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(Citation)

Funkcialaj Ekvacioj, 42(3):325-338

(Issue Date)

1999-12

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100499969>

(URL)

<https://hdl.handle.net/20.500.14094/0100499969>



## Generalized and Classical Almost Periodic Solutions of Lagrangian Systems

By

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### Introduction

A variational method of finding the almost (quasi-) periodic solutions of a Lagrangian system

$$(1) \quad \ddot{x} = \frac{\partial V(t, x)}{\partial x}$$

with almost (quasi-) periodic in  $t$  force function  $V: \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$  consists in finding the extremals of the functional

$$(2) \quad J_0(x(t)) = M\left(\frac{1}{2} \|\dot{x}(t)\|^2 + V(t, x(t))\right),$$

where  $\|\cdot\|$  is the Euclidean norm in  $\mathbf{R}^n$ , generated by a standard scalar product  $\langle \cdot, \cdot \rangle$ ,  $M(\cdot) = \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \cdot dt$ . Such an approach has been developed in detail for periodic systems (for example, see [1]–[5]). Some works of J. Blot are dedicated to the almost periodic case ([7]–[8]). When solving the minimization problem for the functional  $J_0$  one has to extend it from the space  $CAP^1(\mathbf{R}; \mathbf{R}^n)$  of  $\mathbf{R}^n$ -valued continuous almost periodic functions with almost periodic derivatives to the space of Besicovitch-almost periodic functions, possessing generalized (in the sense of Sobolev) Besicovitch-almost periodic derivatives. Such spaces were effectively used in the theory of non-linear oscillations of finite—and infinite dimensional systems in [9], [10]. The use of Besicovitch functions makes it comparatively easy to obtain the generalized solutions of equation (1) under the condition of global (in  $x$ ) convexity of a force function. It is rather difficult to detect whether or not these solutions are classical due to the absence of embedding theorems in the spaces of such functions. The way for overcoming the above described difficulty is shown in this article in the case of quasiperiodic in  $t$  function  $V(t, x)$  (see section 4). On the other hand, it became possible to substitute the condition of global convexity in  $x$  for  $V(t, x)$  with the demand of convexity on a compact subset  $\Omega \subset \mathbf{R}^n$ . At the same time, it enabled us to establish the existence of a bounded Besicovitch-almost periodic minimizing function (section 2) and to generalize the results of [6], obtained for

$n = 1$  and in the special case  $V(t, x) = \cos(x - E(t))$ ,  $E(t) \in CAP(\mathbf{R}; \mathbf{R})$ . Note, that our approach, as distinct from [6], does not need the use of Bochner-Fejer polynomials.

### 1. Range of definition of the functional

Denote by  $B_p(\mathbf{R}; \mathbf{R}^n)$  the space of  $\mathbf{R}^n$ -valued Besicovitch almost periodic functions. Function  $f$  belongs to  $B_p(\mathbf{R}; \mathbf{R}^n)$ , if  $\|f(t)\|^p$  is locally integrable, and for any  $\varepsilon > 0$  there exists such a trigonometric polynomial  $P_\varepsilon(t)$  with coefficients in  $\mathbf{R}^n$ , that  $\overline{M}(\|f(t) - P_\varepsilon(t)\|^p) < \varepsilon$ ,  $\overline{M}\{\cdot\} = \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \cdot dt$ . Let  $\langle f, g \rangle_0 = M(\langle f, g \rangle)$ , where  $\|f\|_0 = M^{1/2}(\|f\|^2)$ . Sobolev generalized derivative of the function  $f \in B_2(\mathbf{R}; \mathbf{R}^n)$  with respect to the product  $\langle \cdot, \cdot \rangle_0$  (if it exists) denote by  $Df(t)$ . Functions from  $B_2(\mathbf{R}; \mathbf{R}^n)$ , possessing generalized derivatives, form the space  $B_2^1(\mathbf{R}; \mathbf{R}^n)$ . Let the seminorm in that space be  $\|\cdot\|_1 = \|D \cdot\|_0 + \|\cdot\|_0$ . Let  $\Omega \subset \mathbf{R}^n$  be a compact convex set;  $C^r(\Omega)$  be the space of  $r$  times continuously differentiable functions in  $\Omega$ . Introduce the space  $B_2(\mathbf{R}; C^r(\Omega))$ , formed by functions  $g: \mathbf{R} \times \Omega \rightarrow \mathbf{R}$  with the following properties: 1) for any fixed  $t \in \mathbf{R}$  the function  $x \rightarrow g(t, x)$  belongs to  $C^r(\Omega)$ ; 2) for any fixed  $x \in \Omega$  and for any  $\rho = (\rho_1, \dots, \rho_n) \in (N \cup \{0\})^n$ ,  $0 \leq \rho_1 + \dots + \rho_n \leq r$  the function  $t \rightarrow \partial^\rho g(t, x)/\partial x^\rho$  is measurable; 3)  $(\max_{x \in \Omega} |\partial^\rho g(t, x)/\partial x^\rho|)^2$  is locally integrable and for any  $\varepsilon > 0$  there exists a trigonometric polynomial (in  $t$ )  $P_\varepsilon^\rho(t, x) = \sum_{k=1}^{N(\varepsilon)} a_k^\rho(x) e^{i\lambda_k t}$ ,  $N(\varepsilon) \in \mathbf{N}$ ,  $\lambda_k \in \mathbf{R}$ ,  $a_k^\rho(x) \in C(\Omega)$ , such that

$$\overline{M}^{1/2} \left( \max_{x \in \Omega} \left| \frac{\partial^\rho g(t, x)}{\partial x^\rho} - P_\varepsilon^\rho(t, x) \right|^2 \right) < \varepsilon.$$

*Remark 1.1.* Without loss of generality we can suppose that  $a_k^\rho(x) \in C^\infty(\Omega)$ . Denote by  $\overline{X}_\Omega$  the closure of the set  $X_\Omega = \{x(t) \in CAP^1(\mathbf{R}; \mathbf{R}^n) : x(\mathbf{R}) \in \Omega\}$  in the seminorm  $\|\cdot\|_1$ , put

$$\hat{X}_\Omega = \{x(t) \in \overline{X}_\Omega : x(\mathbf{R}) \in \Omega\}$$

**Proposition 1.1.** *Let  $V(t, x) \in B_2(\mathbf{R}; C(\Omega))$ . Then the functional*

$$(3) \quad J(x(t)) = M\left(\frac{1}{2} \|Dx(t)\|^2 + V(t, x(t))\right)$$

*is defined and continuous on  $\hat{X}_\Omega$ .*

The proof follows from Lemma 1.3, which is quoted below. Now we introduce a mapping  $P_\Omega: \mathbf{R}^n \rightarrow \Omega$ , where  $P_\Omega x := x$ , if  $x \in \Omega$ ; if  $x \in \mathbf{R}^n \setminus \Omega$ , then  $P_\Omega x$  is the nearest to  $x$  point of  $\partial\Omega$ . The Lemma formulated below follows from convexity of  $\Omega$ .

**Lemma 1.1.** For any  $x, y \in \mathbf{R}^n$  the following inequality holds:

$$\|P_\Omega x - P_\Omega y\| \leq \|x - y\|.$$

**Lemma 1.2.**  $\hat{X}_\Omega$  is a closed convex subset of  $B_2^1(\mathbf{R}; \mathbf{R}^n)$ .

*Proof.* Convexity of  $\hat{X}_\Omega$  stems from convexity of  $\Omega$ . The set  $\bar{X}_\Omega$  is closed, hence it is sufficient to show that for any  $x(t) \in \bar{X}_\Omega$  there exists  $\hat{x}(t) \in \hat{X}_\Omega$  such that  $\|x(t) - \hat{x}(t)\|_0 = 0$ . Such a condition holds for  $\hat{x}(t) = P_\Omega x(t)$ . In fact, let  $x_n(t) \in X_\Omega$ ,  $n = 1, 2, \dots$ , be a sequence, converging to  $x(t)$  in the seminorm  $\|\cdot\|_1$ . Since

$$\|\hat{x}(t) - x_n(t)\| = \|P_\Omega x(t) - P_\Omega x_n(t)\| \leq \|x(t) - x_n(t)\|,$$

we have  $\hat{x}(t) \in \hat{X}_\Omega$  and  $\|\hat{x}(t) - x(t)\|_0 = 0$ .  $\square$

**Lemma 1.3.** If  $V(t, x) \in B_2(\mathbf{R}; C(\Omega))$ , then the operator, defined by the equality  $\mathbf{V}[x(\cdot)](t) = V(t, x(t))$ , maps  $\hat{X}_\Omega$  into  $B_2(\mathbf{R}; \mathbf{R})$  continuously with respect to the seminorm  $\|\cdot\|_0$ .

*Proof.* Let  $x(t) \in \hat{X}_\Omega$ , and let  $x_n(t) \in X_\Omega$ ,  $n = 1, 2, \dots$ , be a sequence such that  $\|x(t) - x_n(t)\|_0 \rightarrow 0$ ,  $n \rightarrow \infty$ . For any  $\varepsilon > 0$  define a polynomial  $P_\varepsilon(t, x) = \sum_{k=1}^{N(\varepsilon)} a_k(x) e^{i\lambda_k t}$  as in the definition of  $B_2(\mathbf{R}; C(\Omega))$ . Then

$$\bar{M}^{1/2} (|V(t, x) - P_\varepsilon(t, x_n(t))|^2) \leq \varepsilon + \sum_{k=1}^{N(\varepsilon)} \max_{x \in \Omega} \left\| \frac{\partial a_k(x)}{\partial x} \right\| \|x(t) - x_n(t)\|_0 \leq 2\varepsilon,$$

if we choose  $n = n(\varepsilon)$  large enough. Hence,  $V(t, x(t)) \in B_2(\mathbf{R}; \mathbf{R})$ . Continuity of  $\mathbf{V}$  follows from the estimate

$$\|V(t, x(t)) - V(t, y(t))\|_0 \leq 2\varepsilon + \sum_{k=1}^{N(\varepsilon)} \max_{x \in \Omega} \left\| \frac{\partial a_k(x)}{\partial x} \right\| \|x(t) - y(t)\|_0. \quad \square$$

Now in order to prove Proposition 1.1 it suffices to note that  $B_2(\mathbf{R}; \mathbf{R}) \subset B_1(\mathbf{R}; \mathbf{R})$  and  $M(|\cdot|) \leq M^{1/2}(|\cdot|^2)$ .

## 2. Existence of minimum

For any set  $\mathcal{M} \subset \mathbf{R}^n$  denote by  $\mathcal{M} + \delta$  its closed  $\delta$ -neighbourhood.

**Theorem 2.1.** Let  $V(t, x) \in B_2(\mathbf{R}; C(\Omega))$ , assume that there exists a closed convex set  $\Theta \subset \mathbf{R}^n$  such that  $\Theta + \delta \subset \Omega$  and for any  $t \in \mathbf{R}$  the following con-

ditions hold:

$$(4) \quad V(t, \lambda x + (1 - \lambda)y) \leq \lambda V(t, x) + (1 - \lambda)V(t, y) \\ \forall x, y \in \Theta, \quad \forall \lambda \in [0, 1];$$

$$(5) \quad V(t, x) \geq V(t, P_\Theta x) \quad \forall x \in \Theta + \delta.$$

Then there exists a function  $x_*(t) \in \hat{X}_\Theta$ , such that

$$(6) \quad J(x_*(t)) = \inf\{J(x) | x \in \hat{X}_{\Theta+\delta}\}.$$

If, in addition, we suppose that  $V(t, x) \in B_2(\mathbf{R}; C^1(\Omega))$ , then  $x_*(t)$  is a Sobolev generalized solution of equation (1), i.e.

$$(7) \quad A[y(\cdot)] := \langle Dx_*(t), \dot{y}(t) \rangle_0 + \left\langle \frac{\partial V(t, x_*(t))}{\partial x}, y(t) \right\rangle_0 = 0 \\ \forall y(t) \in CAP^1(\mathbf{R}; \mathbf{R}^n).$$

*Proof.* Using the constructions, described in [6], it is easy to establish the existence of such a function  $x_*(t) \in \hat{X}_\Theta$  that  $J(x_*(t)) = \inf\{J(x) | x \in \hat{X}_\Theta\} := J_*$ .

To prove equality (6), we use the following lemma.

**Lemma 2.1.** *If  $f(t) \in B_2^1(\mathbf{R}; \mathbf{R}^n)$ , then*

$$(8) \quad \|Df(t)\|_0^2 = \lim_{s \rightarrow 0} \|s^{-1}(f(t+s) - f(t))\|_0^2.$$

*Proof of the lemma.* Let  $\sum f_k e^{i\lambda_k t}$  be Fourier series of the function  $f$ . Then  $\|Df\|_0^2 = \sum |\lambda_k|^2 \|f_k\|^2 < \infty$ , and

$$\begin{aligned} & \|s^{-1}(f(t+s) - f(t))\|_0^2 - \|Df\|_0^2 \\ &= \left| \sum \left( \left| \frac{e^{i\lambda_k s} - 1}{s} \right|^2 - |\lambda_k|^2 \right) \|f_k\|^2 \right| \\ &= \left| \sum |\lambda_k|^2 \left( \left| \int_0^1 e^{i\lambda_k \tau s} d\tau \right|^2 - 1 \right) \|f_k\|^2 \right| \leq 2 \sum |\lambda_k|^2 \|f_k\|^2. \end{aligned}$$

Now we can see, that Lemma 2.1 follows from Weierstrass theorem and continuity in  $s$  of the sum of functional series. The proof of the lemma is complete.  $\square$

Suppose that (6) does not hold. Then there exists a function  $x_0(t) \in \hat{X}_{\Theta+\delta}$ , such that  $J(x_0(t)) < J_*$ . From the Lemmas 1.1 and 2.1 we have

$$\begin{aligned}\|DP_{\theta}x_0(t)\|_0 &= \lim_{s \rightarrow 0} \|s^{-1}(P_{\theta}x_0(t+s) - P_{\theta}x_0(t))\|_0 \\ &\leq \lim_{s \rightarrow 0} \|s^{-1}(x_0(t+s) - x_0(t))\|_0 = \|Dx_0(t)\|_0,\end{aligned}$$

and after condition (5)  $M(V(t, P_{\theta}x_0(t))) \leq M(V(t, x_0(t)))$ . Hence  $J(P_{\theta}x_0(t)) \leq J(x_0(t)) < J_*$ . It is impossible, because  $P_{\theta}x_0(t) \in \hat{X}_{\theta}$ .

Now let us check (7). Note that under the additional condition of the theorem we derive from Lemma 1.3 that  $\partial V(t, x(t))/\partial x \in B_2(\mathbf{R}; \mathbf{R}^n)$  for each  $x(t) \in \hat{X}_{\Omega}$ . For any  $y(t) \in CAP^1(\mathbf{R}; \mathbf{R}^n)$  and for each sufficiently small by module  $s$  an inequality holds:

$$\begin{aligned}J(x_*(t) + sy(t)) - J(x_*(t)) \\ = s(A[y(\cdot)] + M(\langle W(t, sy(t)) - W(t, 0), y(t) \rangle + \frac{1}{2}s\|\dot{y}(t)\|^2)) \geq 0,\end{aligned}$$

where  $W(t, y) = \int_0^1 (\partial V(t, x_*(t) + \tau y)/\partial x) d\tau$ .

Since we get from Lemma 1.3 that  $\lim_{s \rightarrow 0} M(\langle W(t, sy(t)) - W(t, 0), y(t) \rangle) = 0$ , this inequality is possible only when (7) holds.  $\square$

### 3. Besicovitch quasiperiodic functions and functions on tori

Let  $\omega \in \mathbf{R}^m$  be a vector with rationally independent components. Taking the closure of the space of  $\mathbf{R}^n$ -valued polynomials, which are linear combinations of functions like  $e^{i\langle k, \omega \rangle t}$ ,  $k \in \mathbf{Z}^m$ , in the seminorm  $\|\cdot\|_0(\|\cdot\|_1)$  one obtains the space of Besicovitch quasiperiodic functions  $B_2(\mathbf{R}; \mathbf{R}^n; \omega)$  ( $B_2^1(\mathbf{R}; \mathbf{R}^n; \omega)$ ) [10].

Let  $T^m$  be a standard  $m$ -dimensional torus  $\{\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbf{R}^m | \text{mod } 2\pi\}$ . Denote by  $H(T^m, \mathbf{R}^n)$  the space of  $\mathbf{R}^n$ -valued integrable with the square of Euclidean norm functions on  $T^m$ . The seminorm in this space is  $\|\cdot\|_0 := M^{1/2}(\|\cdot\|^2) := (\int_{T^m} \|\cdot\|^2 d\varphi)^{1/2}$ . Generalized (Sobolev) derivative in the direction  $\omega$  of a function  $f \in H(T^m, \mathbf{R}^n)$  is denoted by  $D_{\omega}f$ . Using the method explained in the proof of Lemma 4 one can show that the existence of function  $g \in H(T^m, \mathbf{R}^n)$ , for which

$$(9) \quad \lim_{s \rightarrow 0} \|s^{-1}((f(\varphi + s\omega) - f(\varphi)) - g(\varphi))\|_0 = 0,$$

is equivalent to existence of  $D_{\omega}f$ , and one can put  $D_{\omega}f = g$ .

The set of functions possessing generalized derivatives in the direction  $\omega$  forms the space  $H_{\omega}^1(T^m, \mathbf{R}^n)$  with seminorm  $\|\cdot\|_1 = \|D_{\omega} \cdot\|_0 + \|\cdot\|_0$ . There exists a natural bijection between the classes of equivalent (in the sense of corresponding seminorms) functions from the above described spaces of Besicovitch-quasiperiodic functions, on the one hand, and the spaces of

functions on torus, on the other hand. Definition of the space  $B_2(\mathbf{R}; C(\Omega); \omega)$  does not need explanation. Concerning its analogue on torus, we shall not demand apriori a uniform (in  $x \in \Omega$ ) approximation of corresponding functions by trigonometric polynomials. Function  $F(\varphi, x) : T^m \times \Omega \rightarrow \mathbf{R}$  belongs to  $H(T^m; C(\Omega))$ , if for any  $x \in \Omega$  the mapping  $\varphi \rightarrow F(\varphi, x)$  is measurable, and for almost all  $\varphi \in T^m$  function  $x \rightarrow F(\varphi, x)$  is continuous, and  $\max_{x \in \Omega} |F(\varphi, x)| \in H(T^m; \mathbf{R})$ . The space  $H(T^m; C^r(\Omega))$  is formed of functions from  $H(T^m; C(\Omega))$  such that they possess all partial derivatives in  $x_1, \dots, x_n$  up to the order  $r$  inclusive, and that all these derivatives belong to  $H(T^m; C(\Omega))$ .

Now let  $V : \mathbf{R} \times \Omega \rightarrow \mathbf{R}$  have such property, that for  $x \in \Omega$  the mapping  $t \rightarrow V(t, x)$  belongs to  $B_2(\mathbf{R}; \mathbf{R})$ , and corresponding function on torus  $W(\varphi, x)$  belongs to  $H(T^m; C(\Omega))$ . Instead of minimization problem for functional (3) we obtain an equivalent problem for the functional

$$(10) \quad \mathcal{J}(u(\varphi)) = M\left(\frac{1}{2} \|D_\omega u(\varphi)\|^2 + W(\varphi, u(\varphi))\right),$$

defined on the set  $\hat{U}_\Omega \subset H_\omega^1(T^m; \mathbf{R}^n)$ , which is formed like  $\hat{X}_\Omega$ .

Introduce the space  $H^r(T^m; \mathbf{R}^n)$ , which consists of functions from  $H(T^m; \mathbf{R}^n)$ , having all Sobolev generalized derivatives up to the order  $r$  inclusive, and a subspace of it,  $\hat{H}^r(T^m; \mathbf{R}^n)$ , formed of functions, which are, as well as all of their partial derivatives, essentially bounded.

We shall consider now several technical results.

**Lemma 3.1.** *If  $f \in H(T^m; \mathbf{R}^n)$ , then  $\lim_{s \rightarrow 0} \|f(\varphi + sa) - f(\varphi)\|_0 = 0$  for any  $a \in \mathbf{R}^m$ .*

**Lemma 3.2.** *If  $f \in \hat{H}^1(T^m; \mathbf{R}^n)$ ,  $g \in \hat{H}^1(T^m; \mathbf{R}^n)$ , then  $f \cdot g \in \hat{H}^1(T^m; \mathbf{R}^n)$ , and  $D_a(f \cdot g) = D_a f \cdot g + f \cdot D_a g$  for any  $a \in \mathbf{R}^m$ .*

The proof consists in applying the equality (9).

**Lemma 3.3.** *Let  $F[u(\cdot)](\varphi) := F(\varphi, u(\varphi))$ . If  $F(\varphi, x) \in H(T^m; C(\Omega))$ , then*

$$F : \{u(\varphi) \in H(T^m; \mathbf{R}^n) : u(T^m) \subset \Omega\} \rightarrow H(T^m; \mathbf{R})$$

*and this operator is continuous.*

*Proof.* Defining  $F$  by equality  $F(\varphi, x) = F(x, P_\Omega x)$  for all  $(\varphi, x) \in T^m \times \mathbf{R}^n$ , we can regard  $F$  to be a Nemytski operator defined on all  $H(T^m; \mathbf{R}^n)$ . It is well known that the operator  $F : H(T^m; \mathbf{R}^n) \rightarrow H(T^m; \mathbf{R}^n)$  is continuous (see [11] for details).

**Lemma 3.4.** *Let  $F(\varphi, x) \in C^1(T^m \times \Omega)$ . Then  $F(\varphi, u(\varphi)) \in \hat{H}^1(T^m; \mathbf{R})$  for any function  $u(\varphi) \in \hat{H}^1(T^m; \mathbf{R}^n)$  such that  $u(T^m) \subset \Omega$ , and*

$$(11) \quad D_{e_j}F(\varphi, u(\varphi)) = \frac{\partial F(\varphi, x)}{\partial \varphi_j} \Big|_{x=u(\varphi)} + \left\langle \frac{\partial F(\varphi, u(\varphi))}{\partial x}, D_{e_j}u(\varphi) \right\rangle,$$

$j = 1, \dots, n$ , where  $e_j \in \mathbf{R}^n$ ,  $e_1 = (1, 0, \dots, 0)$ ,  $e_2 = (0, 1, \dots, 0)$ ,  $\dots$ ,  $e_n = (0, 0, \dots, 1)$ .

*Proof.* Denote by  $\Phi_j(\varphi)$  the expression in the right-hand side of (11) and put

$$\begin{aligned} \Delta_j^s u(\varphi) &= u(\varphi + se_j) - u(\varphi); \\ G(\varphi, s, y, z) &= \int_0^1 \frac{\partial F(\varphi + \tau se_j, \tau y + (1 - \tau)z)}{\partial \varphi_j} d\tau; \\ E(\varphi, s, y, z) &= \int_0^1 \frac{\partial F(\varphi + \tau se_j, \tau y + (1 - \tau)z)}{\partial x} d\tau. \end{aligned}$$

Using the uniform continuity of functions  $G$  and  $E$ , and also Lemmas 3.2 and 3.3, we have

$$\begin{aligned} \|s^{-1} \Delta_j^2 F(\varphi, u(\varphi)) - \Phi_j(\varphi)\|_0 &= \|G(\varphi, s, u(\varphi + se_j), u(\varphi)) - G(\varphi, 0, u(\varphi), u(\varphi)) \\ &\quad + \langle E(\varphi, s, u(\varphi + se_j), u(\varphi)), s^{-1} \Delta_j^s u(\varphi) - D_{e_j}u(\varphi) \rangle \\ &\quad + \langle E(\varphi, s, u(\varphi + se_j), u(\varphi)) \\ &\quad - E(\varphi, 0, u(\varphi), u(\varphi)), D_{e_j}u(\varphi) \rangle\|_0 \rightarrow 0, s \rightarrow 0. \quad \square \end{aligned}$$

#### 4. Generalized and classical solutions

The following statement is an analogue of Theorem 2.1.

**Theorem 4.1.** Let  $W(\varphi, x) \in H(T^m; C(\Omega))$  and suppose that there exists a closed convex set  $\Theta \subset \mathbf{R}^n$  such that  $\Theta + \delta \subset \Omega$  and for almost all  $\varphi \in T^m$

$$(12) \quad \begin{aligned} W(\varphi, \lambda x + (1 - \lambda)y) \\ \leq \lambda W(\varphi, x) + (1 - \lambda)W(\varphi, y) \quad \forall x, y \in \Theta, \quad \forall \lambda \in [0, 1], \end{aligned}$$

$$(13) \quad W(\varphi, x) \geq W(\varphi, P_\Theta x) \quad \forall x \in \Theta + \delta.$$

Then there exists a function  $u_*(\varphi) \in \hat{U}_\Theta$ , such that

$$\mathcal{J}(u_*(\varphi)) = \inf\{\mathcal{J}(u) | u \in \hat{U}_{\Theta+\delta}\}.$$

If, in addition,  $W(\varphi, x) \in H(T^m, C^1(\Omega))$ , then  $u_*(\varphi)$  is a Sobolev generalized

solution of equation

$$(14) \quad D_{\omega}^2 u = \frac{\partial W(\varphi, u)}{\partial u}.$$

Further we shall need information about the solution of linear equation.

**Lemma 4.1.** *Let  $B(\varphi) \in \hat{H}(T^m; \text{Hom}(\mathbf{R}^n; \mathbf{R}^n))$ ,  $b(\varphi) \in \hat{H}(T^m; \mathbf{R}^n)$ , and assume that there exists a constant  $\gamma > 0$  such that for almost all  $\varphi \in T^m$*

$$\langle B(\varphi)x, x \rangle \geq \gamma \|x\|^2 \quad \forall x \in \mathbf{R}^n.$$

*Then equation (14) with  $W(\varphi, x) = \langle B(\varphi)x, x \rangle/2 + \langle b(\varphi), x \rangle$  has a unique (in the sense of  $\|\cdot\|_0$ ) generalized solution  $u = w(\varphi) \in H_{\omega}^1(T^m; \mathbf{R}^n)$ , and*

$$\|w(\varphi)\|_0 \leq \gamma^{-1} \|b(\varphi)\|_0, \quad \text{ess sup}_{T^m} \|w(\varphi)\| \leq \gamma^{-1} \text{ess sup}_{T^m} \|b(\varphi)\|$$

*If in addition  $B(\varphi) \in \hat{H}^1(T^m; \text{Hom}(\mathbf{R}^n, \mathbf{R}^n))$ ,  $b(\varphi) \in \hat{H}^1(T^m; \mathbf{R}^n)$ , then  $w(\varphi) \in \hat{H}^1(T^m; \mathbf{R}^n)$ .*

*Proof.* Let  $\Theta$  be a ball with radius  $R$ . Then  $P_{\Theta}x = R\|x\|^{-1}x$ ,  $x \in \mathbf{R}^n \setminus \Theta$ . Condition (12) holds. Condition (13) is met, if  $\langle \partial W(\varphi, x)/\partial x, x \rangle|_{\|x\|=R} > 0$ , i.e., if

$$(\langle B(\varphi)x, x \rangle + \langle b(\varphi), x \rangle)|_{\|x\|=R} > 0.$$

This condition holds for any  $R$  satisfying  $R > \gamma^{-1} \text{ess sup} \|b(\varphi)\|$ . The existence of the sought solution  $w(\varphi)$  follows from Theorem 4.1. Its uniqueness is proved using the fact that  $B(\varphi)$  is positive definite. Since  $w(\varphi)$  is a generalized solution of equation (14), we have

$$\begin{aligned} \gamma \|w(\varphi)\|_0^2 &\leq M(\|D_{\omega} w(\varphi)\|^2 + \langle B(\varphi)w(\varphi), w(\varphi) \rangle) \\ &= -\langle b(\varphi), w(\varphi) \rangle_0 \leq \|b(\varphi)\|_0 \|w(\varphi)\|_0, \end{aligned}$$

and now  $\|w(\varphi)\|_0$  can be easily estimated.

Further, it is seen from the above proved facts, that, if additional conditions of the Lemma are met, equation

$$D_{\omega}^2 u = B(\varphi)u + D_{e_j} B(\varphi) \cdot w(\varphi) + D_{e_j} b(\varphi)$$

has a generalized solution  $w_1(\varphi) \in \hat{H}(T^m; \mathbf{R}^n)$ . Let us show that the function  $y(\varphi, s) = s^{-1} \Delta_j^s w(\varphi) - w_1(\varphi)$  (operator  $\Delta_j^s$  is defined in the proof of Lemma 3.4) has the property  $\|y(\varphi, s)\|_0 \rightarrow 0$ , as  $s \rightarrow 0$ . This function is a generalized

solution of equation

$$\begin{aligned} D_{\omega}^2 y &= B(\varphi)y + [s^{-1}A_j^s B(\varphi) - D_{e_j} B(\varphi)]w(\varphi + se_j) \\ &\quad + D_{e_j} B(\varphi)A_j^s w(\varphi) + s^{-1}A_j^s b(\varphi) - D_{e_j} b(\varphi). \end{aligned}$$

It follows from the above established facts that

$$\begin{aligned} \|y(\varphi, s)\|_0 &\leq \gamma^{-1}(\text{ess sup}\|w(\varphi)\| \|s^{-1}A_j^s B(\varphi) - D_{e_j} B(\varphi)\|_0 \\ &\quad + \text{ess sup}\|D_{e_j} B(\varphi)\| \|A_j^s w(\varphi)\|_0 + \|s^{-1}A_j^s b(\varphi) - D_{e_j} b(\varphi)\|_0) \rightarrow 0 \\ &\quad (s \rightarrow 0). \end{aligned}$$

It means that  $D_{e_j} w(\varphi) = w_1(\varphi) \in \hat{H}(T^m; \mathbf{R}^n)$ . □

Now it is not difficult to find sufficient conditions for the existence of classical solutions of equation (14).

**Theorem 4.2.** *Let a function  $W(\varphi, x)$  be such that  $\partial W(\varphi, x)/\partial x \in C^r(T^m \times \Omega)$ , where  $r > (m/2) + 2$ . Suppose that there exists a set  $\Theta$ , having the same properties as in Theorem 4.1 with the condition of convexity of  $W$  replaced by the stronger one: assume that there exists  $r > 0$  such that for almost all  $\varphi \in T^m$*

$$\left\langle \frac{\partial^2 W(\varphi, x)}{\partial x^2} y, y \right\rangle \geq \gamma \|y\|^2 \quad \forall x \in \Theta, \quad \forall y \in \mathbf{R}^n.$$

Then equation (14) has a classical solution.

*Proof.* Let  $u_*(\varphi)$  be a function from Theorem 4.1,  $u_1(\varphi)$  is a generalized solution of equation in variations

$$(15) \quad D_{\omega}^2 u = B(\varphi)u + b(\varphi),$$

where  $B(\varphi) = \partial^2 W(\varphi, u_*(\varphi))/\partial x^2$ ,  $b(\varphi) = \partial^2 W(\varphi, u_*(\varphi))/\partial \varphi_j \partial x$ . Existence and essential boundedness of the latter are guaranteed by Lemma 1.2. Function  $y(\varphi, s) = s^{-1}A_j^s u_*(\varphi) - u_1(\varphi)$  is a generalized solution of the equation

$$D_{\omega}^2 y = B(\varphi, s)y + (B(\varphi, s) - B(\varphi))u_1(\varphi) + (b(\varphi, s) - b(\varphi)),$$

where

$$\begin{aligned} B(\varphi, s) &= \int_0^1 \frac{\partial^2 W(\varphi + \tau se_j, \tau u_*(\varphi + se_j) + (1 - \tau)u_*(\varphi))}{\partial x^2} d\tau; \\ b(\varphi, s) &= \int_0^1 \frac{\partial^2 W(\varphi + \tau se_j, \tau u_*(\varphi + se_j) + (1 - \tau)u_*(\varphi))}{\partial \varphi_j \partial x} d\tau. \end{aligned}$$

Since  $\|B(\varphi, s) - B(\varphi)\|_0 \rightarrow 0$ ,  $\|b(\varphi, s) - b(\varphi)\|_0 \rightarrow 0$  when  $s \rightarrow 0$  (see the proof of Lemma 3.4), it follows from Lemma 1.2 that  $\|y(\varphi, s)\|_0 \rightarrow 0$ ,  $s \rightarrow 0$ . It means that  $D_{e_j} u_*(\varphi)$  exists and is a generalized solution of (15). Using Lemmas 3.2, 3.4, we can successively apply Lemma 1.2 to (15) and prove the existence of generalized derivatives of the function  $u_*(\varphi)$  up to the order  $r$  inclusive. The proof is completed using the embedding theorem.  $\square$

Further we show that this result can be proven under a considerably milder restriction on the smoothness of  $W(\varphi, x)$ .

**Theorem 4.3.** *Let  $W(\varphi, x) \in C^3(T^m \times \Omega)$ . Suppose that there exists a compact convex set  $\Theta \subset \Omega$  such that its boundary  $\partial\Theta$  is a differentiable manifold (of class  $C^1$ ), and that for any  $\varphi \in T^m$*

$$(16) \quad \left\langle \frac{\partial^2 W(\varphi, x)}{\partial x^2} y, y \right\rangle \geq \gamma \|y\|^2 \quad \forall x \in \Theta, \quad \forall y \in \mathbf{R}^n,$$

and

$$(17) \quad \left\langle \frac{\partial W(\varphi, x)}{\partial x}, n(x) \right\rangle > 0, \quad \forall \varphi \in T^m, \quad \forall x \in \partial\Theta,$$

where  $n(x)$  is an external normal unit vector to  $\partial\Theta$  at the point  $x$ . Then the equation

$$(18) \quad \frac{d^2 x}{dt^2} = \frac{\partial W(\omega t, x)}{\partial x}$$

has a classical quasiperiodic solution with frequency  $\omega$ .

*Proof.* Since  $\Omega$  is a compact set, function  $W$  can be extended periodically to  $\mathbf{R}^{m+n}$ . Further we shall make no difference between the function  $W$  and its extension. It is a well known fact of analysis that  $W$  can be approximated with trigonometric in  $\varphi$  and  $x$  polynomials  $W_0(\varphi, x)$  in the usual norm of  $C^2(T^m \times \mathbf{R}^n)$ . Since  $W_0$  can be chosen sufficiently close to  $W$  in the sense of that norm, we can assume without loss of generality that conditions (16) and (17) hold for  $W_0$ . It follows from Theorem 4.2 that the equation

$$D_\omega^2 u = \frac{\partial W_0(\varphi, u)}{\partial u}$$

has a solution  $u_0 \in C^2(T^m)$ .

The equation (18) can be written equivalently in the form

$$(19) \quad \frac{dz}{dt} = X(t, z),$$

where  $z = (x, \dot{x})^T$ ,  $X(t, z) = (\dot{x}, \partial W(\omega t, x)/\partial x)^T$ . For an arbitrary small  $\varepsilon > 0$  one can write:

$$(20) \quad X(t, z) = X_0(t, z) + \varepsilon X_1(t, z),$$

where  $X_0(t, z) = (\dot{x}, \partial W_0(\omega t, x)/\partial x)^T$ ,  $W_0$  is a trigonometric polynomial, and  $X_1(t, z) \in C^2(\mathbf{R}^{2n+1})$ . Using the result of Theorem 4.2, it is easy to see that the equation  $\dot{z} = X_0(t, z)$  has a quasiperiodic solution  $z_0(t)$ . We shall seek for the solution of (18) in the form:

$$(21) \quad z(t) = z_0(t) + \sqrt{\varepsilon} y(t).$$

Substituting (20) and (21) in (19) yields:

$$(22) \quad \dot{y} = \frac{\partial X_0(t, z_0)}{\partial z} y + \sqrt{\varepsilon} Y_1(t, y, \varepsilon),$$

where  $Y_1(t, y, \varepsilon) \in C^2(\mathbf{R}^{2n+1} \times [0, 1])$ . Note that

$$\frac{\partial X_0(t, z_0(t))}{\partial z} = \begin{pmatrix} 0 & E \\ \frac{\partial^2 W_0(\omega t, x_0(t))}{\partial x^2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & E \\ \tilde{A}(\omega t) & 0 \end{pmatrix} = \hat{A}(\omega t),$$

where  $\tilde{A} > \gamma > 0$ , since  $x_0(t) = (z_{0,1}(t), \dots, z_{0,n}(t)) \in \Theta$  for each  $t \in \mathbf{R}$ , and  $E$  denotes an identity  $n \times n$  matrix. Consider the equation

$$(23) \quad \frac{dy}{dt} = \hat{A}(\omega t) y,$$

where  $\hat{A}(\varphi) \in C^1(T^m)$ . Let

$$S = - \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$$

Then the derivative of a quadratic form  $\langle S y, y \rangle$  along the solution  $y(t)$  of (23) can be written in the form  $\langle \hat{S}(\omega t) y, y \rangle$ , where

$$\hat{S}(\varphi) = S \hat{A}(\varphi) + \hat{A}^*(\varphi) S = -2 \begin{pmatrix} \tilde{A}(\varphi) & 0 \\ 0 & E \end{pmatrix}$$

is a negative definite matrix. Hence the homogeneous equation (23) is exponentially dichotomic, and it has a unique Green function  $G(t, \tau)$  (see [10]). Therefore, the problem of finding the quasiperiodic solution of (22) can be reduced to the following integral equation:

$$(24) \quad y(t) = \sqrt{\varepsilon} \int_{-\infty}^{\infty} G(t, \tau) Y_1(\tau, y(\tau), \varepsilon) d\tau.$$

Observing that  $\varepsilon$  is small and applying to the equation (24) the fixed point theorem, we come to conclusion that the quasiperiodic solution  $y(t)$  of equation (22) with frequency  $\omega$  exists. Hence we can assert that equation (19) has a quasiperiodic solution  $z(t) = z_0(t) + \sqrt{\varepsilon}y(t)$ , and the proof is complete.  $\square$

## 5. Corollaries

Consider a special case of system (1)

$$(25) \quad \frac{d^2x}{dt^2} = \nabla F(x) + g(t),$$

where  $x \in \mathbf{R}^n$ ,  $g(t) : \mathbf{R} \rightarrow \mathbf{R}^n$  is a quasiperiodic function:  $g(t) = G(\omega t)$ ,  $G \in C^3(T^m; \mathbf{R}^n)$ ,  $F \in C^3(\Omega)$ . Under such conditions it is not difficult to obtain from Theorem 4.3 the following.

**Proposition 5.1.** *Let  $\Theta$  be a subset of  $\mathbf{R}^n$  such that  $\Theta + \delta \subset \Omega$  and its boundary  $\partial\Theta$  is a differentiable manifold (of class  $C^1$ ), and assume that there exists  $\gamma > 0$  such that*

$$(26) \quad \left\langle \frac{\partial^2 F(x)}{\partial x^2} y, y \right\rangle > \gamma \|y\|^2, \quad \forall x \in \Theta, \quad \forall y \in \mathbf{R}^n,$$

and also that

$$(27) \quad \langle \nabla F(x) + G(\varphi), n(x) \rangle > 0, \quad \forall x \in \partial\Theta, \quad \forall \varphi \in T^m,$$

where  $n(x)$  is an external normal unit vector to  $\partial\Theta$  at the point  $x$ . Then system (25) has a classical quasiperiodic solution with frequency  $\omega$ .

**Remark 5.1.** A set, bounded by a level surface of function  $F(x)$ , can be chosen as  $\Theta$  (if it belongs to  $\Omega$  together with its certain neighbourhood). Then, as far as  $n(x) = \nabla F(x) / \|\nabla F(x)\|$ , condition (27) holds if

$$(28) \quad \|\nabla F(x)\| > \sup_{\varphi \in T^m} \|G(\varphi)\|, \quad \forall x \in \partial\Theta.$$

**Remark 5.2.** In case when  $\Omega$  coincides with  $\mathbf{R}^n$ , if condition (26) holds, then all the level surfaces of  $F(x)$  are bounded, and  $\|\nabla F(x)\| \rightarrow \infty$ ,  $\|x\| \rightarrow \infty$ . Then, as far as  $\sup_{\varphi \in T^m} \|G(\varphi)\| < \infty$ , there exists a unique classical quasiperiodic solution of equation (25).

## 6. Examples

Now we shall illustrate the use of the above described theory. Let us consider the quasiperiodically forced pendulum equation:

$$(29) \quad \ddot{x} + \sin x(t) = e(t),$$

where  $e(t) = L(\omega t)$ ,  $L \in C(T^m)$ . It was proved in [7] that there exists a generalized Besicovitch-quasiperiodic solution  $x(t)$  of (29) such that  $\pi/2 < x(t) < 3\pi/2$ , if  $\text{osc } E(t) = \sup E(t) - \inf E(t) < \pi/2$ , where  $\dot{E}(t) = e(t)$ ,  $E(t)$  is a quasiperiodic function (if such a function exists). Using Theorem 2.1, we can get the same result. Moreover, if  $E(t)$  is smooth enough, Theorem 4.3 guarantees that this solution is a classical quasiperiodic one. Furthermore, applying Proposition 5.1 we can establish the existence of a classical quasiperiodic solution under conditions that are completely different from those of paper [7]. Namely, we can state that equation (29) has a classical quasiperiodic solution  $x(t)$ , such that  $\pi/2 < x(t) < 3\pi/2$ , if  $\sup_{t \in \mathbf{R}} |e(t)| < 1$ ,  $e(t) = L(\omega t)$ ,  $L \in C^3(T^m)$ .

Consider another example. Equation

$$(30) \quad \ddot{u} - Au - \nabla Q(u) = f(t)$$

where  $u \in \mathbf{R}^n$ ,  $A$  is a positive definite symmetric  $n \times n$  matrix,  $f = F(\omega t)$ ,  $F \in C(T^m)$ , was studied in [12], where existence and uniqueness of generalized quasiperiodic solution of the equation was shown under assumption that  $Q(u)\|u\|^{-2} \rightarrow +\infty$ ,  $\|u\| \rightarrow \infty$ .

Let  $F \in C^3(T^m)$ , and let  $Q: \mathbf{R}^n \rightarrow \mathbf{R}$  be a convex function,  $Q \in C^3(\mathbf{R}^n)$ . Then it is possible to apply Proposition 5.1 and establish the existence and uniqueness of classical quasiperiodic solution of (30).

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(Ricevita la 8-an de julio, 1998)

(Reviziita la 17-an de majo, 1999)