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Nonoscillatory Solutions of Functional Difference Equations Involving Quasi-Differences

By

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1. Introduction

Let $\mathbf{R}^*$ denote the extended real line, i.e., $\mathbf{R}^* = \mathbf{R} \cup \{\infty, -\infty\}$. For integers a, b ($> a$), we shall denote the discrete interval $[a, b] = \{a, a+1, \dots, b\}$. Further, for $k \geq 0$ we define the usual factorial expression $j^{(k)} = j(j-1) \cdots (j-k+1)$ with $j^{(0)} = 1$.

Let r_i , $0 \leq i \leq n$ be functions such that $r_i : [\ell_0, \infty) \rightarrow \mathbf{R}^+$. For an integer $k \in [0, n]$ and a function y defined on $[\ell_0, \infty)$, we define the k th quasi-difference of y , also known as the k th r -difference of y as

$$(1.1) \quad \Delta_r^k y = r_k(r_{k-1}(r_{k-2}(\cdots(r_1(r_0 y)^{\Delta})^{\Delta} \cdots)^{\Delta})^{\Delta})^{\Delta}$$

where $(\cdot)^{\Delta} \equiv \Delta(\cdot)$. It is clear from (1.1) that

$$(1.2) \quad \Delta_r^0 y = r_0 y$$

and

$$(1.3) \quad \Delta_r^k y = r_k(\Delta_r^{k-1} y)^{\Delta} = r_k \Delta(\Delta_r^{k-1} y).$$

Further, if $r_i = 1$, $0 \leq i \leq k$, then $\Delta_r^k y = \Delta^k y$.

In this paper we shall consider the difference equation

$$(E_m) \quad \Delta_r^n y(\ell) + g(\ell)F(y(a_1(\ell)), y(a_2(\ell)), \dots, y(a_m(\ell))) = h(\ell), \quad \ell \geq \ell_0$$

where $F : \mathbf{R}^m \rightarrow \mathbf{R}$ is continuous, $a_i : [\ell_0, \infty) \rightarrow \mathbf{Z}$, $1 \leq i \leq m$ and $g, h : [\ell_0, \infty) \rightarrow \mathbf{R}$. Throughout, it is assumed that

$$(A1) \quad r_n = 1;$$

$$(A2) \quad \text{for each } 1 \leq i \leq m, \lim_{\ell \rightarrow \infty} a_i(\ell) = \infty;$$

$$(A3) \quad \text{if } u_i > 0, 1 \leq i \leq m, \text{ then } F(u_1, \dots, u_m) > 0; \text{ and if } u_i < 0, 1 \leq i \leq m, \text{ then } F(u_1, \dots, u_m) < 0.$$

By a solution y of (E_m) , we mean a nontrivial sequence $y(\ell)$ satisfying (E_m) for $\ell \geq \ell_0$. A solution is said to be *nonoscillatory* if it is eventually positive or negative, and *oscillatory* otherwise. We shall investigate the asymptotic behavior of a nonoscillatory solution of (E_m) . To be specific, we shall develop

criteria such that a nonoscillatory solution of (E_m) is bounded and/or tends to zero as $\ell \rightarrow \infty$.

The introduction of quasi-difference operator is motivated by the work of several authors [4–6] in which differential equations involving quasi-derivatives are discussed. In fact, our results extend the work of Staikos and Philos [6] to discrete case, and naturally generalize/complement several other known criteria for the special case of (E_m) when $r_i = 1$, $0 \leq i \leq n$, e.g., see the monographs [1, 2] and the hundreds of references therein.

The paper is organized as follows. In Section 2, we shall define a function of type $r[k]$ and give four examples of such a function. Two lemmas which are needed later are also presented. Five main results are established in Section 3 and they lead to 11 corollaries in Section 4. Finally, in Section 5 we include five examples to dwell upon the importance of the results obtained.

2. Functions of type $r[k]$

Let x be a function such that $x : [\ell_0, \infty) \rightarrow \mathbf{R}^+$, and let T_i , $0 \leq i \leq n$ be defined by

$$(2.1) \quad T_n = x,$$

$$(2.2) \quad T_j(\ell + 1) = r_j(\ell) \Delta T_{j+1}(\ell), \quad j = n-1, n-2, \dots, 0.$$

Let $k \in [0, n-1]$. We say that x is of type $r[k]$ if

(B1) T_j , $k+1 \leq j \leq n$ are defined on $[\ell_0 + n - j, \infty)$;

(B2) T_{k+1} is a constant nonzero function on $[\ell_0 + n - k - 1, \infty)$;

(B3) if $k \leq n-2$, then for each $k+2 \leq j \leq n$,

$$\lim_{\ell \rightarrow \infty} T_j(\ell) \in \{0, \infty, -\infty\};$$

(B4) if $k \leq n-3$, then for each $k+2 \leq j \leq n-1$, $T_j(\ell) \neq 0$ for $\ell \geq \ell_1$ where ℓ_1 is some integer.

Some examples of functions of type $r[k]$ are given below. The verification is direct and hence is omitted. These functions shall be needed in later sections.

Example 2.1. For $0 \leq i \leq j \leq n-1$, define

$$(2.3) \quad \rho_{ij}(\ell) = \begin{cases} 1, & \text{if } i = j \\ \sum_{\ell_j=\ell_0}^{\ell-1} \frac{1}{r_j(\ell_j)} \sum_{\ell_{j-1}=\ell_0}^{\ell_j-1} \frac{1}{r_{j-1}(\ell_{j-1}+1)} \sum_{\ell_{j-2}=\ell_0}^{\ell_{j-1}-1} \frac{1}{r_{j-2}(\ell_{j-2}+2)} \\ \quad \dots \sum_{\ell_{i+1}=\ell_0}^{\ell_{i+2}-1} \frac{1}{r_{i+1}(\ell_{i+1}+j-i-1)}, & \text{if } i \leq j-1. \end{cases}$$

In the case $k \leq n-2$, suppose that

$$(2.4) \quad \sum_{j=s}^{\infty} \frac{1}{r_j(s)} = \infty, \quad k+1 \leq j \leq n-1.$$

Then, $\rho_{k,n-1}$ is of type $r[k]$ for any $k \in [0, n-1]$.

Example 2.2. For $0 \leq k \leq n-1$, define

$$(2.5) \quad \sigma_k(\ell) = \begin{cases} 1, & \text{if } k = n-1 \\ \sum_{\ell_{n-1}=\ell}^{\infty} \frac{1}{r_{n-1}(\ell_{n-1})} \sum_{\ell_{n-2}=\ell_{n-1}}^{\infty} \frac{1}{r_{n-2}(\ell_{n-2}+1)} \sum_{\ell_{n-3}=\ell_{n-2}}^{\infty} \frac{1}{r_{n-3}(\ell_{n-3}+2)} \\ \quad \cdots \sum_{\ell_{k+1}=\ell_{k+2}}^{\infty} \frac{1}{r_{k+1}(\ell_{k+1}+n-k-2)}, & \text{if } k \leq n-2. \end{cases}$$

In the case $k \leq n-2$, suppose that

$$(2.6) \quad \sum_{j=s}^{\infty} \frac{1}{r_j(s)} < \infty, \quad k+1 \leq j \leq n-1.$$

Then, σ_k is of type $r[k]$ for any $k \in [0, n-1]$.

Example 2.3. Let $r_j = 1$ for $j \neq 0, n-1$ and let $r_{n-1} = r$. For $0 \leq k \leq n-2$, define

$$(2.7) \quad \theta_k(\ell) = \sum_{s=\ell}^{\infty} \frac{(s-\ell_0)^{(n-2-k)}}{r(s)}.$$

Suppose that

$$(2.8) \quad \sum_{s=\ell_0}^{\infty} \frac{s^{(n-2-k)}}{r(s)} < \infty.$$

Then, θ_k is of type $r[k]$ for any $k \in [0, n-2]$.

Example 2.4. Let $N \in [2, n-1]$ be fixed, $r_j = 1$ for $j \neq 0, n-N$ and $r_{n-N} = r$. For $0 \leq k \leq n-N-1$ and $1 \leq j \leq N-1$, define

$$(2.9) \quad \eta_{kj}(\ell) = \sum_{s=\ell_0}^{\ell-1} (\ell-s)^{(j-1)} \sum_{u=s}^{\infty} \frac{(u-\ell_0)^{(n-N-1-k)}}{r(u+1)}.$$

Suppose that

$$(2.10) \quad \sum_{s=\ell_0}^{\infty} \frac{s^{(n-N-1-k)}}{r(s+1)} < \infty$$

and

$$(2.11) \quad \lim_{\ell \rightarrow \infty} \eta_{kj}(\ell) \in \{0, \infty\}, \quad 1 \leq j \leq N-1.$$

Then, $\eta_{k,N-1}$ is of type $r[k]$ for any $k \in [0, n-N-1]$.

Lemma 2.1. *Consider the difference equation*

$$(2.12) \quad \Delta u(\ell) - \frac{\Delta T(\ell)}{T(\ell)} u(\ell) + \frac{\Delta T(\ell)}{T(\ell)} H(\ell) = 0.$$

where

(a) $T(\ell)$, $\Delta T(\ell) \neq 0$ for $\ell \geq L$;

(b) $\lim_{\ell \rightarrow \infty} T(\ell) \in \{0, \infty, -\infty\}$.

Let y be the solution of (2.12) such that $y(L) = 0$. If $\lim_{\ell \rightarrow \infty} H(\ell)$ exists in $\mathbf{R}^*$, so does $\lim_{\ell \rightarrow \infty} y(\ell)$. Further, $\lim_{\ell \rightarrow \infty} |H(\ell)| = \infty$ implies $\lim_{\ell \rightarrow \infty} |y(\ell)| = \infty$.

Proof. It can be verified that the solution y is given by

$$(2.13) \quad y(\ell) = -T(\ell) \sum_{s=L}^{\ell-1} \frac{H(s) \Delta T(s)}{T(s) T(s+1)}.$$

Case 1. $\lim_{\ell \rightarrow \infty} T(\ell) = 0$. If $\lim_{\ell \rightarrow \infty} H(\ell) \neq 0$, then $H(\ell)$ is of fixed sign for large ℓ . So it is obvious that

$$(2.14) \quad \sum_{s=L}^{\infty} \frac{H(s) \Delta T(s)}{T(s) T(s+1)} = \infty \quad \text{or} \quad -\infty.$$

Applying discrete l'Hospital's rule [1, p. 28], we find

$$(2.15) \quad \begin{aligned} \lim_{\ell \rightarrow \infty} y(\ell) &= \lim_{\ell \rightarrow \infty} \frac{-\sum_{s=L}^{\ell-1} \frac{H(s) \Delta T(s)}{T(s) T(s+1)}}{\frac{1}{T(\ell)}} \\ &= \lim_{\ell \rightarrow \infty} \frac{-\frac{H(\ell) \Delta T(\ell)}{T(\ell) T(\ell+1)}}{\frac{1}{T(\ell+1)} - \frac{1}{T(\ell)}} = \lim_{\ell \rightarrow \infty} H(\ell). \end{aligned}$$

If $\lim_{\ell \rightarrow \infty} H(\ell) = 0$, then it is clear that $\sum_{s=L}^{\infty} (H(s) \Delta T(s)) / (T(s) T(s+1))$ exists in $\mathbf{R}^*$. In the case that $\sum_{s=L}^{\infty} (H(s) \Delta T(s)) / (T(s) T(s+1))$ is infinite (i.e., (2.14)), we proceed as above and obtain (2.15). If $\sum_{s=L}^{\infty} (H(s) \Delta T(s)) /$

$(T(s)T(s+1))$ is finite, then it follows from (2.13) that

$$(2.16) \quad \lim_{\ell \rightarrow \infty} y(\ell) = -\lim_{\ell \rightarrow \infty} T(\ell) \cdot \sum_{s=L}^{\infty} \frac{H(s)\Delta T(s)}{T(s)T(s+1)} = 0 = \lim_{\ell \rightarrow \infty} H(\ell).$$

Case 2. $\lim_{\ell \rightarrow \infty} T(\ell) = \infty$ or $-\infty$. In view of condition (a), $T(\ell)$ and $\Delta T(\ell)$ are of the same fixed sign for large ℓ . If $\lim_{\ell \rightarrow \infty} H(\ell) \neq 0$, then $H(\ell)$ is of fixed sign for large ℓ . Hence, we see that $\sum_{s=L}^{\infty} (H(s)\Delta T(s))/(T(s)T(s+1))$ exists in $\mathbf{R}^*$. If $\lim_{\ell \rightarrow \infty} H(\ell) = 0$, then it is also obvious that $\sum_{s=L}^{\infty} (H(s)\Delta T(s))/(T(s)T(s+1))$ exists in $\mathbf{R}^*$.

Suppose that $\sum_{s=L}^{\infty} (H(s)\Delta T(s))/(T(s)T(s+1)) \neq 0$. Then, it follows from (2.13) that

$$(2.17) \quad \lim_{\ell \rightarrow \infty} y(\ell) = -\lim_{\ell \rightarrow \infty} T(\ell) \cdot \sum_{s=L}^{\infty} \frac{H(s)\Delta T(s)}{T(s)T(s+1)} = \infty \quad \text{or} \quad -\infty.$$

If $\sum_{s=L}^{\infty} (H(s)\Delta T(s))/(T(s)T(s+1)) = 0$, then by using discrete l'Hospital's rule again we find (2.15).

In both cases, we have (from (2.15) and (2.16))

$$\lim_{\ell \rightarrow \infty} y(\ell) = \lim_{\ell \rightarrow \infty} H(\ell)$$

or (2.17). Hence, the conclusion of the lemma follows.

Lemma 2.2. *Let x be a function of type $r[k]$ for some $k \in [0, n-1]$. If $\sum_{s=L}^{\infty} x(s+1)\Delta_r^n y(s)$ exists in $\mathbf{R}^*$, so does $\lim_{\ell \rightarrow \infty} \Delta_r^k y(\ell)$. Further, $|\sum_{s=L}^{\infty} x(s+1)\Delta_r^n y(s)| = \infty$ implies $\lim_{\ell \rightarrow \infty} |\Delta_r^k y(\ell)| = \infty$.*

Proof. *Case 1* $k = n-1$. By (2.1) and (B2), $x = T_n = T_{k+1} = c$ for some constant $c > 0$. It follows that

$$\lim_{\ell \rightarrow \infty} \Delta_r^{n-1} y(\ell) = \Delta_r^{n-1} y(L) + \frac{1}{c} \sum_{s=L}^{\infty} x(s+1)\Delta_r^n y(s)$$

from which the conclusion is immediate.

Case 2 $k \leq n-2$. For $k+1 \leq j \leq n$, we define

$$(2.18) \quad q_j(\ell) = \sum_{s=L}^{\ell-1} T_j(s+1)\Delta(\Delta_r^{j-1} y(s)) = \sum_{s=L}^{\ell-1} T_j(s+1) \frac{\Delta_r^j y(s)}{r_j(s)}$$

where T_j 's are related to x as in the definition of a function of type $r[k]$.

On summing (2.18) by parts and using (2.2), it follows that

$$\begin{aligned}
 q_j(\ell) &= T_j(s) \Delta_r^{j-1} y(s) \Big|_{s=L}^{\ell} - \sum_{s=L}^{\ell-1} \Delta_r^{j-1} y(s) \Delta T_j(s) \\
 &= \frac{T_j(\ell)}{r_{j-1}(\ell)} \left[\frac{T_{j-1}(\ell+1) \Delta_r^{j-1} y(\ell)}{r_{j-1}(\ell)} \right] - T_j(L) \Delta_r^{j-1} y(L) \\
 &\quad - \sum_{s=L}^{\ell-1} \Delta_r^{j-1} y(s) \frac{T_{j-1}(s+1)}{r_{j-1}(s)} \\
 &= \frac{T_j(\ell)}{\Delta T_j(\ell)} \Delta q_{j-1}(\ell) - T_j(L) \Delta_r^{j-1} y(L) - q_{j-1}(\ell)
 \end{aligned}$$

or equivalently

$$\Delta q_{j-1}(\ell) - \frac{\Delta T_j(\ell)}{T_j(\ell)} q_{j-1}(\ell) + \frac{\Delta T_j(\ell)}{T_j(\ell)} H_j(\ell) = 0$$

where

$$(2.19) \quad H_j(\ell) = -q_j(\ell) - T_j(L) \Delta_r^{j-1} y(L).$$

Therefore, q_{j-1} is a solution of the difference equation (2.12)| $_{T=T_j, H=H_j}$ and we note that $q_{j-1}(L) = 0$. It is also obvious from (2.19) that if $\lim_{\ell \rightarrow \infty} q_j(\ell)$ exists in $\mathbf{R}^*$, so does $\lim_{\ell \rightarrow \infty} H_j(\ell)$. Applying Lemma 2.1, we see that $\lim_{\ell \rightarrow \infty} q_{j-1}(\ell)$ exists in $\mathbf{R}^*$. Further, if $\lim_{\ell \rightarrow \infty} |q_j(\ell)| = \infty$, then (2.19) provides $\lim_{\ell \rightarrow \infty} |H_j(\ell)| = \infty$. So by Lemma 2.1 again, we have $\lim_{\ell \rightarrow \infty} |q_{j-1}(\ell)| = \infty$.

Now, since

$$\lim_{\ell \rightarrow \infty} q_n(\ell) = \sum_{s=L}^{\infty} T_n(s+1) \frac{\Delta_r^n y(s)}{r_n(s)} = \sum_{s=L}^{\infty} x(s+1) \Delta_r^n y(s)$$

exists in $\mathbf{R}^*$ (given assumption), the above arguments can be applied for $j = n, n-1, \dots, k+2$. Consequently, we have

$$(2.20) \quad \lim_{\ell \rightarrow \infty} q_{k+1}(\ell) \text{ exists in } \mathbf{R}^*.$$

In addition,

$$(2.21) \quad \left| \sum_{s=L}^{\infty} x(s+1) \Delta_r^n y(s) \right| = \infty \quad \text{implies} \quad \lim_{\ell \rightarrow \infty} |q_{k+1}(\ell)| = \infty.$$

Next, since $T_{k+1} = c$ for some constant $c > 0$, from (2.18) we find

$$q_{k+1}(\ell) = \sum_{s=L}^{\ell-1} T_{k+1}(s+1) \Delta(\Delta_r^k y(s)) = c[\Delta_r^k y(\ell) - \Delta_r^k y(L)]$$

which leads to

$$(2.22) \quad \lim_{\ell \rightarrow \infty} \Delta_r^k y(\ell) = \frac{1}{c} \lim_{\ell \rightarrow \infty} q_{k+1}(\ell) + \Delta_r^k y(L).$$

Hence, in view of (2.20) and (2.21), it follows from (2.22) that $\lim_{\ell \rightarrow \infty} \Delta_r^k y(\ell)$ exists in $\mathbf{R}^*$ and $|\sum_{s=L}^{\infty} x(s+1) \Delta_r^n y(s)| = \infty$ implies $\lim_{\ell \rightarrow \infty} |\Delta_r^k y(\ell)| = \infty$.

3. Main results

Theorem 3.1. *Let*

$$(C1) \quad \limsup_{\ell \rightarrow \infty} r_0(\ell) < \infty.$$

Further, suppose that there exists a function x of type $r[k]$ for some $k \in [0, n-1]$ such that

$$(C2) \quad \text{if } k \geq 1, \text{ then}$$

$$\sum_{i=1}^{\infty} \frac{1}{r_i(\ell)} = \infty, \quad 1 \leq i \leq k;$$

$$(C3) \quad \text{for all } 0 < \mu \leq 1 \text{ and } v > 0,$$

$$\sum_{\ell=0}^{\infty} x(\ell+1)[\mu g^+(\ell) - g^-(\ell) - v|h(\ell)|] = \infty$$

or

$$\sum_{\ell=0}^{\infty} x(\ell+1)[\mu g^-(\ell) - g^+(\ell) - v|h(\ell)|] = \infty$$

where $g^+(\ell) = \max\{g(\ell), 0\}$ and $g^-(\ell) = \max\{-g(\ell), 0\}$. Then, any bounded solution y of (E_m) satisfies $\liminf_{\ell \rightarrow \infty} |y(\ell)| = 0$.

Proof. Suppose the contrary and let y be a bounded solution of (E_m) such that $\liminf_{\ell \rightarrow \infty} |y(\ell)| > 0$. Then, y is nonoscillatory. We can assume that y is eventually positive, since if otherwise, in view of (A3), the substitution $z = -y$ will transform (E_m) into an equation of the same form satisfying the conditions of the theorem. So let $y(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$.

By (A2) and the boundedness of y , there exist positive constants M_1 and M_2 such that for some $L > \ell_1$,

$$(3.1) \quad M_1 \leq y(a_i(\ell)) \leq M_2, \quad 1 \leq i \leq m, \quad \ell \geq L.$$

Then, in view of (A3) there exist positive constants c_1 and c_2 such that

$$(3.2) \quad c_1 \leq F(y(a_1(\ell)), y(a_2(\ell)), \dots, y(a_m(\ell))) \leq c_2, \quad \ell \geq L.$$

Now, we rewrite (E_m) as

$$(3.3) \quad \begin{aligned} \Delta_r^n y(\ell) &= h(\ell) - g^+(\ell)F(y(a_1(\ell)), \dots, y(a_m(\ell))) \\ &\quad + g^-(\ell)F(y(a_1(\ell)), \dots, y(a_m(\ell))). \end{aligned}$$

Define $\mu = c_1 c_2^{-1} \in (0, 1]$ and $v = c_2^{-1} > 0$. Using (3.2), it follows from (3.3) that

$$(3.4) \quad \begin{aligned} x(\ell + 1)\Delta_r^n y(\ell) &\geq x(\ell + 1)[h(\ell) - c_2 g^+(\ell) + c_1 g^-(\ell)] \\ &= c_2 x(\ell + 1)[\mu g^-(\ell) - g^+(\ell) + v h(\ell)], \quad \ell \geq L \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} x(\ell + 1)\Delta_r^n y(\ell) &\leq x(\ell + 1)[h(\ell) - c_1 g^+(\ell) + c_2 g^-(\ell)] \\ &= -c_2 x(\ell + 1)[\mu g^+(\ell) - g^-(\ell) - v h(\ell)], \quad \ell \geq L. \end{aligned}$$

Noting (C3), inequalities (3.4) and (3.5) yield

$$(3.6) \quad \sum_{\ell=L}^{\infty} x(\ell + 1)\Delta_r^n y(\ell) = \infty \quad \text{or} \quad -\infty.$$

By Lemma 2.2, relation (3.6) implies

$$(3.7)_k \quad \lim_{\ell \rightarrow \infty} \Delta_r^k y(\ell) = \infty \quad \text{or} \quad -\infty.$$

Finally, we shall show that

$$\lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \infty \quad \text{or} \quad -\infty,$$

i.e., $(3.7)_0$ holds. This leads to a contradiction since by (C1) and the boundedness of y , $\Delta_r^0 y(\ell) = (r_0 y)(\ell)$ is bounded. The proof of the theorem is then complete. Obviously, $(3.7)_0$ is trivial if $k = 0$. Suppose that $k \geq 1$. In view of $(3.7)_k$, there exists $L_1 \geq L$ such that for $\ell \geq L_1$,

$$\Delta_r^k y(\ell) \geq 1 \quad \text{or} \quad \Delta_r^k y(\ell) \leq -1.$$

This is equivalent to

$$(3.8) \quad \Delta(\Delta_r^{k-1} y)(\ell) \geq \frac{1}{r_k(\ell)} \quad \text{or} \quad \Delta(\Delta_r^{k-1} y)(\ell) \leq -\frac{1}{r_k(\ell)}, \quad \ell \geq L_1.$$

Summing (3.8) from L_1 to $(\ell - 1)$, we get

$$(3.9) \quad \begin{aligned} \Delta_r^{k-1} y(\ell) &\geq \Delta_r^{k-1} y(L_1) + \sum_{s=L_1}^{\ell-1} \frac{1}{r_k(s)} \quad \text{or} \\ \Delta_r^{k-1} y(\ell) &\leq \Delta_r^{k-1} y(L_1) - \sum_{s=L_1}^{\ell-1} \frac{1}{r_k(s)}. \end{aligned}$$

By (C2), letting $\ell \rightarrow \infty$ in (3.9) leads to

$$\lim_{\ell \rightarrow \infty} \Delta_r^{k-1} y(\ell) = \infty \quad \text{or} \quad -\infty$$

which is the same as $(3.7)_{k-1}$. Hence, by repeating the above procedure we finally obtain $(3.7)_0$. The proof of the theorem is complete.

Theorem 3.2. *Let (C1) holds. Further, suppose that there exists a function x of type $r[0]$ such that*

$$(C4) \quad \sum_{\ell=L}^{\infty} x(\ell+1)|h(\ell)| < \infty;$$

$$(C5) \quad \sum_{\ell=L}^{\infty} x(\ell+1)g^+(\ell) = \infty \quad \text{and} \quad \sum_{\ell=L}^{\infty} x(\ell+1)g^-(\ell) < \infty$$

or

$$\sum_{\ell=L}^{\infty} x(\ell+1)g^+(\ell) < \infty \quad \text{and} \quad \sum_{\ell=L}^{\infty} x(\ell+1)g^-(\ell) = \infty.$$

Then, any bounded nonoscillatory solution y of (E_m) satisfies $\lim_{\ell \rightarrow \infty} r_0(\ell)y(\ell) = 0$. If, in addition,

$$(C6) \quad \liminf_{\ell \rightarrow \infty} r_0(\ell) > 0,$$

then any bounded nonoscillatory solution y of (E_m) satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Let y be a bounded nonoscillatory solution of (E_m) . As in the proof of Theorem 3.1, we assume that y is eventually positive and $y(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$. As before, we have (3.1)–(3.3). It follows from (3.3) that

$$\begin{aligned} (3.10) \quad & \sum_{\ell=L}^{\tau} x(\ell+1)\Delta_r^n y(\ell) \\ &= \sum_{\ell=L}^{\tau} x(\ell+1)h(\ell) - \sum_{\ell=L}^{\tau} x(\ell+1)g^+(\ell)F(y(a_1(\ell)), \dots, y(a_m(\ell))) \\ & \quad + \sum_{\ell=L}^{\tau} x(\ell+1)g^-(\ell)F(y(a_1(\ell)), \dots, y(a_m(\ell))), \quad \tau \geq L. \end{aligned}$$

Noting (3.2), (C4) and (C5), from (3.10) we see that $\sum_{\ell=L}^{\infty} x(\ell+1)\Delta_r^n y(\ell)$ exists in $\mathbf{R}^*$. So by Lemma 2.2, $\lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell)$ also exists in $\mathbf{R}^*$.

Next, since conditions (C4) and (C5) imply (C3), an application of Theorem 3.1 (with $k = 0$) provides

$$(3.11) \quad \liminf_{\ell \rightarrow \infty} y(\ell) = 0.$$

In view of (3.11) and (C1), we find

$$(3.12) \quad \liminf_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \liminf_{\ell \rightarrow \infty} r_0(\ell) y(\ell) = \liminf_{\ell \rightarrow \infty} r_0(\ell) \cdot \liminf_{\ell \rightarrow \infty} y(\ell) = 0.$$

As $\lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell)$ exists, using (3.12) we immediately get

$$\lim_{\ell \rightarrow \infty} r_0(\ell) y(\ell) = \lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \liminf_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = 0.$$

The last part of the theorem is obvious.

The next result is for the difference equation (E_1) . Here, we shall denote $a_1 = a$.

Theorem 3.3. *Let (C1) hold and*

$$(C7) \quad \sum_{\ell=1}^{\infty} \frac{1}{r_1(\ell)} = \infty;$$

$$(C8) \quad \liminf_{\ell \rightarrow \infty} r_1(\ell) > 0.$$

Suppose that there exists a function x of type $r[1]$ such that (C4) holds and (C9) for some positive integer δ ,

$$\liminf_{\ell \rightarrow \infty} \sum_{s=\ell}^{\ell+\delta} x(s+1)g^+(s) > 0 \quad \text{and} \quad \sum_{s=\ell}^{\infty} x(s+1)g^-(s) < \infty$$

or

$$\liminf_{\ell \rightarrow \infty} \sum_{s=\ell}^{\ell+\delta} x(s+1)g^-(s) > 0 \quad \text{and} \quad \sum_{s=\ell}^{\infty} x(s+1)g^+(s) < \infty.$$

Further, assume that a is nondecreasing and Δa is bounded on $[\ell_0, \infty)$. Then, any bounded nonoscillatory solution y of (E_1) satisfies

$$\lim_{\ell \rightarrow \infty} r_1(\ell) \Delta(r_0 y)(\ell) = \lim_{\ell \rightarrow \infty} \Delta(r_0 y)(\ell) = \lim_{\ell \rightarrow \infty} r_0(\ell) y(\ell) = 0.$$

Proof. Let y be a bounded nonoscillatory solution of (E_1) . Once again y is assumed to be eventually positive and $y(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$. As before, we have (3.1)–(3.3) and (3.10) (with $m = 1$). In view of (3.2), (C4) and (C9), from (3.10) we see that $\sum_{\ell=L}^{\infty} x(\ell+1) \Delta_r^n y(\ell)$ exists in $\mathbf{R}^*$. So by Lemma 2.2, $\lim_{\ell \rightarrow \infty} \Delta_r^1 y(\ell)$ also exists in $\mathbf{R}^*$. We shall first show that

$$(3.13) \quad \lim_{\ell \rightarrow \infty} \Delta_r^1 y(\ell) = \lim_{\ell \rightarrow \infty} r_1(\ell) \Delta(r_0 y)(\ell) = 0.$$

Suppose the contrary. Then, there exists a positive constant c such that for

some $L_1 \geq L$,

$$\Delta_r^1 y(\ell) \geq c \quad \text{or} \quad \Delta_r^1 y(\ell) \leq -c, \quad \ell \geq L_1.$$

This is the same as

$$(3.14) \quad \Delta(r_0 y)(\ell) \geq \frac{c}{r_1(\ell)} \quad \text{or} \quad \Delta(r_0 y)(\ell) \leq -\frac{c}{r_1(\ell)}, \quad \ell \geq L_1.$$

On summing (3.14) from L_1 to $(\ell - 1)$ and taking (C7) into account, we get $\lim_{\ell \rightarrow \infty} r_0(\ell)y(\ell) = \infty$ or $-\infty$. This is a contradiction because $r_0 y$ is bounded by virtue of (C1) and the boundedness of y . Hence, (3.13) is proved.

Next, in view of (C8), relation (3.13) immediately provides

$$(3.15) \quad \lim_{\ell \rightarrow \infty} \Delta(r_0 y)(\ell) = 0$$

which is the second conclusion of the theorem.

It remains to show that

$$(3.16) \quad \lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \lim_{\ell \rightarrow \infty} r_0(\ell)y(\ell) = 0.$$

For this, noting that (C7) is actually (C2) when $k = 1$, and also (C4) and (C9) imply (C3), we apply Theorem 3.1 (with $k = 1$) to get (3.11) which, in view of (C1), leads to (3.12). To complete the proof, we need to show further that

$$(3.17) \quad \limsup_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = 0.$$

Suppose (3.17) does not hold. Then, taking note of (A2) and (3.12) (which has been proved), we have

$$\liminf_{\ell \rightarrow \infty} \Delta_r^0 y(a(\ell)) = 0 \quad \text{and} \quad \limsup_{\ell \rightarrow \infty} \Delta_r^0 y(a(\ell)) \geq d$$

where d is some positive constant. Using the arguments in [3], there exist three integer sequences $\{\alpha_j\}$, $\{\beta_j\}$, $\{\gamma_j\}$ with $\lim_{j \rightarrow \infty} \alpha_j = \infty$ such that for each $j \geq 1$,

$$L \leq \alpha_j < \gamma_j < \beta_j \leq \alpha_{j+1};$$

$$(3.18) \quad \Delta_r^0 y(a(\alpha_j)) = \frac{d}{2} = \Delta_r^0 y(a(\beta_j));$$

$$(3.19) \quad \Delta_r^0 y(a(\gamma_j)) > d;$$

$$\Delta_r^0 y(a(\ell)) > \frac{d}{2}, \quad \alpha_j < \ell < \beta_j.$$

Now, by discrete mean value theorem [1, p. 24], we have

$$(3.20) \quad \frac{\Delta_r^0 y(a(\gamma_j)) - \Delta_r^0 y(a(\alpha_j))}{a(\gamma_j) - a(\alpha_j)} \leq \Delta[\Delta_r^0 y(a(\xi_j))] \quad \text{or} \quad \nabla[\Delta_r^0 y(a(\xi_j))]$$

where $a(\xi_j) \in [a(\alpha_j) + 1, a(\beta_j) - 1]$ and $\lim_{j \rightarrow \infty} \xi_j = \infty$. Using the fact that a is nondecreasing and also (3.18) and (3.19) in (3.20), we obtain

$$(3.21) \quad \frac{d - (d/2)}{a(\beta_j) - a(\alpha_j)} < \Delta[(r_0 y)(a(\xi_j))] \quad \text{or} \quad \nabla[(r_0 y)(a(\xi_j))].$$

Since the right side of (3.21) tends to zero as $j \rightarrow \infty$ (by (3.15)), it is necessary that

$$(3.22) \quad \lim_{j \rightarrow \infty} [a(\beta_j) - a(\alpha_j)] = \infty.$$

Once again, by mean value theorem we have

$$(3.23) \quad a(\beta_j) - a(\alpha_j) \leq (\beta_j - \alpha_j) \Delta a(\theta_j) \quad \text{or} \quad (\beta_j - \alpha_j) \nabla a(\theta_j)$$

where $\theta_j \in [\alpha_j + 1, \beta_j - 1]$. Since Δa is bounded (so is ∇a), in view of (3.22), it is clear from (3.23) that

$$(3.24) \quad \lim_{j \rightarrow \infty} (\beta_j - \alpha_j) = \infty.$$

Next, we observe that for each $j \geq 1$,

$$\Delta_r^0 y(a(\ell)) \geq \frac{d}{2}, \quad \ell \in [\alpha_j, \beta_j].$$

Consequently, using (C1) we find

$$y(a(\ell)) \geq \frac{d}{2r_0(a(\ell))} \geq \frac{d}{2 \sup_{\ell \geq \ell_0} r_0(\ell)} > 0, \quad \ell \in [\alpha_j, \beta_j].$$

Hence, $y(a(\ell))$ has a positive lower bound on the set $\bigcup_{j=1}^{\infty} [\alpha_j, \beta_j]$. So in view of (A3), there exists a positive constant M such that

$$F(y(a(\ell))) \geq M, \quad \ell \in \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j].$$

Then, it is clear that

$$(3.25) \quad \begin{aligned} \sum_{s=L}^{\infty} x(s+1)g^{\pm}(s)F(y(a(s))) &\geq \sum_{j=1}^{\infty} \sum_{s=\alpha_j}^{\beta_j} x(s+1)g^{\pm}(s)F(y(a(s))) \\ &\geq M \sum_{j=1}^{\infty} \sum_{s=\alpha_j}^{\beta_j} x(s+1)g^{\pm}(s). \end{aligned}$$

On the other hand, by (3.24) and (C9) we have

$$\sum_{j=1}^{\infty} \sum_{s=\alpha_j}^{\beta_j} x(s+1)g^{+}(s) = \infty \quad \text{or} \quad \sum_{j=1}^{\infty} \sum_{s=\alpha_j}^{\beta_j} x(s+1)g^{-}(s) = \infty.$$

Therefore, it follows from (3.25) that

$$(3.26) \quad \sum_{s=L}^{\infty} x(s+1)g^+(s)F(y(a(s))) = \infty \quad \text{or} \quad \sum_{s=L}^{\infty} x(s+1)g^-(s)F(y(a(s))) = \infty.$$

In view of (3.26) and (C4), we see from $(3.10)|_{m=1}$ that $\sum_{s=L}^{\infty} x(s+1)A_r^n y(s) = \infty$ or $-\infty$. By Lemma 2.2, this implies that $\lim_{\ell \rightarrow \infty} A_r^1 y(\ell) = \infty$ or $-\infty$ which is a contradiction to (3.13). Hence, we have shown (3.17) and this completes the proof of the theorem.

The next two results are for the difference equation (E_m) where g is nonnegative. They are discrete generalization of the results of Kusano and Onose [4, Theorem 3; 5, Theorem 3].

Theorem 3.4. *Let (C1) and (C6) hold. Further, suppose that*

$$(C10) \quad \sum_{r_i(\ell)}^{\infty} \frac{1}{r_i(\ell)} = \infty, \quad 1 \leq i \leq n-1;$$

$$(C11) \quad g(\ell) \geq 0, \quad \ell \geq \ell_0 \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} g(\ell) = \infty;$$

$$(C12) \quad \liminf_{\substack{u_i \rightarrow \infty \\ 1 \leq i \leq m}} F(u_1, \dots, u_m) > 0 \quad \text{and} \quad \limsup_{\substack{u_i \rightarrow -\infty \\ 1 \leq i \leq m}} F(u_1, \dots, u_m) < 0;$$

$$(C13) \quad \sum_{\ell=\ell_0}^{\infty} x(\ell+1)|h(\ell)| < \infty$$

where x is a function of type $r[0]$ given by

$$\begin{aligned} x(\ell) &= \rho_{0,n-1}(\ell) \quad (\text{see (2.3)}) \\ &= \sum_{\ell_{n-1}=\ell_0}^{\ell-1} \frac{1}{r_{n-1}(\ell_{n-1})} \sum_{\ell_{n-2}=\ell_0}^{\ell_{n-1}-1} \frac{1}{r_{n-2}(\ell_{n-2}+1)} \\ &\quad \times \sum_{\ell_{n-3}=\ell_0}^{\ell_{n-2}-1} \frac{1}{r_{n-3}(\ell_{n-3}+2)} \cdots \sum_{\ell_1=\ell_0}^{\ell_2-1} \frac{1}{r_1(\ell_1+n-2)}. \end{aligned}$$

Then, any nonoscillatory solution y of (E_m) is bounded and satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Let y be a nonoscillatory solution of (E_m) . Again y is assumed to be eventually positive and $y(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$. Let L be such that for $\ell \geq L$, $a_i(\ell) \geq \ell_1$ for every $1 \leq i \leq m$. It is clear from (E_m) that

$$\begin{aligned}
(3.27) \quad & \sum_{\ell=L}^{\tau} x(\ell+1) \Delta_r^n y(\ell) \\
&= \sum_{\ell=L}^{\tau} x(\ell+1) h(\ell) - \sum_{\ell=L}^{\tau} x(\ell+1) g(\ell) F(y(a_1(\ell)), \dots, y(a_m(\ell))), \\
&\ell \geq L.
\end{aligned}$$

In view of (A3) and (C13), from (3.27) we see that $\sum_{\ell=L}^{\infty} x(\ell+1) \Delta_r^n y(\ell)$ exists in $\mathbf{R}^*$. It follows from Lemma 2.2 that $\lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell)$ also exists in $\mathbf{R}^*$. First, we shall show that

$$(3.28) \quad \lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \lim_{\ell \rightarrow \infty} r_0(\ell) y(\ell) < \infty.$$

Suppose the contrary, i.e.,

$$(3.29) \quad \lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = \infty.$$

Then, taking note of (C1) it is obvious that

$$\lim_{\ell \rightarrow \infty} y(\ell) = \lim_{\ell \rightarrow \infty} \frac{\Delta_r^0 y(\ell)}{r_0(\ell)} = \infty.$$

Consequently, by (A2) and (C12), the function $F(y(a_1(\ell)), \dots, y(a_m(\ell)))$ has positive lower bound. This, together with (C11), yields

$$(3.30) \quad \sum_{\ell=L}^{\infty} g(\ell) F(y(a_1(\ell)), \dots, y(a_m(\ell))) = \infty.$$

Further, it is clear that (C13) leads to

$$(3.31) \quad \sum_{\ell=L}^{\infty} |h(\ell)| < \infty.$$

Now, noting that $\Delta_r^n y = \Delta_r(\Delta_r^{n-1} y)$, we sum (E_m) from L to $(\ell-1)$ to get

$$(3.32) \quad \Delta_r^{n-1} y(\ell) = \Delta_r^{n-1} y(L) - \sum_{s=L}^{\ell-1} g(s) F(y(a_1(s)), \dots, y(a_m(s))) + \sum_{s=L}^{\ell-1} h(s).$$

Letting $\ell \rightarrow \infty$ in (3.32) and applying (3.30) and (3.31), we obtain

$$(3.33) \quad \lim_{\ell \rightarrow \infty} \Delta_r^{n-1} y(\ell) = -\infty.$$

Using (3.33) and (C10), a similar argument as in the proof of Theorem 3.1 gives

$$\lim_{\ell \rightarrow \infty} \Delta_r^0 y(\ell) = -\infty$$

which is a contradiction to (3.29). Hence, we have proved (3.28).

Next, it follows from (3.28) and (C6) that

$$\lim_{\ell \rightarrow \infty} y(\ell) = \lim_{\ell \rightarrow \infty} \frac{A_r^0 y(\ell)}{r_0(\ell)} < \infty,$$

i.e., y is bounded. Finally, noting that (C11) implies (C5) and (C13) is (C4), we apply Theorem 3.2 to get $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Theorem 3.5. *Let (C1) and (C6) hold. Further, suppose that*

$$(C14) \quad \sum_{\ell_i(\ell)}^{\infty} \frac{1}{r_i(\ell)} < \infty, \quad 1 \leq i \leq n-1;$$

$$(C15) \quad \sum_{\ell}^{\infty} |h(\ell)| < \infty;$$

$$(C16) \quad g(\ell) \geq 0, \quad \ell \geq \ell_0 \quad \text{and} \quad \sum_{\ell}^{\infty} x(\ell+1)g(\ell) = \infty$$

where x is a function of type $r[0]$ given by

$$\begin{aligned} x(\ell) &= \sigma_0(\ell) \quad (\text{see (2.5)}) \\ &= \sum_{\ell_{n-1}=\ell}^{\infty} \frac{1}{r_{n-1}(\ell_{n-1})} \sum_{\ell_{n-2}=\ell_{n-1}}^{\infty} \frac{1}{r_{n-2}(\ell_{n-2}+1)} \\ &\quad \times \sum_{\ell_{n-3}=\ell_{n-2}}^{\infty} \frac{1}{r_{n-3}(\ell_{n-3}+2)} \cdots \sum_{\ell_1=\ell_2}^{\infty} \frac{1}{r_1(\ell_1+n-2)}. \end{aligned}$$

Then, any nonoscillatory solution y of (E_m) is bounded and satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Let y be a nonoscillatory solution of (E_m) . We shall assume that y is eventually positive and $y(\ell) > 0$ for $\ell \geq \ell_1 \geq \ell_0$. From (E_m) we have (3.32) which, in view of (A3) and (C15), leads to

$$(3.34) \quad A_r^{n-1} y(\ell) \leq M_{n-1}, \quad \ell \geq L$$

where M_{n-1} is some positive constant. Summing (3.34) from L to $(\ell-1)$ and taking into account (C14), we get

$$A_r^{n-2} y(\ell) \leq A_r^{n-2} y(L) + M_{n-1} \sum_{s=L}^{\ell-1} \frac{1}{r_{n-1}(s)} \leq M_{n-2}, \quad \ell \geq L$$

where once again M_{n-2} is some positive constant. By continuing the process, eventually we obtain $A_r^0 y (= r_0 y)$ is bounded. By (C6), this implies that y is also bounded. Finally, noting that (C15) and (C16) respectively leads to (C4) and (C5), we use Theorem 3.2 to get $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

4. Applications

To show the importance of the results obtained in Section 3, we shall investigate the asymptotic behavior of nonoscillatory solutions of the difference equation

$$(\mathbf{D}_{mN}) \quad \Delta^N[r(\ell)\Delta^{n-N}y(\ell)] \\ + g(\ell)F(y(a_1(\ell)), y(a_2(\ell)), \dots, y(a_m(\ell))) = h(\ell), \quad \ell \geq \ell_0$$

where $N \in [1, n-1]$ is fixed, $r: [\ell_0, \infty) \rightarrow \mathbf{R}^+$ and (A2), (A3) are satisfied.

We note that $(\mathbf{D}_{mN})$ is a special case of (E_m) when $r_j = 1$ for $j \neq n-N$ and $r_{n-N} = r$.

Corollary 4.1. *Suppose that*

$$(4.1) \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{r(\ell)} = \infty.$$

Further, there exists a real number p and an integer $k \in [0, N-1]$ such that for all $0 < \mu \leq 1$ and $v > 0$,

$$(4.2) \quad \sum_{\ell=\ell_0}^{\infty} (\ell+1-p)^{(k)} [\mu g^+(\ell) - g^-(\ell) - v|h(\ell)|] = \infty$$

or

$$\sum_{\ell=\ell_0}^{\infty} (\ell+1-p)^{(k)} [\mu g^-(\ell) - g^+(\ell) - v|h(\ell)|] = \infty.$$

Then, any bounded solution y of $(\mathbf{D}_{mN})$ satisfies $\liminf_{\ell \rightarrow \infty} |y(\ell)| = 0$.

Proof. Without loss of generality, we assume that $\ell_0 \geq p+k$. It can be verified easily that the function

$$x(\ell) = (\ell-p)^{(k)}$$

is of type $r[n-k-1]$. Noting that $1 \leq n-N \leq n-1$, we have $r_0 = 1$ and so (C1) is satisfied. Next, since $n-N \leq n-k-1 \leq n-1$, condition (C2) (with k replaced by $(n-k-1)$) reduces to (4.1). Finally, (4.2) is exactly (C3) with k replaced by $(n-k-1)$. Hence, the conclusion follows immediately from Theorem 3.1.

Corollary 4.2. *Suppose that there exists an integer $k \in [0, n-N-1]$ such that*

$$(4.3) \quad \sum_{\ell=\ell_0}^{\infty} \frac{\ell^{(n-N-1-k)}}{r(\ell+N-1)} = \infty.$$

Further, let (C3) be satisfied with

$$(4.4) \quad x(\ell) = \sum_{s=\ell_0}^{\ell-1} \frac{(\ell-s-1)^{(N-1)}(s-\ell_0)^{(n-N-1-k)}}{r(s+N-1)}.$$

Then, any bounded solution y of (D_{mN}) satisfies $\liminf_{\ell \rightarrow \infty} |y(\ell)| = 0$.

Proof. By direct computation, we have

$$x(\ell) = (N-1)!(n-N-1-k)!\rho_{k,n-1}(\ell)$$

where $\rho_{k,n-1}$ is given in (2.3). Using (4.3), it can be verified that the function x is of type $r[k]$. Moreover, it is noted in the proof of Corollary 4.1 that (C1) is fulfilled. Finally, since $k \leq n-N-1$, we observe that (C2) obviously holds. Hence, the conclusion is immediate from Theorem 3.1.

Corollary 4.3. Suppose that there exists an integer $k \in [0, n-N-1]$ such that

- (a) $\sum^{\infty} \frac{s^{(n-2-k)}}{r(s)} < \infty$ if $N=1$; $\sum^{\infty} \frac{s^{(n-N-1-k)}}{r(s+1)} < \infty$ if $N>1$;
- (b) if $N>1$, then $\lim_{\ell \rightarrow \infty} \eta_{kj}(\ell) \in \{0, \infty\}$ for every $1 \leq j \leq N-1$; and
- (c) condition (C3) is satisfied with

$$x(\ell) = \begin{cases} \theta_k(\ell), & N=1 \\ \eta_{k,N-1}(\ell), & N>1 \end{cases}$$

where θ_k and η_{kj} are respectively given in (2.7) and (2.9). Then, any bounded solution y of (D_{mN}) satisfies $\liminf_{\ell \rightarrow \infty} |y(\ell)| = 0$.

Proof. Clearly, the function x is of type $r[k]$. Moreover, as noted in the proof of Corollary 4.2, conditions (C1) and (C2) are satisfied. Therefore, the conclusion follows immediately from Theorem 3.1.

Corollary 4.4. Suppose that there exists a real number p such that

$$(4.5) \quad \sum^{\infty} (\ell+1-p)^{(n-1)} |h(\ell)| < \infty;$$

$$(4.6) \quad \sum^{\infty} (\ell+1-p)^{(n-1)} g^+(\ell) = \infty \quad \text{and} \quad \sum^{\infty} (\ell+1-p)^{(n-1)} g^-(\ell) < \infty$$

or

$$\sum^{\infty} (\ell+1-p)^{(n-1)} g^+(\ell) < \infty \quad \text{and} \quad \sum^{\infty} (\ell+1-p)^{(n-1)} g^-(\ell) = \infty.$$

Then, any bounded nonoscillatory solution y of (D_{mN}) satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Since $r_0 = 1$, conditions (C1) and (C6) are fulfilled. As in the proof of Corollary 4.1, it is noted that the function

$$x(\ell) = (\ell - p)^{(n-1)}$$

is of type $r[0]$. Thus, (4.5) and (4.6) correspond to (C4) and (C5) respectively. We use Theorem 3.2 to get the conclusion.

Corollary 4.5. *Suppose that*

$$(4.7) \quad \sum_{\ell}^{\infty} \frac{\ell^{(n-N-1)}}{r(\ell + N - 1)} = \infty.$$

Further, let (C4) and (C5) be satisfied with

$$(4.8) \quad x(\ell) = \sum_{s=\ell_0}^{\ell-1} \frac{(\ell - s - 1)^{(N-1)}(s - \ell_0)^{(n-N-1)}}{r(s + N - 1)}.$$

Then, any bounded nonoscillatory solution y of (D_{mN}) satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Once again conditions (C1) and (C6) are satisfied. Next, we observe that

$$x(\ell) = (N - 1)!(n - N - 1)!\rho_{0,n-1}(\ell)$$

where $\rho_{0,n-1}$ is given in (2.3), and also (4.7) ensures that x is of type $r[0]$. Theorem 3.2 is then used to get the conclusion.

Corollary 4.6. *Suppose that*

- (a) $\sum^{\infty} \frac{s^{(n-2)}}{r(s)} < \infty$ if $N = 1$; $\sum^{\infty} \frac{s^{(n-N-1)}}{r(s+1)} < \infty$ if $N > 1$;
- (b) if $N > 1$, then $\lim_{\ell \rightarrow \infty} \eta_{0j}(\ell) \in \{0, \infty\}$ for every $1 \leq j \leq N - 1$; and
- (c) conditions (C4) and (C5) are satisfied with

$$x(\ell) = \begin{cases} \theta_0(\ell), & N = 1 \\ \eta_{0,N-1}(\ell), & N > 1 \end{cases}$$

where θ_0 and η_{0j} are respectively given in (2.7) and (2.9). Then, any bounded nonoscillatory solution y of (D_{mN}) satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. Since $r_0 = 1$, conditions (C1) and (C6) are satisfied. Next, noting that the function x is of type $r[0]$, the conclusion is clear from Theorem 3.2.

Corollary 4.7. *Consider the equation $(D_{1,n-1})$ and denote $a_1 = a$. Let*

$$(4.9) \quad \sum_{\ell}^{\infty} \frac{1}{r(\ell)} = \infty,$$

$$(4.10) \quad \liminf_{\ell \rightarrow \infty} r(\ell) > 0$$

and

$$(4.11) \quad \sum_{\ell}^{\infty} \ell^{(n-2)} |h(\ell)| < \infty.$$

Suppose that for some positive integer δ ,

$$(4.12) \quad \liminf_{\ell \rightarrow \infty} \sum_{s=\ell}^{\ell+\delta} s^{(n-2)} g^+(s) > 0 \quad \text{and} \quad \sum_{s=\ell}^{\infty} s^{(n-2)} g^-(s) < \infty$$

or

$$\liminf_{\ell \rightarrow \infty} \sum_{s=\ell}^{\ell+\delta} s^{(n-2)} g^-(s) > 0 \quad \text{and} \quad \sum_{s=\ell}^{\infty} s^{(n-2)} g^+(s) < \infty.$$

Further, assume that a is nondecreasing and Δa is bounded on $[\ell_0, \infty)$. Then, any bounded nonoscillatory solution y of $(D_{1,n-1})$ satisfies

$$\lim_{\ell \rightarrow \infty} r(\ell) \Delta y(\ell) = \lim_{\ell \rightarrow \infty} \Delta y(\ell) = \lim_{\ell \rightarrow \infty} y(\ell) = 0.$$

Proof. Here, $r_0 = 1$ and $r_1 = r$. Thus, (C1), (C7) (= (4.9)) and (C8) (= (4.10)) hold. Next, we note that the function

$$x(\ell) = (\ell - \ell_0)^{(n-2)}$$

is of type $r[1]$. Subsequently, (4.11) and (4.12) respectively imply that (C4) and (C9) are satisfied for this x . The conclusion is now immediate from Theorem 3.3.

Corollary 4.8. Consider the equation $(D_{1,N})$ where $N \leq n-2$, and denote $a_1 = a$. Suppose that

$$(4.13) \quad \sum_{\ell}^{\infty} \frac{\ell^{(n-N-2)}}{r(\ell + N - 1)} = \infty.$$

Let (C4) and (C9) be satisfied with

$$(4.14) \quad x(\ell) = \sum_{s=\ell_0}^{\ell-1} \frac{(\ell - s - 1)^{(N-1)} (s - \ell_0)^{(n-N-2)}}{r(s + N - 1)}.$$

Further, assume that a is nondecreasing and Δa is bounded on $[\ell_0, \infty)$. Then, any bounded nonoscillatory solution y of $(D_{1,N})$ satisfies

$$\lim_{\ell \rightarrow \infty} \Delta y(\ell) = \lim_{\ell \rightarrow \infty} y(\ell) = 0.$$

Proof. Since $2 \leq n - N \leq n - 1$, it is clear that $r_0 - r_1 = 1$. Therefore, (C1), (C7) and (C8) are fulfilled. Next, it is noted that

$$x(\ell) = (N-1)!(n-N-2)!\rho_{1,n-1}(\ell)$$

where $\rho_{1,n-1}$ is given in (2.3). Using (4.13), it can be verified that the function x is of type $r[1]$. Hence, the conclusion is immediate from Theorem 3.3.

Corollary 4.9. *Consider the equation $(D_{1,N})$ where $N \leq n-2$, and denote $a_1 = a$. Suppose that*

- (a) $\sum^{\infty} \frac{s^{(n-3)}}{r(s)} < \infty$ if $N = 1$; $\sum^{\infty} \frac{s^{(n-N-2)}}{r(s+1)} < \infty$ if $N > 1$;
- (b) if $N > 1$, then $\lim_{\ell \rightarrow \infty} \eta_{1j}(\ell) \in \{0, \infty\}$ for every $1 \leq j \leq N-1$; and
- (c) conditions (C4) and (C9) are satisfied with

$$x(\ell) = \begin{cases} \theta_1(\ell), & N = 1 \\ \eta_{1,N-1}(\ell), & N > 1 \end{cases}$$

where θ_1 and η_{1j} are respectively given in (2.7) and (2.9). Further, assume that a is nondecreasing and Δa is bounded on $[\ell_0, \infty)$. Then, any bounded nonoscillatory solution y of $(D_{1,N})$ satisfies

$$\lim_{\ell \rightarrow \infty} \Delta y(\ell) = \lim_{\ell \rightarrow \infty} y(\ell) = 0.$$

Proof. As noted in the proof of Corollary 4.8, $r_0 = r_1 = 1$ and so (C1), (C7) and (C8) are satisfied. Moreover, the function x is of type $r[1]$. Thus, the conclusion follows immediately from Theorem 3.3.

Corollary 4.10. *Suppose that (4.1), (4.7), (C11) and (C12) hold. Further, let (C13) be satisfied with x defined in (4.8). Then, any nonoscillatory solution y of (D_{mN}) is bounded and satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.*

Proof. Once again conditions (C1) and (C6) are satisfied. Next, since $1 \leq n-N \leq n-1$, condition (C10) reduces to (4.1). Noting that the function x given in (4.8) is of type $r[0]$, we apply Theorem 3.4 to get the conclusion.

Corollary 4.11. *Consider the equation (D_{m1}) with $n-2$, i.e.,*

$$(4.15) \quad \Delta[r(\ell)\Delta y(\ell)] + g(\ell)F(y(a_1(\ell)), y(a_2(\ell)), \dots, y(a_m(\ell))) = h(\ell), \quad \ell \geq \ell_0.$$

Let (C15) hold and let (C16) be satisfied with

$$(4.16) \quad x(\ell) = \sum_{s=\ell}^{\infty} \frac{1}{r(s)}.$$

Further, suppose that

$$(4.17) \quad \sum_{\ell}^{\infty} \frac{1}{r(\ell)} < \infty.$$

Then, any nonoscillatory solution y of (4.15) is bounded and satisfies $\lim_{\ell \rightarrow \infty} y(\ell) = 0$.

Proof. It is obvious that $r_0 = 1$ and $r_1 = r$. Thus, (C1) and (C6) are fulfilled, and (C14) reduces to (4.17). Noting that here $x = \sigma_0$, the conclusion is immediate from Theorem 3.5.

5. Examples

The first four examples illustrate Corollaries 4.1–4.11 and the last example is a counter example.

Example 5.1. Consider the difference equation

$$\begin{aligned} (D_{1,3}) \quad \Delta^3[(\ell+2)\Delta y(\ell)] + \frac{1}{(\ell+1)(\ell+2)(\ell+3)} y(\ell+3) \\ = \frac{7}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)}, \quad \ell \geq 0. \end{aligned}$$

Here, $n = 4$, $N = 3$, $m = 1$, $\ell_0 = 0$, $r(\ell) = \ell + 2$, $g(\ell) = [(\ell+1)(\ell+2)(\ell+3)]^{-1}$, $h(\ell) = 7[(\ell+1)(\ell+2)(\ell+3)(\ell+4)]^{-1}$, $a(\ell) = \ell + 3$ and $F(u) = u$. Obviously, the hypothesis (A2) and (A3) are satisfied.

It can be easily verified that all the conditions of Corollaries 4.1 (take $k = 2$) and 4.7 are fulfilled. However, Corollaries 4.4 and 4.10 *cannot* be used as conditions (4.5) and (C11) are not met. Moreover, Corollaries 4.3 and 4.6 and also *not* applicable as their condition (a) is not satisfied. Consequently, we conclude from Corollaries 4.1 and 4.7 that

- (a) if y_1 is a bounded solution of $(D_{1,3})$, then $\liminf_{\ell \rightarrow \infty} |y_1(\ell)| = 0$;
- (b) if y_2 is a bounded nonoscillatory solution of $(D_{1,3})$, then $\lim_{\ell \rightarrow \infty} (\ell+2)\Delta y_2(\ell) = \lim_{\ell \rightarrow \infty} \Delta y_2(\ell) = \lim_{\ell \rightarrow \infty} y_2(\ell) = 0$.

In fact, a bounded as well as nonoscillatory solution is given by $y(\ell) = (\ell+1)^{-1}$ and we note that it does satisfy the above conclusions.

Example 5.2. Consider the difference equation

$$(D_{1,1}) \quad \Delta[\ell \Delta^2 y(\ell)] + 4y(\ell+3)^3 = -\frac{4(3\ell+7)}{(\ell+1)(\ell+2)(\ell+3)^3}, \quad \ell \geq 1.$$

Here, $n = 3$, $N = 1$, $m = 1$, $\ell_0 = 1$, $r(\ell) = \ell$, $g(\ell) = 4$, $h(\ell) = -4(3\ell+7) \cdot [(\ell+1)(\ell+2)(\ell+3)^3]^{-1}$, $a(\ell) = \ell + 3$ and $F(u) = u^3$. Once again, the hypothesis (A2) and (A3) are satisfied.

We note that Corollaries 4.3, 4.6 and 4.9 are *not* applicable here as their condition (a) is violated. However, it can be easily checked that all the conditions of Corollary 4.8 are fulfilled. In fact, from (4.14) we have $x(\ell+1) = \sum_{s=1}^{\ell} 1/s$ and (C4) is satisfied as

$$\begin{aligned} \sum_{\ell=0}^{\infty} x(\ell+1)|h(\ell)| &= \sum_{\ell=0}^{\infty} x(\ell+1) \frac{4(3\ell+7)}{(\ell+1)(\ell+2)(\ell+3)^3} \\ &< \sum_{\ell=0}^{\infty} x(\ell+1) \frac{28(\ell+1)}{(\ell+1)^5} < \sum_{\ell=0}^{\infty} \left(\sum_{s=1}^{\ell} 1 \right) \frac{28}{(\ell+1)^4} < \infty. \end{aligned}$$

It can also be verified that the conditions of Corollaries 4.1, 4.2 (take $k = 0$), 4.4, 4.5 and 4.10 are met. Hence, we conclude that

- (a) if y_1 is a bounded solution of $(D_{1,1})$, then $\liminf_{\ell \rightarrow \infty} |y_1(\ell)| = 0$;
- (b) if y_2 is a bounded nonoscillatory solution of $(D_{1,1})$, then $\lim_{\ell \rightarrow \infty} \Delta y_2(\ell) = \lim_{\ell \rightarrow \infty} y_2(\ell) = 0$;
- (c) if y_3 is a nonoscillatory solution of $(D_{1,1})$, then it is bounded and satisfies $\lim_{\ell \rightarrow \infty} y_3(\ell) = 0$.

It is noted that a bounded as well as nonoscillatory solution is given by $y(\ell) = \ell^{-1}$ and indeed it does fulfill the above conclusions.

Example 5.3. Consider the difference equation

$$\begin{aligned} (D_{5,1}) \quad & \Delta[\ell(\ell+1)\Delta y(\ell)] + \ell(\ell+1)(\ell+2)y(\ell)y(\ell+1)y(\ell+2)y(\ell+3)y(\ell+4) \\ &= \frac{1}{(\ell+3)(\ell+4)}, \quad \ell \geq 1. \end{aligned}$$

Here, $n = 2$, $N = 1$, $m = 5$, $\ell_0 = 1$, $r(\ell) = \ell(\ell+1)$, $g(\ell) = \ell(\ell+1)(\ell+2)$, $h(\ell) = [(\ell+3)(\ell+4)]^{-1}$, $a_i(\ell) = \ell + i - 1$, $1 \leq i \leq 5$ and $F(u_1, \dots, u_5) = \prod_{i=1}^5 u_i$. Once again, the hypothesis (A2) and (A3) are satisfied.

It can easily be verified that the conditions of Corollaries 4.3, 4.6 and 4.11 are fulfilled. On the other hand, conditions (4.1), (4.3), (4.5) and (4.7) are not met. Thus, we *cannot* employ Corollaries 4.1, 4.2, 4.4, 4.5 and 4.10 here. By Corollaries 4.3, 4.6 and 4.11, we conclude that

- (a) if y_1 is a bounded solution of $(D_{5,1})$, then $\liminf_{\ell \rightarrow \infty} |y_1(\ell)| = 0$;
- (b) if y_2 is a bounded nonoscillatory solution of $(D_{5,1})$, then $\lim_{\ell \rightarrow \infty} y_2(\ell) = 0$;
- (c) if y_3 is a nonoscillatory solution of $(D_{5,1})$, then it is bounded and satisfies $\lim_{\ell \rightarrow \infty} y_3(\ell) = 0$.

In fact, a bounded as well as nonoscillatory solution is given by $y(\ell) = \ell^{-1}$ and we note that it does satisfy the above conclusions.

Example 5.4. Consider the difference equation

$$\begin{aligned} (D_{1,1}) \quad & \Delta[(\ell+1)(\ell+2)\Delta^2 y(\ell)] + (\ell+2)(\ell+3)^2 \frac{1}{y(\ell+3)^5} \\ &= -\frac{\ell^3 + 15\ell^2 + 52\ell + 54}{\ell(\ell+1)(\ell+3)^3}, \quad \ell \geq 1. \end{aligned}$$

Here, $n = 3$, $N = 1$, $m = 1$, $\ell_0 = 1$, $r(\ell) = (\ell + 1)(\ell + 2)$, $g(\ell) = (\ell + 2)(\ell + 3)^2$, $h(\ell) = -(\ell^3 + 15\ell^2 + 52\ell + 54)[\ell(\ell + 1)(\ell + 3)^3]^{-1}$, $a(\ell) = \ell + 3$ and $F(u) = u^{-5}$. Clearly, the hypothesis (A2) and (A3) are satisfied.

It can be checked that the conditions of Corollaries 4.3 (take $k = 1$) and 4.9 are met. However, conditions (4.1), (4.5), (4.13) and condition (a) of Corollary 4.6 are violated. As such, we *cannot* apply Corollaries 4.1, 4.4, 4.6, 4.8 and 4.10 here. By Corollaries 4.3 and 4.9, we conclude that

- (a) if y_1 is a bounded solution of $(D_{1,1})$, then $\liminf_{\ell \rightarrow \infty} |y_1(\ell)| = 0$;
- (b) if y_2 is a bounded nonoscillatory solution of $(D_{1,1})$, then $\lim_{\ell \rightarrow \infty} \Delta y_2(\ell) = \lim_{\ell \rightarrow \infty} y_2(\ell) = 0$.

Indeed, a bounded as well as nonoscillatory solution is given by $y(\ell) = \ell^{-1}$ and we note that it does fulfill the above conclusions.

Example 5.5. Consider the difference equation

$$\begin{aligned} (D_{2,1}) \quad & \Delta[\ell(\ell + 1)\Delta y(\ell)] + (\ell + 2)(\ell + 3)(\ell + 4)y(\ell + 1)y(\ell + 2)^2 \\ & = (\ell + 1)[2 + (\ell + 2)^3(\ell + 3)(\ell + 4)], \quad \ell \geq 0. \end{aligned}$$

Here, $n = 2$, $N = 1$, $m = 2$, $\ell_0 = 0$, $r(\ell) = \ell(\ell + 1)$, $g(\ell) = (\ell + 2)(\ell + 3)(\ell + 4)$, $h(\ell) = (\ell + 1)[2 + (\ell + 2)^3(\ell + 3)(\ell + 4)]$, $a_i(\ell) = \ell + i$, $i = 1, 2$ and $F(u_1, u_2) = u_1 u_2^2$. Obviously, the hypothesis (A2) and (A3) are satisfied.

It can be checked that (C11) and (C12) are fulfilled whereas (4.1), (4.7) and (C13) (with x defined in (4.8)) are not satisfied. Hence, the conditions of Corollary 4.10 are *violated*. Indeed, $(D_{2,1})$ has a nonoscillatory solution given by $y(\ell) = \ell$ and contrary to the conclusion of Corollary 4.10, this solution is not bounded and $\lim_{\ell \rightarrow \infty} y(\ell) \neq 0$.

Moreover, (C16) (with x defined in (4.16)) and (4.17) are satisfied but (C15) is not met. Therefore, the conditions of Corollary 4.11 are *violated*. As noted above, $(D_{2,1})$ has a nonoscillatory solution that contradicts the conclusion of Corollary 4.11.

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