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Rapid Oscillations in Sublinear Problems

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1. Introduction

We consider the equation

$$(1) \quad x'' + g(x) = f(t, x, x'),$$

where $g \in C(\mathbf{R}, \mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\}, \mathbf{R})$ and $f \in C^1([a, b] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$, $xg(x) > 0$ for x small.

We are motivated by sublinear equation

$$(2) \quad x'' + x^{1/3} = 0,$$

which has small amplitude periodical solutions with arbitrarily small periods. Integrating this equation and taking into account the initial values

$$(3) \quad x(a) = A > 0, \quad x'(a) = 0,$$

one gets the expression

$$(4) \quad T = \frac{4\sqrt{6}}{3} A^{1/3} \int_0^1 \frac{ds}{\sqrt{1 - s^{4/3}}}$$

for the period.

Thus, $T(A) \rightarrow 0$ as $A \rightarrow 0$.

This property is characteristic for sublinear equations

$$(5) \quad x'' + g(x) = 0,$$

with a nonlinearity satisfying the condition

$$(6) \quad \lim_{|x| \rightarrow 0} \frac{g(x)}{x} = +\infty.$$

In case $|x| \rightarrow \infty$ the nonlinearity is referred to as superlinear.

Intensive literature is devoted to problems of this kind (see, for instance, [1], [2], [6], [5], [3]).

In [1], for instance, it was shown that the equation (1) with the Sturm–

Liouville boundary conditions

$$\begin{aligned}
 (7) \quad & a_1 x(0) + a_2 x'(0) = A, \\
 & b_1 x(T) + b_2 x'(T) = B, \\
 & a_1^2 + a_2^2 > 0, \quad b_1^2 + b_2^2 > 0
 \end{aligned}$$

has infinite number of solutions if $g(x)$ is superlinear and

$$(8) \quad |f(t, x, y)| < c_1 |x| + c_2 |y| + c_3$$

for any (t, x, y) considered.

It can be shown by geometrical reasoning that sublinear equations of the form (5) with a nonlinearity $g(x)$ of odd type have infinite number of solutions, satisfying the Sturm–Liouville boundary conditions (7). In the case of equation (2) those solutions are represented in a phase plane by closed trajectories

$$x'^2 + \frac{3}{2}x^{4/3} = \text{const.}$$

Thus the property of an equation to be sublinear is important for boundary value problems and should be described formally.

2. SB-property

We say that a nonlinear equation (1) possesses the SB-property (sublinearity) if

- (A1) zero solution is unique;
- (A2) for any N there exists $\delta > 0$ such that any solution with the initial data satisfying

$$(9) \quad |x(a)| + |x'(a)| < \delta$$

has at least N zeros in the interval (a, b) .

For example, the equation

$$x'' + x^{1/3} = x'$$

possesses the SB-property unlikely the equation

$$x'' + x^{1/3} = x'^{1/3},$$

which does not possess this property.

Moreover, the equation

$$x'' + k^2 x^{1/3} = c |x'|^{1/2} \text{sign } x'$$

may possess or not to possess the SB-property, depending on the choice of k and c .

Thus the problem arises: if $g(x)$ in (1) is sublinear, what restrictions on the right side $f(t, x, x')$ ensure that the equation still possesses the SB-property.

Our goal in this paper is to provide sharp in some sense conditions for the equation (1) to possess the SB-property.

3. First order auxiliary equation

In what follows we need some results on the first order equation

$$(10) \quad \frac{1}{2} \frac{dz}{dx} = cx^{\gamma-\varepsilon} z^{\varepsilon/(\gamma+1)} - k^2 x^\gamma,$$

where ε, γ, c and k are positive numbers and $0 < \varepsilon \leq \gamma < 1$.

Lemma 3.1. Equation (10) has solutions of the type $z = Dx^{\gamma+1}$, if

$$(11) \quad k^2 \leq c_* := c \frac{\gamma+1-\varepsilon}{\gamma+1} \left[\frac{2\varepsilon c}{(\gamma+1)^2} \right]^{\varepsilon/(\gamma+1-\varepsilon)}.$$

Proof. The function $z = Dx^{\gamma+1}$ solves the equation (10), if D is a solution of the finite equation

$$(12) \quad 2k^2 + (\gamma+1)D = 2cD^{\varepsilon/(\gamma+1)}.$$

This equation has two different solutions D_1 and D_2 if the inequality (11) is strict. Functions $z_1 = D_1 x^{\gamma+1}$ and $z_2 = D_2 x^{\gamma+1}$ are then solutions of (10). In case $k^2 = c_*$ the equation (12) has exactly one solution which generates a unique solution of (10). ■

Lemma 3.2. If $k^2 > c_*$ in (10), then any solution of the equation (10), which satisfies the initial value condition

$$(13) \quad z(x_0) = 0, \quad x_0 > 0$$

is positive in $[0, x_0)$.

Proof. A zero isocline $i(x)$ of the equation (10) is

$$(14) \quad i(x) = \left(\frac{k^2}{c} \right)^{(\gamma+1)/\varepsilon} x^{\gamma+1}$$

for any $0 < \varepsilon \leq \gamma$. This isocline emanates from the origin and separates the quadrant $x \geq 0, z \geq 0$ so that in the upper part solutions increase and in the lower part decrease. Hence $z(x)$ is positive in $(0, x_0)$ for any x_0 . One has

to prove only that

$$(15) \quad z(0) > 0.$$

The equation (10) is reduced to

$$(16) \quad x \frac{du}{dx} = -(2k^2 + (\gamma + 1)u - 2cu^{\varepsilon/(\gamma+1)})$$

by the variable change

$$(17) \quad z = x^{\gamma+1}u(x), \quad u \geq 0.$$

We obtain, integrating (16) in the interval $(0, x_1)$, where $0 < x_1 < x_0$, $z(x_1) > z(x_0) = 0$, that

$$(18) \quad \int_{u(x_1)}^{u(x)} \frac{ds}{2k^2 + (\gamma + 1)s - 2cs^{\varepsilon/(\gamma+1)}} = \ln \frac{x_1}{x}.$$

The equation with respect to D

$$(19) \quad 2k^2 + (\gamma + 1)D = 2cD^{\varepsilon/(\gamma+1)}$$

does not have real solutions, if $k^2 > c_*$. Under the integral sign, the denominator is positive. Then (18) implies that $u(x) \rightarrow +\infty$ as $x \rightarrow 0$.

Rewrite (18) as

$$(20) \quad \int_{u(x_1)}^{u(x)} \frac{ds}{s} + 2 \int_{u(x_1)}^{u(x)} \frac{(-k^2 + cs^{\varepsilon/(\gamma+1)})}{s(2k^2 + (\gamma + 1)s - 2cs^{\varepsilon/(\gamma+1)})} ds = (\gamma + 1) \ln \frac{x_1}{x}.$$

Since the denominator in the second integral is positive and $0 < \varepsilon/(\gamma + 1) < 1$, this integral tends to a finite limit L as $u(x)$ tends to $+\infty$.

Then (20) can be written as

$$(21) \quad \ln \frac{u(x)}{u(x_1)} + L + \delta(x) = (\gamma + 1) \ln \frac{x_1}{x},$$

where $\delta(x) \rightarrow 0$ as $x \rightarrow 0$, or, making use of the expression $z(x) = x^{\gamma+1}u(x)$,

$$(22) \quad \ln \left[\frac{z(x)}{z(x_1)} \cdot \left(\frac{x_1}{x} \right)^{\gamma+1} \right] + L + \delta(x) = (\gamma + 1) \ln \frac{x_1}{x}.$$

Finally,

$$\lim_{x \rightarrow 0} \ln \frac{z(x)}{z(x_1)} = -L$$

or $z(x) \rightarrow z(x_1) \exp\{-L\} > 0$ as $x \rightarrow 0$. ■

4. Main results

Theorem 4.1. Suppose that a number $0 < r < 1$ and a positive valued continuous function $\omega(x)$ exist such that the following conditions are fulfilled for (1):

- 1) $|g(x)| \geq k^2|x|^\gamma + \omega(|x|)$ for $|x| < r$, $0 < \gamma < 1$;
- 2) $|f(t, x, y)| \leq \omega(|x|) + c|x|^{\gamma-\varepsilon}|y|^{2\varepsilon/(\gamma+1)}$ for $t \in [a, b]$ and $|x| + |y| < r$, $0 < \varepsilon \leq \gamma$;
- 3) $k^2 > c_*$, where c_* is given in (11);
- 4) the zero solution is unique, that is there does not exist a nontrivial solution of (1) which has zero initial values.

Then equation (1) possesses the SB-property.

Remark. In what follows we assume that solutions of the equation (1) extend to the interval $[a, b]$. This assumption does not affect our considerations. Indeed, we are looking for small amplitude solutions. So redefine functions $g(x)$ and $f(t, x, y)$ outside a bounded (t, x, y) -domain so that modified functions $G(x)$ and $F(t, x, y)$ are of the same smoothness class and bounded. Then a function $m(r)$ exists such that solutions of the modified equation $x'' + G(x) = F(t, x, x')$, which satisfy the initial condition $|x(a)| + |x'(a)| \leq r$, are bounded, that is $|x(t)| + |x'(t)| \leq m(r)$ in the interval $[a, b]$. It will be shown below that $m(r) \rightarrow 0$ as $r \rightarrow 0$ and hence solutions of the modified equation, which has small enough initial values, are at the same time solutions of the equation (1).

We need the following auxiliary results to prove the theorem.

Lemma 4.1. The equation

$$(23) \quad x'' + k^2|x|^\gamma \operatorname{sign} x = c|x|^{\gamma-\varepsilon}(x'^2)^{\varepsilon/(\gamma+1)} \operatorname{sign} x,$$

where $0 < \varepsilon \leq \gamma < 1$, $k, c > 0$ are real numbers, has not oscillatory solutions if

$$(24) \quad k^2 \leq c_*,$$

where c_* is given in (11).

If $k^2 > c_*$, then for any $x_0 \neq 0$ a solution of (23) with the initial data

$$(25) \quad x(a) = x_0, \quad x'(a) = 0$$

is periodic and the period is

$$(26) \quad T(x_0) = x_0^{(1-\gamma)/2} T(1).$$

Proof. Rewrite (23) as

$$(27) \quad \begin{aligned} x' &= y, \\ y' &= cy^{2\varepsilon/(\gamma+1)}|x|^{\gamma-\varepsilon} \operatorname{sign} x - k^2|x|^\gamma \operatorname{sign} x \end{aligned}$$

and

$$(28) \quad \frac{dy}{dx} = \frac{cy^{2\varepsilon/(\gamma+1)}|x|^{\gamma-\varepsilon} \operatorname{sign} x - k^2|x|^\gamma \operatorname{sign} x}{y}.$$

The new variable $z = y^2$ satisfies then the equation

$$(29) \quad \frac{dz}{dx} = 2c|x|^{\gamma-\varepsilon}z^{\varepsilon/(\gamma+1)} \operatorname{sign} x - 2k^2|x|^\gamma \operatorname{sign} x.$$

The function $z = Dx^{\gamma+1}$ solves this equation, where D can be found from

$$(30) \quad 2k^2 + (\gamma + 1)D = 2cD^{\varepsilon/(\gamma+1)}.$$

If the finite equation (30) is solvable which is equivalent, by Lemma 3.1, to the condition that $k^2 \leq c_*$, then do not exist solutions of (23), which satisfy the initial conditions

$$(31) \quad x(a) = x_0 > 0, \quad x'(a) = 0$$

and one has at some point $t_1 > a$ that $x'(t_1) \neq 0$, $x(t_1) = 0$. Then no oscillatory solutions exist.

Consider the case $k^2 > c_*$. Solutions satisfying the initial conditions (31) are uniformly bounded in C^1 -norm by a constant $\rho(x_0)$ since they extend to the interval $[a, b]$. Then trajectories of solutions of the equation (23) cross the axis $x = 0$ with $x' \neq 0$. Otherwise a corresponding $z(x)$ attains zero value at $x = 0$ which contradicts Lemma 3.2.

To prove (26) consider a solution $x(t)$ of (23) with the initial values $x(0) = 1$, $x'(0) = 0$. Let $T(1)$ stand for its period. Consider now a solution $u(t)$ of (23) with the initial data $u(0) = x_0$, $u'(0) = 0$. It can be verified that

$$u(t) = x_0 \cdot x(x_0^{(\gamma-1)/2}t).$$

Therefore $u(t)$ is periodic with a period $T(x_0)$ given in (26). Since the equation (23) is autonomous the same is true for a solution satisfying the initial conditions (25). It is obvious that $T(x_0) \rightarrow 0$ as $x_0 \rightarrow 0$. ■

Remark. The right side of (23) is discontinuous with respect to x , if $\varepsilon = \gamma$. Then trajectories in the phase plane may not be smooth at the points of intersection with the x' -axis.

Lemma 4.2. *Suppose that conditions of Theorem 4.1 are fulfilled. Then given $\delta > 0$ a number $r > 0$ exists such that any solution of (1), which satisfies the initial conditions*

$$(32) \quad x(t_0) = x_0 < r, \quad x'(t_0) = 0, \quad t_0 \in (a + \delta, b - \delta)$$

has zeros in both intervals $(t_0 - \delta, t_0)$ and $(t_0, t_0 + \delta)$. A number r can be chosen uniformly with respect to t_0 .

Proof. To be specific, consider the case of $x_0 > 0$. A solution $x(t)$ of the problem (1), (32) in the interval of positivity satisfies

$$(33) \quad x''(t) + k^2 x^\gamma(t) = cx^{\gamma-\varepsilon}(t)(x'^2)^{\varepsilon/(\gamma+1)}(t) - E(t),$$

where $E(t) \geq 0$. We will compare $x(t)$ with a solution $y(t)$ of

$$(34) \quad y''(t) + k^2 y^\gamma(t) = cy^{\gamma-\varepsilon}(t)(y'^2)^{\varepsilon/(\gamma+1)}(t),$$

satisfying the same initial conditions $y(t_0) = x_0$, $y'(t_0) = 0$. The solution $x(t)$ decreases in some right neighbourhood of t_0 since $x'(t_0) = 0$ and $x''(t_0) < 0$. In the interval (t_0, t_M) , in which both functions $x(t)$ and $y(t)$ are monotone, the new variables $u = x'^2$ and $v = y'^2$ satisfy the first order equations

$$(35) \quad \frac{du}{dx} = 2cx^{\gamma-\varepsilon}u^{\varepsilon/(\gamma+1)} - 2k^2x^\gamma - e(x),$$

$$(36) \quad \frac{dv}{dx} = 2cx^{\gamma-\varepsilon}v^{\varepsilon/(\gamma+1)} - 2k^2x^\gamma,$$

and the initial conditions

$$(37) \quad u(x_0) = v(x_0) = 0.$$

The function $e(x)$ in (35) is $e(x) = E(t(x))$, $t(x)$ is the inverse function for $x(t)$, defined for $x \in (x(t_M), x(t_0))$.

Suppose that $u(x_1) < v(x_1)$ at some point x_1 . In view of (37), $x_2 \in (x_1, x_0]$ exists such that

$$(38) \quad u(x) < v(x) \quad \forall x \in [x_1, x_2], \quad u(x_2) = v(x_2).$$

One concludes from (35) that

$$\frac{du}{dx} < \frac{dv}{dx}, \quad x \in [x_1, x_2].$$

This contradicts the inequality in (38).

Thus $u(x) \geq v(x)$ for $x \in [0, x_0]$ and

$$(39) \quad \left| \frac{dx}{dt} \right| \Big|_{t=t_x} \geq \left| \frac{dy}{dt} \right| \Big|_{t=t_y},$$

where t_x and t_y are such that $x(t_x) = y(t_y)$. The inequalities (39) for derivatives of inverse functions $t(x)$ and $\tau(x)$ (inverse to $x(t)$ and $y(t)$ respectively) take the

form

$$\frac{d\tau}{dx} \leq \frac{dt}{dx} < 0$$

and, therefore, taking into account that $t(x_0) = \tau(x_0) = t_0$, one obtains the inequality $t(0) \leq \tau(0)$, thus completed the proof. ■

Lemma 4.3. *A positive valued function $m(r)$ exists such that*

- 1) *if $x(t)$ is a solution of the equation (1) and $\|x(a)\| := |x(a)| + |x'(a)| \leq r$ then $\|x(t)\| := |x(t)| + |x'(t)| \leq m(r)$ for any $t \in (a, b)$;*
- 2) *$m(r) \rightarrow 0$ as $r \rightarrow 0$.*

Proof. We may assume that all solutions of (1) extend to the whole interval $[a, b]$. The explanation is given in the remark following the formulation of the Theorem 4.1. The existence of a function $m(r)$, satisfying the condition 1 of Lemma 4.3, follows then from Fukuhara's theorem on compactness of the integral funnel (cf. [9, Lemma 15.1]). One have to show only that $m(r) \rightarrow 0$ as $r \rightarrow 0$. Following [9, Lemma 15.1], construct

$$m(r) = \sup_{t \in [a, b]} \{\|x(t)\| : x \in S_r\},$$

where S_r is the set of solutions of (1), satisfying the initial condition $\|x(a)\| \leq r$.

Since the set S_r is compact in $C^1[a, b]$ (by Fukuhara's theorem, cf. [9, Theorem 15.1]), the supremum above evidently tends to zero as $r \rightarrow 0$. ■

Lemma 4.4. *Suppose that conditions of Theorem 4.1 are fulfilled. Then given $\delta > 0$ there exists $r > 0$ such that any solution $x(t)$ of (1) with the initial values*

$$x(a) = x_0, \quad x'(a) = x'_0,$$

has a zero $t_1 \in (a, a + \delta)$, if $\|x(a)\| \leq r$.

Proof: To be specific, suppose that $x_0 \geq 0$, $x'_0 > 0$. The case of $x'_0 = 0$ was discussed in Lemma 4.2. We will show that $x(t)$ has an extremum (positive maximum) to the right of $t = a$ if $\|x(a)\|$ is sufficiently small.

Notice that, by preceding Lemma, a function $m(r)$ exists described above. Choose $y_0 > 0$ so that a solution $y(t)$ of the equation (23), which satisfies the initial conditions

$$(40) \quad y(t) = y_0, \quad y'(t) = 0,$$

has zeros in the intervals $(t - \delta/3, t)$ and $(t, t + \delta/3)$. This is possible, by Lemma 4.1. Choose $r > 0$ so small that any solution $x(t)$ satisfying the initial

condition $\|x(a)\| \leq r$ satisfies also $\|x(t)\| \leq m(r) \leq y_0$. This is possible, by preceding lemma.

If $x(t)$ attains its maximum at some point $t_* > a$, then set $t = t_*$ in (40). Obviously $x(t_*) \leq y_0$. Then $y_1(t)$, satisfying the equation (23) and the initial conditions

$$y_1(t_*) = x(t_*) \leq y_0, \quad y'_1(t_*) = 0,$$

has a zero in $(t_*, t_* + \delta/3)$. A solution $x(t)$ then also has a zero in this interval, by Lemma 4.2.

Suppose that $x(t)$ is monotone (monotonically increasing) in $(a, a + \delta/2)$. Then even at $t = a + 2\delta/3$ one has $x(t) \leq y_0$. Pick $t_2 \in (a + \delta/3, a + 2\delta/3)$ such that $x'(t_2) > 0$. Consider also a solution of (23) $y(t)$, which satisfies the initial value conditions

$$y(t_2) = x(t_2) \leq y_0, \quad y'(t_2) = 0.$$

By the choice of y_0 , $y(t)$ has a zero t_1 in the interval $(t_2 - \delta/3, t_2)$.

Graphs of $x(t)$ and $y(t)$ intersect at least twice, at the point $t = t_2$ and at some $a_1 \in (t_1, t_2)$, $x'(t) > 0$, $y'(t) > 0$, $t \in [a_1, t_2]$.

We will show now that this is impossible. Since the functions $x(t)$ and $y(t)$ are monotone in $[a_1, t_2]$, the respective inverse functions $t(x)$ and $\tau(x)$ exist. Their graphs intersect also in the interval $[\mu, \alpha]$, where $\mu = x(a_1) = y(a_1)$ ($t(\alpha) = \tau(\alpha) = t_2$, $t(\mu) = \tau(\mu) = a_1$, $t(x) > \tau(x)$ for $x \in (\mu, \alpha)$, $t'_x, \tau'_x > 0$ for $x \in (\mu, \alpha)$).

The differential equations for $t(x)$ and $\tau(x)$, derived from (33) and (34) respectively, are:

$$(41) \quad -\frac{t''_{xx}}{t'^3_x} + k^2 x^\gamma = cx^{\gamma-\varepsilon} \frac{1}{t'^{2\beta}_x} - E(t(x))$$

$$(42) \quad -\frac{\tau''_{xx}}{\tau'^3_x} + k^2 x^\gamma = cx^{\gamma-\varepsilon} \frac{1}{\tau'^{2\beta}_x},$$

where $\beta = \varepsilon/(\gamma + 1)$. Subtracting (42) from (41) we get

$$(43) \quad \frac{1}{2} \left(\frac{1}{t'^2_x(x)} - \frac{1}{\tau'^2_x(x)} \right)' = cx^{\gamma-\varepsilon} z(x) \left(\frac{1}{t'^2_x(x)} - \frac{1}{\tau'^2_x(x)} \right) - E(t(x)),$$

where

$$z(x) = \left(\frac{1}{t'^{2\beta}_x(x)} - \frac{1}{\tau'^{2\beta}_x(x)} \right) \left(\frac{1}{t'^2_x(x)} - \frac{1}{\tau'^2_x(x)} \right)^{-1}$$

and $z(x)$ is defined by continuity ($z(x) = \beta t'^{-2(\beta-1)}_x(x)$) at the points where $t'_x(x) = \tau'_x(x)$.

Set $u = t_x'^{-2}(x) - \tau_x'^{-2}(x)$ and $q(x) = x^{\gamma-\varepsilon}z(x)$. Then the equation (43) takes the form

$$(44) \quad u'_x = 2cq(x)u - 2E(t(x)).$$

Notice that at some $x_* \in [\mu, \alpha]$

$$(45) \quad u(x_*) = 0.$$

Then, resolving (44) and making use of (45), one gets

$$(46) \quad u(x) = -2 \int_{x_*}^x e^{v(x)-v(s)} E(t(s)) ds,$$

where $v(x) = 2c \int_{x_*}^x q(s) ds$. One concludes from (46) that $u(\alpha) \leq 0$. On the other hand,

$$\lim_{x \rightarrow \alpha} u(x) = \lim_{x \rightarrow \alpha} \left(\frac{1}{t_x'^2(x)} - \frac{1}{\tau_x'^2(x)} \right) = \lim_{t \rightarrow t_2} (x'^2(t) - y'^2(t)) = x'^2(t_2) > 0,$$

and we arrive at the contradiction.

Other cases can be treated similarly.

We have proved that $x(t)$ necessarily has an extremum at a point, which is sufficiently close to the starting point $t = a$, if the initial values are small enough. The proof is completed then by application of Lemma 4.2. ■

Proof of the theorem. Let an integer N be given. Define $\delta = (b - a)/(N + 1)$. We will show that $x_0 > 0$ can be chosen so that any solution of (1), satisfying the condition

$$\|x(a)\| \leq x_0,$$

has zeros and the maximal distance between consecutive zeros is less than δ .

Choose r_1 so that a solution of (1) with the initial data, satisfying the inequality $\|x(a)\| < r_1$, vanishes at some point t_1 to the right of a and $t_1 - a < \delta$. This is possible by Lemma 4.4.

Choose r_2 so that any solution of (1) with an extremum $|x(t_0)| \leq r_2$, $x'(t_0) = 0$, $t_0 \in (a + \delta, b - \delta)$, has zeros $t_2 \in (t_0 - \delta, t_0)$ and $t_3 \in (t_0, t_0 + \delta)$. This is possible, by Lemma 4.3, if the initial data at $t = a$ are sufficiently small. More precisely, a positive number $r_3(r_2)$ exists such that any solution of (1), satisfying the condition $\|x(a)\| < r_3$, possesses the property above.

Then, taking into account the condition 4 of the Theorem 4.1, one concludes that any solution of (1), which satisfies

$$\|x(a)\| < \max\{r_1, r_3\},$$

has zeros and the distance between two consecutive zeros is less than δ . By the choice of δ the total number of zeros exceeds N . ■

Remark. The condition 3 of Theorem 4.1 cannot be improved since, by Lemma 4.1, the equation

$$(47) \quad x'' + k^2|x|^\gamma \operatorname{sign} x = c(x'^2)^{\gamma/(\gamma+1)}$$

does not possess the SB-property, if $k^2 \leq c_*$.

Moreover, power indices in (47) cannot be improved also as the proposition below shows.

Proposition. *The equation*

$$(48) \quad x'' + k^2|x|^{\gamma+\varepsilon} \operatorname{sign} x = c(x'^2)^{\gamma/(\gamma+1)},$$

where $0 < \gamma < \varepsilon + \gamma < 1$, does not possess the SB-property irrespectively of the choice of the constants k and $c > 0$.

Proof. Consider a solution $x(t)$ of (48) with the initial data

$$(49) \quad x(a) = x_0, \quad x'(a) = 0.$$

In the interval of monotonicity of $x(t)$ the new variable $z = x'^2$ satisfies the first order equation

$$(50) \quad \frac{dz}{dx} = 2cz^{\gamma/(\gamma+1)} - 2k^2x^{\gamma+\varepsilon}.$$

The zero isocline $i(x)$ of the equation (50) is

$$(51) \quad i(x) = \left(\frac{k^2}{c}\right)^{(\gamma+1)/\gamma} x^{((\gamma+\varepsilon)/\gamma)(\gamma+1)}.$$

Thus the respective $z(x)$ either is positive for any $x \in [0, x_0)$ or enters the origin along with the zero isocline. In both cases $x(t)$ is monotonically decreasing until it is zero.

Let a number K satisfy the inequality

$$(52) \quad K^2 < c_* := \frac{c}{\gamma+1} \left[\frac{2\gamma c}{(\gamma+1)^2} \right]^\gamma.$$

Then, by Lemma 3.1, a solution of the form $v(x) = Ax^{\gamma+1}$ exists of the equation

$$(53) \quad \frac{dv}{dx} = 2cv^{\gamma/(\gamma+1)} - 2K^2x^\gamma.$$

Then a particular solution $v(x)$, satisfying the initial condition $v(2x_0) = 0$, attains its maximum at the point of intersection with the respective zero isocline and vanishes at $x = 0$.

Suppose that the equation under consideration (48) possesses the SB-property. Then the respective $z(x)$ is positive at $x = 0$. It appears that the graph of $z(x)$ crosses the one of $v(x)$ from above at some point $x_* \in (0, x_0)$.

On the other hand,

$$(54) \quad \left. \frac{dz}{dx} \right|_{x=x_*} = 2cz^{\gamma/(\gamma+1)}(x_*) - 2K^2x_*^\gamma \cdot \left(\frac{k^2}{K^2}x_*^\varepsilon \right).$$

The respective expression for dv/dx is

$$(55) \quad \left. \frac{dv}{dx} \right|_{x=x_*} = 2cv^{\gamma/(\gamma+1)}(x_*) - 2K^2x_*^\gamma.$$

Assume that x_0 in (49) is so small that the expression $(k^2/K^2)x_*^\varepsilon$, where, evidently, $x_* < x_0$, is less than the unity. Then, comparing the right sides in (54) and (55) and taking into account that $z(x_*) = v(x_*)$, one concludes that

$$\left. \frac{dz}{dx} \right|_{x=x_*} > \left. \frac{dv}{dx} \right|_{x=x_*}.$$

Therefore the equation (48) does not possess the SB-property. ■

Remark. One can show also that the equation

$$(56) \quad x'' + k^2|x|^\gamma \operatorname{sign} x = c(x'^2)^p$$

does not possess the SB-property if

$$(57) \quad 0 < p < \frac{\gamma}{\gamma + 1}.$$

This case can be reduced to the previous one by the following arguments. Let μ be defined by the equality $p = \mu/(\mu + 1)$. Since $p < \gamma/(\gamma + 1)$, $\mu < \gamma$.

The equation (56) may be rewritten now as

$$(58) \quad x'' + k^2|x|^{\mu+(\gamma-\mu)} \operatorname{sign} x = c(x'^2)^{\mu/(\mu+1)},$$

and the application of the previous Proposition proves that the equation (56) does not possess the SB-property.

5. Example

In this section we provide an example of the equation of the form (1), which has a nontrivial solution with the zero initial values. Usually such solutions are referred to as singular ones ([3], [2]).

Consider the equation

$$(59) \quad x'' = -k^2|x|^\gamma \operatorname{sign} x - c(x'^2)^{\gamma/(\gamma+1)} \operatorname{sign} x',$$

where $0 < \gamma < 1$, $k^2 > c_* = [c/(\gamma+1)][2\gamma c/(\gamma+1)^2]^\gamma$. The latter expression for c_* is derived from (11), letting $\varepsilon = \gamma$.

Rewrite (59) as

$$(60) \quad \begin{aligned} x' &= y, \\ y' &= -c|y|^{2\gamma/(\gamma+1)} \operatorname{sign} y - k^2|x|^\gamma \operatorname{sign} x. \end{aligned}$$

Let $(x(t), y(t))$ be a solution of (60), Consider the function

$$(61) \quad V(t) = \frac{y^2(t)}{2} + \frac{k^2}{\gamma+1} |x|^{(\gamma+1)}(t).$$

Full derivative for V with respect to t is

$$(62) \quad \begin{aligned} V'(t) &= yy' + k^2 x' |x|^\gamma \operatorname{sign} x \\ &= y[-c|y|^{2\gamma/(\gamma+1)} \operatorname{sign} y - k^2|x|^\gamma \operatorname{sign} x] \\ &\quad + k^2 y |x|^\gamma \operatorname{sign} x = -c|y|^{(3\gamma+1)/(\gamma+1)} < 0, \quad \text{if } y \neq 0. \end{aligned}$$

Consider a solution of (59), satisfying the initial conditions

$$(63) \quad x(0) = 0, \quad x'(0) = 1.$$

We will show that this solution is continuable in a self-similar fashion and it vanishes in finite time, that is, it is a singular solution.

First, show that $x(t)$ has a zero $\tau_1 > 0$, where x' is positive. Obviously, $x(t)$ either has a positive maximum at some point $t_M > 0$ or it is monotone. The second case is excluded by the observation that then $x''(t) < -k^2 x(\varepsilon) < 0$ for any $t > \varepsilon$, where ε is a positive number.

The equation (59) takes the form

$$x'' = -k^2 x^\gamma + c(x'^2)^{\gamma/(\gamma+1)}$$

for values of $t > t_M$, where $x(t)$ is positive and $x'(t)$ is negative. Taking in account the choice of k^2 , one concludes then that Lemma 4.1 is applicable and hence $x(t)$ has a zero $t_0 > t_M > 0$, where x' is negative.

It can be shown similarly that $x(t)$ has a zero $\tau_1 > t_0$, where x' is positive. It is important for further considerations that, in view of (62),

$$(64) \quad 0 < x'(\tau_1) =: \sigma < x'(0) = 1.$$

Our construction is based on the following observation. Let $x(t)$ be a solution of (59). Then the function $x_r(t) = r^{2/(1+\gamma)} x(r^{-(1-\gamma)/(1+\gamma)} t)$ also is a solution and

$$(65) \quad x_r(0) = r^{2/(1+\gamma)} x(0),$$

$$(66) \quad x'_r(0) = r x'(0).$$

Define a function $x_\sigma(t) = \sigma^{2/(1+\gamma)} x(\sigma^{-(1-\gamma)/(1+\gamma)} t)$. Notice that $x_\sigma(t)$ has its second zero to the right of $t = 0$ at the point

$$(67) \quad t_1 = \sigma^{(1-\gamma)/(1+\gamma)} \tau_1 < \tau_1.$$

By virtue of (66),

$$(68) \quad x'_\sigma(0) = \sigma x'(0) = \sigma.$$

Since the equation (59) is autonomous, the function $x_\sigma(t - \tau_1)$ is a solution and, moreover, it extends the solution $x(t)$ to the interval $[0, \tau_1] \cup [\tau_1, \tau_2]$. Smoothness at the point of junction is ensured by (68). The length of the second interval is

$$(69) \quad \tau_2 - \tau_1 = t_1 = \sigma^{(1-\gamma)/(1+\gamma)} \tau_1.$$

Analogically one can extend $x(t)$ to the interval $[0, \tau_3]$ and so on. One has then that

$$(70) \quad \tau_3 - \tau_2 = \sigma^{(1-\gamma)/(1+\gamma)} (\tau_2 - \tau_1) = \sigma^{2(1-\gamma)/(1+\gamma)} \tau_1.$$

Proceeding in this way, one gets a sequence of intervals $[\tau_{i-1}, \tau_i]$, each of the length $l_i = \sigma^{(i-1)(1-\gamma)/(1+\gamma)} \tau_1$. The sum

$$(71) \quad \tau_1 (1 + \sigma^{(1-\gamma)/(1+\gamma)} + \dots + \sigma^{i(1-\gamma)/(1+\gamma)} + \dots) = \frac{\tau_1}{1 - \sigma^{(1-\gamma)/(1+\gamma)}}$$

is finite. This means that $x(t)$ vanishes in finite time and hence it is a singular solution.

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