v(\tau, \xi)\|_{L_t^1} \\
 &= 2^{-(2J+3)/2} \iint \delta(\tau - |\xi - \eta| - |\eta|) \sum_{j=2J}^{-5} a_j E_j(\eta) d\eta d\tau \\
 &= 2^{-(2J+3)/2} \sum_{j=2J}^{-5} a_j \text{meas}(E_j) \\
 &\geq \sum_{j=2J}^{-5} a_j 2^{2j+J} / C.
 \end{aligned}$$

Thus we have an estimate for uv from below,

$$\begin{aligned}
 (5.6) \quad \|uv\|_{H^{\sigma_0}(\mathbf{R}^4)} &\geq \|\tilde{u}v\|_{L_\xi^2(G; L_t^2)} / C \\
 &\geq \text{meas}(G)^{1/2} \sum_{j=2J}^{-5} a_j 2^{2j+J} / C \\
 &\geq \sum_{j=2J}^{-5} a_j 2^{2j+2J} / C.
 \end{aligned}$$

On the other hand, we can estimate the right hand side of (5.1) as

$$\begin{aligned}
 (5.7) \quad \|\varphi\|_{\dot{B}_{2,r}^1(\mathbf{R}^3)} \|\psi\|_{H^{\sigma_1}(\mathbf{R}^3)} &\leq C \|2^j a_j \text{meas}(E_j)^{1/2}\|_{\ell^r(\mathbf{Z})} \text{meas}(F)^{1/2} \\
 &\leq C \|a_j 2^{2j+2J}\|_{\ell^r(2J \leq j \leq -5)}.
 \end{aligned}$$

Thus, to have (5.1) for any $J \leq -5$ and $\{a_j\}$, we need $r \leq 1$. $\square$

Next we examine the second norm $\dot{B}_{2,1}^\sigma(\mathbf{R}^3)$. By the real interpolation, it follows from (3.1) that

$$(5.8) \quad \|uv\|_{\dot{B}_{2,r'}^\sigma(\mathbf{R}^4)} \leq C \|\varphi\|_{\dot{B}_{2,1}^1(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^3)},$$

for $r' \geq r \geq 1$ and $-1/2 < \sigma < 1$. We see that $r' \geq r$ is necessary.

Proposition 5.2. *For any $\sigma, \sigma' \in \mathbf{R}$, $1 \leq r' < r$, the following estimate does not hold, even if φ and ψ are spherically symmetric.*

$$(5.9) \quad \|uv\|_{\dot{B}_{2,r'}^\sigma(\mathbf{R}^4)} \leq C \|\varphi\|_{H^{\sigma'}(\mathbf{R}^3)} \|\psi\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^3)}.$$

Proof. This is a consequence of the optimality of (3.4), which can be seen by an easy example as follows. Let $j \leq k-2$, $u = e^{i\omega t} \chi_j^3$ and $v = e^{i\omega t} \chi_k^3$. Let

$A = \{\xi | 2^{k-1} + 2^{k-2} < |\xi| < 2^k - 2^{k-2}\}$. If $\xi \in A$ and $\eta \in \text{supp } \tilde{\chi}_j^3$, then $\xi - \eta \in \text{supp } \tilde{\chi}_k^3$. So, by (3.6), we have for $\xi \in A$

$$(5.10) \quad \tilde{u}v(\tau, \xi) = \int \delta(\tau - |\eta| - |\xi - \eta|) \tilde{\chi}_j^3(\eta) d\eta.$$

For $\tau \in \text{supp } \tilde{u}v(\cdot, \xi)$, we have

$$(5.11) \quad |\xi| \leq \tau \leq |\xi| + 2|\eta| \leq |\xi| + 2^{j+1}.$$

By (5.10) and (5.11) we have for $\xi \in A$

$$(5.12) \quad \begin{aligned} \|\tilde{u}v(\tau, \xi)\|_{L_t^2} &\geq 2^{-(j+1)/2} \|\tilde{u}v(\tau, \xi)\|_{L_t^1} \\ &= 2^{-(j+1)/2} \iint \delta(\tau - |\eta| - |\xi - \eta|) \tilde{\chi}_j^3(\eta) d\eta d\tau \\ &= 2^{-(j+1)/2} \text{meas}(\{2^{j-1} < |\xi| < 2^j\}) \\ &\geq 2^{5j/2}/C. \end{aligned}$$

Thus we have

$$(5.13) \quad \begin{aligned} \|uv\|_2 &\geq \text{meas}(A)^{1/2} 2^{5j/2}/C \\ &\geq 2^{3/2k+5j/2}/C \\ &\geq 2^j \|\chi_j^3\|_2 \|\chi_k^3\|_2 / C. \end{aligned}$$

Now let $\{a_k\}_{k=1}^\infty$ be a sequence of nonnegative numbers, $\varphi = \chi_0^3$ and $\psi = \sum_{k=2}^\infty a_k \chi_k^3$. Then, for $u = e^{i\omega t} \varphi$ and $v = e^{i\omega t} \psi$ we have

$$(5.14) \quad \begin{aligned} \|uv\|_{\dot{B}_{2,r}^\sigma(\mathbf{R}^4)} &\geq \|2^{\sigma k} u e^{i\omega t} a_k \chi_k^3\|_2 \|_{\ell^r} / C \\ &\geq \|2^{\sigma k} \|\chi_0^3\|_2 a_k \|\chi_k^3\|_2\|_{\ell^r} / C \\ &\geq \|\chi_0^3\|_{H^{\sigma'}(\mathbf{R}^3)} \|2^{\sigma k} a_k \|\chi_k^3\|_2\|_{\ell^r(\mathbf{Z})} / C. \end{aligned}$$

Thus we need $r' \geq r$ for (5.9). $\square$

6. Illposedness for the equation of wave map type

In [9], the following result was obtained.

Theorem 6.1. *Let $n \geq 2$, $f \in C^{[n/2+1]}(\mathbf{R})$ and $X := \dot{B}_{2,1}^{n/2}(\mathbf{R}^n) \oplus \dot{B}_{2,1}^{n/2-1}(\mathbf{R}^n)$. Then, there exists $\delta > 0$ such that for any $(\varphi, \psi) \in X$ with $\|(\varphi, \psi)\|_X < \delta$, (1.4) has a unique global solution u satisfying $(u, \dot{u}) \in C_B(\mathbf{R}; X)$ and the correspondence $(\varphi, \psi) \mapsto (u, \dot{u})$ is Lipschitz continuous from X to $C_B(\mathbf{R}; X)$.*

The same result was obtained in [11] in the case $n \geq 4$ for the general system of equations for wave maps. On the other hand, it was also proved in [9] that (1.4) is illposed in $B_{2,r}^{n/2} \oplus B_{2,r}^{n/2-1}$ for any $r > 1$ and $n \geq 1$, even time-locally and in small neighborhood of 0 in X , provided f does not satisfy

$$(6.1) \quad \int_0^\infty \exp\left(\int_0^s f(r)dr\right)ds = \infty \quad \text{and} \quad \int_{-\infty}^0 \exp\left(\int_0^s f(r)dr\right)ds = \infty.$$

Thus, only the case $n = 1$ and $r = 1$ remains to be investigated. It was conjectured in [9] that we can not extend Theorem 6.1 to the case $n = 1$. In this section we show that the conjecture is true. In fact, we have the following illposedness result.

Theorem 6.2. *Let $n = 1$ and $X := B_{2,1}^{1/2}(\mathbf{R}) \oplus \dot{B}_{2,1}^{-1/2}(\mathbf{R})$. Suppose $f \in C^{\ell-1}$ and $f^{(\ell-1)}(0) \neq 0$ for some $\ell \in \mathbf{N}$. Then, for any $\varepsilon > 0$ there exists $(\varphi, \psi) \in X$ with $\|(\varphi, \psi)\|_X < \varepsilon$ such that (1.4) has a unique global solution u with small L^∞ -norm, but $u(t) \notin B_{2,r}^{1/2}(\mathbf{R})$ for any $t \neq 0$ and $r < \infty$.*

The singularity is much weaker for the solutions constructed in the proof of the above theorem than that constructed in [9] in the case $r > 1$. Nevertheless the above result is interesting, since we have illposedness for any generic f and for solutions which stay near 0, and so it is meaningful for the theory of wave maps. In [4], another negative result was obtained against the wellposedness in $H^{1/2}(\mathbf{R})$ for the wave maps.

Proof of Theorem 6.2. Let

$$(6.2) \quad G(u) := \int_0^u \exp\left(\int_0^s f(r)dr\right)ds.$$

If u satisfies (1.4), then $v := G(u)$ satisfies the linear wave equation

$$(6.3) \quad \begin{cases} \square v = 0, \\ v(0) = G(\varphi), \quad \dot{v}(0) = G'(\varphi)\psi. \end{cases}$$

We may assume without loss of generality that

$$(6.4) \quad f(0) = \dots = f^{(\ell-2)}(0) = 0 \neq f^{(\ell-1)}(0).$$

Then, denoting $Z(\varphi) := G'(\varphi) - 1$, we have

$$(6.5) \quad Z(0) = \dots = Z^{(\ell-1)}(0) = 0 \neq Z^{(\ell)}(0), \quad Z \in C^\ell,$$

and $Z(u) \neq 0$ for small $u \neq 0$. Let $\Omega := Z^{1/\ell}$. Since $\Omega(0) = 0 \neq \Omega'(0)$ and $G(0) = 0 \neq G'(0)$, there exists $\delta > 0$ dependent only on f such that $\Omega, \Omega^{-1}, G,$

$G^{-1} \in C^1(-\delta, \delta)$. If v satisfies (6.3) and $|v| < \delta$, then $u := G^{-1}(v)$ satisfies (1.4). Moreover, if $u(t) \in B_{2,r}^{1/2} \cap L^\infty$ for some t and r , we have also $v(t) = G(u(t)) \in B_{2,r}^{1/2}$, which follows from Lemma 6.3 below. So, it suffices to find a solution v of (6.3) satisfying

- (i) $\|(\varphi, \psi)\|_X < \varepsilon$,
- (ii) $|v(t, x)| < \delta$ for any (t, x) ,
- (iii) $v(t) \notin B_{2,r}^{1/2}$ for any $t \neq 0$ and $r < \infty$,

for any small $\varepsilon > 0$. Let $\Psi := G'(\varphi)\psi$. By Lemma 6.3, it follows $\|(G(\varphi), \psi)\|_X < C\varepsilon$ from (i). So, we need $\Psi - \psi \notin \dot{B}_{2,1}^{-1/2}$ to have (iii). Let $a \in \ell^1(N)$ satisfy $\|a\|_{\ell^1(N)} \leq 1$ and $g \in C_0^\infty(\mathbf{R})$ satisfy $0 \leq g \leq 1$, $g(\xi) = g(-\xi)$, $g \neq 0$ and $\text{supp } g \in (-2, -1) \cup (1, 2)$. Define the initial data as

$$\begin{aligned} \tilde{\psi}(\xi) &:= \varepsilon^2 \sum_{k \in N} a_k g(2^{-k}\xi), \\ \tilde{\Phi}(\xi) &:= \varepsilon^2 \sum_{k \in N} a_k 2^{-k} g(2^{-k}\xi), \\ \varphi &:= \Omega^{-1}(\Phi). \end{aligned} \tag{6.6}$$

The above definition for φ is possible provided ε is sufficiently small relative to δ , since $\|\Phi\|_{\dot{B}_{2,1}^{1/2}} \leq C\varepsilon^2 \|a\|_{\ell^1}$ and $\dot{B}_{2,1}^{1/2} \hookrightarrow L^\infty$. Similarly we have $\|\psi\|_{\dot{B}_{2,1}^{-1/2}} \leq C\varepsilon^2$ and by Lemma 6.3, we have $\|(\varphi, \psi)\|_X < C\varepsilon^2 < \varepsilon$. By the definition of Ω , we have $\Psi - \psi = \Phi^\ell \psi = Z(\varphi)\psi$. Then we estimate

$$\widetilde{\Phi^\ell \psi}(\xi) \geq \varepsilon^{2(\ell+1)} \sum_{k \in N} a_k^{2(\ell+1)} \underbrace{(g * \cdots * g)}_{\ell+1 \text{ times}}(2^{-k}\xi) / C. \tag{6.7}$$

Let

$$\hat{g} := \underbrace{g * \cdots * g}_{\ell+1 \text{ times}}. \tag{6.8}$$

Since g is positive even function, there exists $\alpha > 0$ depending only on g , such that $\hat{g}(\xi) > \alpha$ for $|\xi| < \alpha$. Then we have

$$\widetilde{\Phi^\ell \psi}(\xi) \geq \varepsilon^{2(\ell+1)} \alpha \sum_{|\xi| < 2^k \alpha} a_k^{2(\ell+1)} / C. \tag{6.9}$$

Denote

$$b_j := \sum_{2^j < 2^k \alpha} a_k^{2(\ell+1)}. \tag{6.10}$$

Now we can estimate the component in $v(t)$ coming from $\Psi - \psi = \Phi^\ell \psi$ as follows.

$$\begin{aligned}
(6.11) \quad \left\| \frac{\sin \omega t}{\omega} \Phi^\ell \psi \right\|_{\dot{B}_{2,r}^{1/2}} &\geq \varepsilon^{2(\ell+1)} \alpha \left\| 2^{-j/2} b_j \sqrt{\int_{2^{j-1} < |\xi| < 2^j} \sin^2 |\xi| t d\xi} \right\|_{\ell^r} / C \\
&\geq \varepsilon^{2(\ell+1)} \alpha \left(\sum_{|2^j t| \gg 1} b_j^r \right)^{1/r} / C.
\end{aligned}$$

It is easy to construct $a \in \ell^1$ satisfying $\|a\|_{\ell^1} \leq 1$ and $b \notin \ell^r$ for any $r < \infty$. For example, let

$$(6.12) \quad a_k = \begin{cases} 2^{-j-1}, & \text{if } k = 2^{2^j}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $v(t) \notin B_{2,r}^{1/2}$ for any $t \neq 0$ and $r < \infty$. Finally, we estimate $\|v\|_{L^\infty}$. It suffices to estimate the component coming from $\Psi - \psi = Z(\varphi)\psi$.

$$\begin{aligned}
(6.13) \quad \left| \int_{x-t}^{x+t} Z(\varphi)\psi(y) dy \right| &\leq C \|\chi_{(x-t, x+t)} Z(\varphi)\|_{\dot{B}_{2,\infty}^{1/2}} \|\psi\|_{\dot{B}_{2,1}^{-1/2}} \\
&\leq C \|Z(\varphi)\|_{\dot{B}_{2,1}^{1/2}} \|\psi\|_{\dot{B}_{2,1}^{-1/2}} \\
&\leq C\varepsilon^4 < \delta,
\end{aligned}$$

where $\chi_{(x-t, x+t)}$ denotes the characteristic function of the interval $(x-t, x+t)$, and the second inequality follows from Lemma 6.4 below. Thus we obtain (i)–(iii) and the desired result. $\square$

Lemma 6.3. *Let $G \in C^1(\mathbf{R})$, $G(0) = 0$ and $\varphi \in B_{2,r}^{1/2} \cap L^\infty$ for some $1 \leq r \leq \infty$. Then we have*

$$(6.14) \quad \|G(\varphi)\|_{B_{2,r}^{1/2}} \leq C \sup_{|\lambda| < \|\varphi\|_{L^\infty}} |G'(\lambda)| \|\varphi\|_{B_{2,r}^{1/2}},$$

where C is a positive universal constant. The same estimate holds for the homogeneous Besov norm.

Proof. By the well-known representation of the Besov norm (see, e.g., [2] [9, Lemma A.5]), we can estimate the left hand side of (6.14) as

$$\begin{aligned}
(6.15) \quad \cdots &\leq C \|G(\varphi)\|_{L^2} + C \|2^{-j/2} \|G(\varphi(x-2^j)) - G(\varphi(x))\|_{L^2}\|_{\ell^r(\mathbf{Z})} \\
&\leq C \sup_{|\lambda| < \|\varphi\|_{L^\infty}} |G'(\lambda)| (\|\varphi\|_{L^2} + \|2^{-j/2} \|\varphi(x-2^j) - \varphi(x)\|_{L^2}\|_{\ell^r(\mathbf{Z})}),
\end{aligned}$$

which is estimated by the right hand side of (6.14). $\square$

Lemma 6.4. *Let $\varphi \in \dot{B}_{2,1}^{1/2}$ and H be the Heaviside function. Then we have*

$$(6.16) \quad \|H\varphi\|_{\dot{B}_{2,\infty}^{1/2}} \leq C\|\varphi\|_{\dot{B}_{2,1}^{1/2}},$$

where C is a positive universal constant.

Proof. Again by the difference representation of the Besov norm, we can estimate the left hand side of (6.16) as

$$(6.17) \quad \begin{aligned} \dots &\leq C \sup_{k \in \mathbb{Z}} 2^{-k/2} \|(H\varphi)(x - 2^k) - (H\varphi)\|_{L^2} \\ &\leq C \sup_{k \in \mathbb{Z}} 2^{-k/2} \|H(x - 2^k) - H(x)\|_{L^2} \|\varphi\|_{L^\infty} + 2^{-k/2} \|\varphi(x - 2^k) - \varphi\|_{L^2} \\ &\leq C\|\varphi\|_{L^\infty} + C\|\varphi\|_{\dot{B}_{2,1}^{1/2}}, \end{aligned}$$

and the desired result follows from the embedding $\dot{B}_{2,1}^{1/2} \hookrightarrow L^\infty$. $\square$

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