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The Nonlinear Boundary Value Problem for the Equation of Mixed Type

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1. Introduction

We consider in the cylinder $Q = D \times (0, T) : t \in (0, T), x \in D \subset R^n$, the linear equation of mixed type

$$(1.1) \quad K(x, t)u_{tt} - \Delta u + \alpha u_t = f(x, t)$$

arising in the asymptotic theory of transonic flows, [1]. A smooth function $K(x, t)$ can change its sign or be equal to zero on arbitrary sets of points in Q ;

$K(x, 0) < 0, K(x, T) \geq 0$; α is a positive constant, $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$. Assumptions $K(x, 0) < 0, K(x, T) \geq 0$ imply that (1.1) is elliptic at $t = 0$ and hyperbolic-parabolic at $t = T$.

It is known, [2–5], that a type of boundary conditions on surfaces $t = 0, t = T$ depends on a sign of $K(x, t)$. In our case

$$(1.2) \quad u(x, 0) = 0,$$

$$(1.3) \quad \frac{\partial u}{\partial \nu} + F(x, t, u_t)|_{\partial D \times (0, T)} = 0,$$

where $\partial/\partial \nu$ denotes the derivative in the direction of an outward normal on ∂D . We assume that D is a bounded domain with a sufficiently smooth boundary ∂D .

The problem (1.1)–(1.3) models transonic flows in a channel with perforated walls under the nonlinear Darcy's law when a velocity of a gas changes from subsonic values at $t = 0$ to supersonic ones at $t = T$, [5].

A big number of papers is devoted to solvability of linear boundary value problems for (1.1), see [2–6] and their references. As a rule, the Dirichlet condition or the Neumann condition was given at the lateral surface $S = \partial D \times (0, T)$. The Neumann condition corresponds to impermeability of a solid wall of a channel. On the contrary, the condition (1.3) allows gas to penetrate through perforated walls. Earlier, [7], we have considered the following

problem which models a viscous transonic flow in perforated channels:

$$\begin{aligned} -\mu u_{xxx} + u_x u_{xx} - u_{yy} &= 0, \\ u|_{x=0} &= u_0(y), \quad u_x|_{x=0} = u_1(y), \quad u_{xx}|_{x=T} = 0; \\ u_y + u_x|_{y=1} &= 0, \quad u_y|_{y=0} = \phi(x). \end{aligned}$$

Existence and uniqueness of regular solutions to this problem were proved provided appropriate norms of u_0 , u_1 , ϕ were sufficiently small.

In the present paper we establish the existence and uniqueness results for (1.1)–(1.3) in a class of generalized solutions. To prove existence of solutions, we use an analogue of the Faedo-Galerkin method and regularization by highest derivatives.

In Faedo-Galerkin's procedure, basis functions depend on x only. In order to find unknown coefficients depending on t , one has to solve globally on $(0, T)$ a nonlinear system of ordinary differential equations which in our case cannot be transformed into the normal system on the whole interval $(0, T)$. This fact does not permit to use directly standard existence theorems of ODE. On the other hand, regularization by a system of a higher order with additional boundary conditions allows to use Galerkin's method and to prove existence results for the regularized system. Passing to the limit on a parameter of regularization, we get an approximate solution (Faedo-Galerkin's approximations) to (1.1)–(1.3). Having sufficient a priori estimates, it is possible to prove convergence of approximate solutions to the exact one. We used the similar technique in [8].

Our paper has the following structure.

In section 2, we give principal notations and assumptions.

In section 3, Faedo-Galerkin's approximations are constructed. Here we use regularization of the system of nonlinear ordinary differential equations of the second order by derivatives of the third order.

In section 4, a priori estimates of approximate solutions are proven.

In section 5, we prove solvability of the problem (1.1)–(1.3).

2. Notations and assumptions

We introduce the following notations:

$$\begin{aligned} (u, v)(t) &= \int_D u(x, t) v(x, t) dx, \quad |u(t)|^2 = (u, u)(t); \\ \|u\|^2 &= \int_0^T |u(t)|^2 dt, \quad S = \partial D \times (0, T), \quad \Gamma = \partial D. \end{aligned}$$

Through $L^p(D)$, $H^1(D)$, $H^1(0, T; L^2(D))$, we denote analogously to [9] the Lebesgue spaces of p -integrable in D functions and the Sobolev spaces.

Assumptions.

A1). $F(x, t, u_t) \in C^1(\bar{S} \times R^1)$.

For all $(x, t) \in \bar{S}$ the following inequalities hold

A2). $|F(x, t, u_t)| + |F_t(x, t, u_t)| \leq C(1 + |u_t|^{\rho+1})$.

A3). $F(x, t, u_t)u_t \geq C_0|u_t|^{\rho+2}$.

A4). $\int_0^{u_t} F(x, t, s)ds \geq 0$.

A5). $(\partial F / \partial u_t)(x, t, u_t) \geq C_0(1 + |u_t|^\rho)$,

where C , C_0 are positive constants; ρ is a nonnegative number,

A6). The function $K(x, t) \in C^1(\bar{Q})$ satisfies the following inequalities:

$$2\alpha - |K_t(x, t)| > 0 \quad \text{in } \bar{Q},$$

A7). $K(x, 0) < 0$,

A8). $K(x, T) \geq 0$, $x \in D$.

3. Construction of approximate solutions

Definition 3.1. A function $u(x, t) : u \in H^1(Q)$, $u(x, 0) = 0$; $u_t \in L^{\rho+2}(S) \cap H^1(Q)$, is called a generalized solution to (1.1)–(1.3) if for a.e. $t \in (0, T)$ and for an arbitrary $v \in L^{\rho+2}(\Gamma) \cap H^1(D)$ the following identity holds

$$(3.1) \quad (Ku_t, v)(t) + (\nabla u, \nabla v)(t) + \int_{\Gamma} F(x, t, u_t)vd\Gamma + (\alpha u_t, v)(t) = (f, v)(t).$$

An approximate solution $u^N(x, t)$ to (1.1)–(1.3) will be sought in the form

$$u^N(x, t) = \sum_{j=1}^N g_{jN}(t)w_j(x).$$

Here $\{w_j(x)\}$ is a basis in $H^1(D) \cap L^{\rho+2}(\Gamma)$, orthonormal in $L^2(D)$.

Unknown functions $g_{jN}(t)$ are solutions to the following boundary value problem for the system of N ordinary differential equations

$$(3.2) \quad (Ku_t^N, w_j) + (\nabla u^N, \nabla w_j)(t) + (\alpha u_t^N, w_j)(t) + \int_{\Gamma} (Fx, t, u_t^N)w_j d\Gamma = (f, w_j)(t),$$

$$(3.3) \quad u^N(x, 0) \equiv g_{jN}(0) = 0, \quad j = 1, \dots, N.$$

The system (3.2) is not a standard ODE system, respectively, (3.2), (3.3) is not the classical Cauchy problem, since at $t = 0$ only one boundary condition is posed for the system of the second order. It is a consequence of the fact that,

due to Assumptions A7, A8, the function $K(x, t)$ at least once changes its sign in Q .

It means that (3.2) cannot be rewritten as a normal ODE system simultaneously on the whole interval $(0, T)$, and that (3.2), (3.3) is the nonlocal boundary value problem. Solutions to this problem have to be constructed globally on $(0, T)$.

Regularized problem. In order to construct $g_{jN}(t)$, we will find at first solutions to the following regularized problem

$$(3.4) \quad \begin{aligned} L_\mu g_{j\mu} &= -\mu g_{\mu j\mu\mu} + (Ku_{\mu\mu}^N, w_j)(t) + (\nabla u_\mu^N, \nabla w_j)(t) \\ &\quad + \alpha(u_{\mu\mu}^{NM}, w_j)(t) + \int_\Gamma F(x, t, u_{\mu\mu}^N) w_j d\Gamma \\ &= (f, w_j)(t), \end{aligned}$$

$$(3.5) \quad g_{\mu j}(0) = 0, \quad g_{\mu j\mu}(0) = g_{\mu j\mu}(T) = 0, \quad j = 1, \dots, N,$$

where μ is a positive number.

In turn, we find $g_{\mu j}(t)$ by Galerkin's method as follows:

$$(3.6) \quad \begin{aligned} g_{\mu j}^M &= \sum_{i=1}^M \beta_{ij} Z_i(t), \quad g_{\mu j}^M(0) = 0, \\ Z_{i\mu\mu} + v_i Z_i &= 0; \quad v_1 = 0, \quad v_i > 0, \quad i = 2, 3, \dots; \\ Z_{i\mu}(0) &= Z_{i\mu}(T) = 0, \quad i = 2, \dots; Z_1 = \frac{1}{\sqrt{T}}, \\ \int_0^T Z_i(t) Z_l(t) dt &= \delta_{il}; \quad l, i = 1, \dots, M; \quad j = 1, \dots, N. \end{aligned}$$

Taking into account (3.6), $u^{NM}(x, t)$ has the following form:

$$(3.7) \quad u^{NM}(x, t) = \int_0^t \sum_{j=1}^N \sum_{i=1}^M \beta_{ij} Z_i(t) w_j(x) dt,$$

where unknown constants β_{ij} are defined as solutions of the following algebraic system of NM nonlinear equations:

$$(3.8) \quad \begin{aligned} -\mu \int_0^T \sum_{i=1}^M \beta_{ij} Z_{i\mu\mu} Z_l dt + \int_0^T ((Ku_{\mu\mu}^{NM}, w_j)(t) + \alpha(u_{\mu\mu}^{NM}, w_j)(t)) Z_l dt \\ + \int_0^T \left\{ (\nabla u_\mu^{NM}, \nabla w_j)(t) + \int_\Gamma F(t, u_{\mu\mu}^{NM}) w_j d\Gamma \right\} Z_l dt \\ = \int_0^T (f, w_j)(t) Z_l dt, \quad j = 1, \dots, N, l = 1, \dots, M. \end{aligned}$$

To establish solvability of (3.8) with respect to β_{ij} , we employ the following lemma, [10], which is also known as Sharp Angle Lemma.

SHARP ANGLE Lemma. *The nonlinear system of equations*

$$A_j(C_1, \dots, C_k) = h_j, \quad j = 1, \dots, k$$

has a solution for each $h = (h_1, \dots, h_k)$ if it satisfies the conditions

(i) *the functions*

$$A_j(C) = A_j(C_1, \dots, C_k) : \mathbf{R}^k \rightarrow \mathbf{R}^k$$

are continuous:

(ii) *there exist constants $a_0 > 0$ and $a_1 > 0$ such that the inequality*

$$(A(C), C) = \sum_{j=1}^k A_j(C) C_j \geq a_0 |C|^{1+\varepsilon} - a_1$$

holds for some $\varepsilon > 0$.

It is clear that the crucial condition of this lemma is an a priori estimate of solutions.

3.2. Solvability of (3.8). First, we will prove two lemmas.

Lemma 3.1. *Assumptions A6, A7 imply the following inequalities*

$$(3.9) \quad 2\alpha - |K_t(x, t)| \geq 2\delta > 0 \quad \text{in } \bar{Q},$$

$$(3.10) \quad K(x, 0) \leq -2\eta < 0 \quad \text{in } \bar{D},$$

where δ, η are positive numbers.

Proof. Since $\bar{D}$ and $\bar{Q}$ are compacts, inequalities (3.9), (3.10) are consequences of Assumptions A6, A7 and the Heine-Borel Lemma.

Lemma 3.2. *There exists a positive number $T_0 > 0$ such that in $\bar{Q}_0 = \bar{D} \times [0, T_0]$ the inequality holds*

$$(3.11) \quad K(x, t) \leq -\eta < 0 \quad \text{in } \bar{Q}_0.$$

Proof. The inequality (3.11) is a consequence of (3.10) and continuity of $K(x, t)$ in $\bar{Q}$.

3.3 The estimate 1. Multiplying the jl -th equation of (3.8) by β_{jl} and summing over l and j , we come, taking into account (3.6), (3.7), to the equality

$$\begin{aligned}
(3.12) \quad & 2\mu \|u_{\mu t}^{NM}\|^2 + \int_Q (2\alpha - K_t)(u_{\mu t}^{NM})^2 dQ + \int_D |\nabla u_{\mu}^{NM}(x, T)|^2 dx \\
& + 2 \int_0^T \int_{\Gamma} F(x, t, u_{\mu t}^{NM}) u_{\mu t}^{NM} d\Gamma dt \\
& = 2 \int_Q f u_{\mu t}^{NM} dQ.
\end{aligned}$$

Using (3.9)–(3.11) and Assumption A8, we reduce (3.12) to the inequality

$$(3.13) \quad \mu \|u_{\mu t}^{NM}\|^2 + \delta \|u_{\mu t}^{NM}\|^2 + C_0 \int_0^T \int_{\Gamma} |u_{\mu t}^{NM}|^{\rho+2} d\Gamma dt \leq C \|f\|^2,$$

where C , C_0 do not depend on μ , N , M . Making use of (3.6) and orthogonality of $\{w_j\}$, $\{Z_i\}$, we have from (3.13)

$$\|u_{\mu t}^{NM}\|^2 = \sum_{j=1}^N \sum_{i=1}^M \beta_{ji}^2 \leq C \|f\|^2.$$

This estimate, (3.12) and (3.13) provide the inequality of the Sharp Angle Lemma with $\varepsilon = 1$ that in turn guarantees solvability of (3.8) with respect to β_{ji} . It means that we have established the existence result in the following

Lemma 3.3. *For all finite integers N , M , there exists a unique function $u_{\mu}^{NM}(x, t)$ having the form (3.7) and satisfying the identity (3.8).*

Proof of uniqueness. Let u_{μ}^{NM} and v_{μ}^{NM} be two distinct solutions to (3.8), then their difference $S = u_{\mu}^{NM} - v_{\mu}^{NM}$ satisfies the identity:

$$\begin{aligned}
& 2\mu \|S_t\|^2 + \int_Q (2\alpha - K_t) S_t^2 dQ + \int_D |\nabla S(x, T)|^2 dx \\
& + 2 \int_0^T \int_{\Gamma} [F(u_{\mu t}^{NM}) - F(v_{\mu t}^{NM})] (u_{\mu t}^{NM} - v_{\mu t}^{NM}) d\Gamma dt \\
& = 0.
\end{aligned}$$

Due to Assumption A5, $F(u_t)$ is a monotonic function. Taking into account this fact, Assumptions A7, A8 and (3.6), we obtain

$$\|S_t\| = 0.$$

Since $S(x, 0) = 0$, then $S(x, t) \equiv 0$ in Q .

This completes the proof of Lemma 3.3. $\square$

Next, we have to pass to the limits as $\mu \rightarrow 0$, $M \rightarrow \infty$. To do this, we must prove additional a priori estimates uniformly in μ , N .

3.4. The estimate 2. Multiplying the jl -th equation of (3.8) by v_l and using properties of our special basis (3.6), we obtain

$$\begin{aligned} \int_0^T \left\{ \mu Z_{llt} \partial_t^3 g_{\mu j}^M - \left[(Ku_{\mu t}^{NM}, w_j)(t) - (\alpha u_{\mu t}^{NM}, w_j)(t) - (\nabla u_{\mu}^{NM}, \nabla w_j)(t) \right. \right. \\ \left. \left. - \int_{\Gamma} F(t, u_{\mu t}^{NM}) w_j d\Gamma \right] Z_{llt} \right\} dt = - \int_0^T (f, w_j)(t) Z_{llt} dt, \\ j = 1, \dots, N, \quad l = 1, \dots, M. \end{aligned}$$

Multiplying the corresponding equation by β_{ji} , summing at first over l then over j ; after integration by parts we come, suppressing indices μ, N, M , to the equality

$$\begin{aligned} (3.14) \quad & \int_Q \{ 2\mu u_{ttt}^2 + (2\alpha + K_t) u_{tt}^2 \} dQ + 2 \int_Q \nabla u_t \nabla u_{tt} dQ \\ & + 2 \int_0^T \int_{\Gamma} \left\{ F_t(t, u_t) u_{tt} + \frac{\partial F}{\partial u_t}(t, u_t) u_{tt}^2 \right\} d\Gamma dt \\ & = 2 \int_Q f_t u_{tt} dQ. \end{aligned}$$

Taking into account (3.6), (3.7), we consider the integral

$$\begin{aligned} I_1 &= \int_Q \nabla u_t^{NM} \nabla u_{tt}^{NM} dx dt \\ &= \sum_{j,l=1}^N \int_Q \sum_{i=1}^M \beta_{ji} Z_i(t) \nabla w_j(x) \sum_{k=1}^M \beta_{jk} Z_{kt}(t) \nabla w_l(x) dt dx \\ &= \sum_{j,l=1}^N \int_D \nabla w_j(x) \nabla w_l(x) dx \int_0^T \sum_{i=1}^M \beta_{ji} Z_i(t) \sum_{k=1}^M Z_{kt}(t) \beta_{lk} dt. \end{aligned}$$

Let us denote

$$a_{jl} = \int_D \nabla w_j(x) \nabla w_l(x) dx,$$

$$A(N) = \max_{j,l \leq N} |a_{jl}|,$$

where $A(N)$ depends only on N .

Using this and (3.7), we estimate I_1 as follows

$$\begin{aligned} |I_1| &\leq A(N) \int_0^T \left\| \sum_{j=1}^N \sum_{i=1}^M \beta_{ji} Z_i(t) \right\| \left\| \sum_{l=1}^N \sum_{k=1}^M \beta_{lk} Z_{kt}(t) \right\| dt \\ &\leq A(N) \times \int_0^T \left\{ \varepsilon_1 \left\| \sum_{l=1}^N g_{llt}^M(t) \right\|^2 + C(\varepsilon_1) \left\| \sum_{j=1}^N g_{jt}^M(t) \right\|^2 \right\} dt, \end{aligned}$$

where ε_1 is an arbitrary positive number. Fixing N , we choose such ε_1 that $0 < A(N)\varepsilon_1 \leq \varepsilon$, where ε is a sufficiently small positive number.

Changing indices, we obtain

$$I_1 \leq \int_0^T \left\{ \varepsilon \sum_{j=1}^N (g_{j\mu}^{M2}) + C(\varepsilon, N) \sum_{j=1}^N (g_{j\mu}^M)^2 \right\} dt \leq \varepsilon \|u_{\mu t}^{NM}\|^2 + C(\varepsilon, N) \|u_{\mu t}^{NM}\|^2.$$

Here the constant $C(\varepsilon, N)$ does not depend on μ, M . Making use of Assumptions A4, A5 and suppressing indices N, M, μ , we estimate the boundary integrals by Young's inequality:

$$\begin{aligned} I_2 &= \left| \int_0^T \int_{\Gamma} F_t(x, t, u_t) u_{tt} d\Gamma dt \right| \leq C \int_0^T \int_{\Gamma} (1 + |u_t|^{\rho+1}) |u_{tt}| d\Gamma dt \\ &\leq \varepsilon \int_0^T \int_{\Gamma} (1 + |u_t|^{\rho}) u_{tt}^2 d\Gamma dt + C(\varepsilon) \int_0^T \int_{\Gamma} (1 + |u_t|^{\rho+2}) d\Gamma dt; \\ I_3 &= \int_0^T \int_{\Gamma} \frac{\partial F}{\partial u_t}(x, t, u_t) u_{tt}^2 d\Gamma dt \geq C_0 \int_0^T \int_{\Gamma} (1 + |u_t|^{\rho}) u_{tt}^2 d\Gamma dt. \end{aligned}$$

Substituting expressions for I_1, I_2, I_3 into (3.14); choosing $\varepsilon > 0$ sufficiently small, and using (3.9) and the estimate (3.13), we come to the inequality

$$(3.15) \quad \|u_{\mu t}^{NM}\|^2 \leq C(N) \|f_t\|^2,$$

whence,

$$(3.16) \quad \int_0^T \sum_{j=1}^N g_{\mu j t t}^{NM}(t)^2 dt \leq C(N),$$

where $C(N)$ depends only on N . Combining (3.13) and (3.16), we get

$$(3.17) \quad \|g_{\mu j}^{NM}\|_{H^2(0, T)} \leq C(N) \|f\|_{H^1(Q)}, \quad j = 1, \dots, N$$

uniformly in μ, M .

The estimate (3.17) allows us to prove

Lemma 3.4. *Under Assumptions A1–A8, for any fixed $N \in \mathbb{N}$ and $\mu > 0$ there exists a unique solution, $g_{\mu j} \in H^3(0, T)$, to (3.4) satisfying (3.17).*

Proof. After integration by parts in the first term of (3.8), we rewrite it, suppressing indices μ, N , in the form

$$\begin{aligned}
& \mu \int_0^T g_{jlt}^M(t) Z_{lt}(t) dt + \int_0^T \left(K \sum_{i=1}^N g_{iit}^M(t) w_i(x), w_j(x) \right) (t) Z_l(t) dt \\
& + \int_0^T \left\{ \left(\sum_{i=1}^N g_i^M(t) \nabla w_i(x), \nabla w_j(x) \right) (t) + \int_{\Gamma} F \left(t, \sum_{i=1}^N g_{it}^M(t) w_i(x) \right) w_j(x) d\Gamma \right. \\
& \left. + \left(\alpha \sum_{i=1}^N g_{iit}^M(t) w_i(x), w_j(x) \right) (t) \right\} Z_l(t) dt = \int_0^T (f, w_j)(t) Z_l(t) dt, \\
& j = 1, \dots, N, \quad l = 1, \dots, M.
\end{aligned}$$

Fixing N , M and passing to the limit as $M \rightarrow \infty$, we obtain, making use of (3.17) and density of $\{Z_l(t)\}$ in $H^1(0, T)$,

$$\begin{aligned}
\mu \int_0^T g_{jlt}(t) \varphi(t) dt &= - \int_0^T \left\{ (Ku_{\mu t}^N, w_j)(t) + (\nabla u_{\mu}^N, \nabla w_j)(t) + \int_{\Gamma} F(t, u_{\mu t}^N) w_j d\Gamma \right. \\
& \left. + (\alpha u_{\mu t}^N, w_j)(t) - (f, w_j)(t) \right\} \varphi(t) dt \\
&\equiv \int_0^T \Phi(t) \varphi(t) dt,
\end{aligned}$$

where $\Phi \in L^2(0, T)$, $\varphi(t)$ is an arbitrary function from $H^1(0, T)$. This implies

$$\mu g_{jlt} \in L^2(0, T), \quad j = 1, \dots, N.$$

It means that we proved existence of solutions to (3.4), (3.5): $g_j \in H^3(0, T)$. Uniqueness can be proved by the same way as in Lemma 3.3.

The next step is to pass to the limit as $\mu \rightarrow 0$.

3.5. Existence of approximate solutions.

Theorem 3.1. *Let N be a fixed positive integer and Assumptions A1–A8 hold. Then for any $f(x, t) : f, f_t \in L^2(Q)$, there exists a function $u^N(x, t) : u^N \in H^2(0, T; L^2(D)) \cap H^1(0, T; H^1(D))$; $Ku_{\mu t}^N \in H^1(0, T; L^2(D))$, satisfying (3.2), (3.3).*

Proof. It follows from (3.13) that at fixed N

$$\lim_{\mu \rightarrow 0} \mu \|u_{\mu t}^N\| = 0,$$

hence,

$$\lim_{\mu \rightarrow 0} \mu \int_0^T g_{ujt} \varphi dt = 0$$

for an arbitrary $\varphi \in H^1(0, T)$.

The inequality (3.17) allows us to pass to the limits, as $\mu \rightarrow 0$, in all of the terms of (3.4); and we obtain

$$(3.18) \quad \begin{aligned} (Ku_{tt}^N, w_j)(t) &= (f, w_j)(t) - (\nabla u^N, \nabla w_j)(t) \\ &\quad - \int_{\Gamma} F(t, u_t^N) w_j d\Gamma - (\alpha u_t^N, w_j)(t) \\ &\equiv Q_j(t), \quad j = 1, \dots, N. \end{aligned}$$

It is easy to see that $Q_j \in H^1(0, T)$ for all $j = 1, \dots, N$; whence, $(Ku_{tt}^N, w_j) \in H^1(0, T)$.

This proves Theorem 3.1. $\square$

Now we must pass to the limit as $N \rightarrow \infty$. With this purpose, we have to prove appropriate estimates which are independent of N .

4. Estimates of Galerkin's approximations

Lemma 4.1. *Let Assumptions A1–A8 hold. Then for any $f \in L^2(Q)$ and $N \in \mathbb{N}$, approximate solutions satisfy the inequality*

$$(4.1) \quad \begin{aligned} &\int_D [K(x, t)(u_t^N)^2 + |\nabla u^N|^2] dx \Big|_{t=0}^{t=T} + \int_D |K(x, 0)|(u_t^N)^2 dx \Big|_{t=0}^{t=T} \\ &\quad + \|u^N\|_{H^1(Q)}^2 + \int_0^T \int_{\Gamma} |u_t^N|^{\rho+2} d\Gamma dt \leq C \|f\|^2, \end{aligned}$$

where the constant C does not depend on N .

Proof. Multiplying the j -th equation of (3.2) by $e^{-\lambda t} g_{jt}(t)$, λ is an arbitrary positive number; summing over $j = 1, \dots, N$; integrating the result over $(0, T)$ and suppressing the index N , we find

$$(4.2) \quad \begin{aligned} &\int_D e^{-\lambda t} K(x, t) u_t^2 dx \Big|_{t=0}^{t=T} + \int_D e^{-\lambda t} |\nabla u|^2 dx \Big|_{t=0}^{t=T} + \int_Q e^{-\lambda t} (2\alpha + \lambda K - K_t) u_t^2 dx dt \\ &\quad + \lambda \int_Q e^{-\lambda t} |\nabla u|^2 dx dt + \int_0^T \int_{\Gamma} e^{-\lambda t} F(x, t, u_t) u_t d\Gamma dt \\ &= \int_Q e^{-\lambda t} f u_t dx dt. \end{aligned}$$

Due to (3.9),

$$2\alpha - K_t \geq 2\delta > 0.$$

Choosing $0 < \lambda \leq \delta / \max_{\bar{Q}} |K(x, t)|$, we get

$$2\alpha + \lambda K - K_t \geq \delta > 0.$$

From Assumptions A7, A8, we have

$$K(x, T) \geq 0, \quad K(x, 0) < 0,$$

whence, the first integral in (4.2) is nonnegative. It follows from the Assumption A3:

$$I_3 = \int_0^T \int_{\Gamma} e^{-\lambda t} F(x, t, u_t) u_t dx dt \geq C_0 \int_0^T \int_{\Gamma} e^{-\lambda t} |u_t|^{\rho+2} d\Gamma dt.$$

Since $\lambda > 0$, then $0 < e_0 \leq e^{-\lambda t} \leq 1$ in $\bar{Q}$, hence,

$$I_1 \geq C_0 e_0 \int_0^T \int_{\Gamma} |u_t|^{\rho+2} dt.$$

Taking into account I_1 and above arguments, we reduce (4.2) to the inequality

$$\begin{aligned} & \int_D \{K(x, t) u_t^2 + |\nabla u|^2\} dx \Big|_{t=0}^{t=T} + \int_D |K(x, 0) u_t^2(x, 0) dx + \|u\|_{H^1(Q)}^2 \\ & + \int_0^T \int_{\Gamma} |u_t|^{\rho+2} d\Gamma dt \leq C \|f\|^2, \end{aligned}$$

where C depends on $\delta, \lambda > 0$, but does not depend on N .

This proves Lemma 4.1. $\square$

Lemma 4.2. *Let Assumptions A1–A8 hold. Then for any $f \in L^2(Q)$ and $N \in \mathbb{N}$*

$$(4.3) \quad \|u_t^N\|_{H^1(Q_1)}^2 \leq C \|f\|_{L^2(Q)}^2,$$

where $Q_1 = D \times (0, T_1)$, T_1 is a positive number; and the constants C , T_1 do not depend on N .

Proof. Due to Lemma 3.2, there exists such $T_0 > 0$ that $K(x, t) \leq -\eta < 0$ in $\bar{Q}_0 = \bar{D} \times [0, T_0]$.

We define on $[0, T]$ a smooth nonnegative function $\xi(t)$ as follows:

$$(4.4) \quad \xi(t) = \begin{cases} 1, & t \in [0, T_1], \\ 0, & t \in [2T_1, T], \end{cases}$$

where $2T_1 = T_0$.

Multiplying the j -th equation of (3.2) by $-\xi(t) g_{jt}(t)$, summing over $j = 1, \dots, N$ and integrating the results over $(0, T)$, we get, suppressing the

index N :

$$(4.5) \quad -\int_Q \xi K u_{tt}^2 dx dt - \int_Q \xi \nabla u \nabla u_{tt} dx dt - \int_0^T \int_\Gamma \xi F(t, u_t) u_{tt} d\Gamma dt \\ - \int_Q \xi \alpha u_t u_{tt} dx dt = - \int_Q \xi f u_{tt} dx dt.$$

We treat all the terms in (4.5) separately:

$$(4.6) \quad I_1 = - \int_Q \xi \nabla u \nabla u_{tt} dx dt = - \int_Q \left\{ \frac{\partial}{\partial t} (\xi \nabla u \nabla u_t) - \xi |\nabla u_t|^2 \right. \\ \left. - \frac{1}{2} \frac{\partial}{\partial t} (\xi_t |\nabla u|^2) + \frac{1}{2} (\xi_{tt} |\nabla u|^2) \right\} dQ.$$

Taking into account (4.4), we find

$$I_1 = \int_Q \xi |\nabla u_t|^2 dQ - \frac{1}{2} \int_Q \xi_{tt} |\nabla u|^2 dQ \geq \int_Q \xi |\nabla u_t|^2 dQ - C \|u\|_{H^1(Q)}^2.$$

Analogously,

$$I_2 = - \int_0^T \int_\Gamma \xi F(t, u_t) u_{tt} d\Gamma dt = - \int_0^T \int_\Gamma \left\{ \frac{\partial}{\partial t} \left(\xi \int_0^{u_t} F(t, s) ds \right) \right. \\ \left. - \xi_t \int_0^{u_t} F(t, s) ds - \xi \int_0^{u_t} F_t(t, s) ds \right\} d\Gamma dt.$$

Making use of (4.4) and Assumptions A2–A4, we obtain

$$(4.7) \quad |I_2| \leq C \int_0^T \int_\Gamma (1 + |u_t|^{\rho+2}) d\Gamma dt.$$

By Schwartz inequality,

$$(4.8) \quad |I_3| = \left| \int_Q \xi \alpha u_t u_{tt} dQ \right| \leq \varepsilon \int_Q \xi u_{tt}^2 dQ + C(\varepsilon) \int_Q \xi u_t^2 dQ \\ \leq \varepsilon \int_Q \xi u_{tt}^2 dQ + C(\varepsilon) \|u\|_{H^1(Q)}^2,$$

where ε is an arbitrary positive number.

Analogously,

$$(4.9) \quad |I_4| = \left| \int_Q \xi f u_{tt} dQ \right| \leq \varepsilon \int_Q \xi u_{tt}^2 dQ + C(\varepsilon) \|f\|^2.$$

Using (4.6)–(4.9), we transform (4.5) into the following inequality

$$\begin{aligned} \int_Q \xi \{(-K - 2\varepsilon)u_t^2 + |\nabla u_t|^2\} dQ &\leq C_\varepsilon (\|f\|^2 + \|u\|_{H^1(Q)}^2) \\ &+ C \int_0^T \int_\Gamma (1 + |u_t|^{\rho+2}) d\Gamma dt. \end{aligned}$$

Choosing $0 < 4\varepsilon \leq \eta$ and taking into account (4.1) and properties of $\xi(t)$, we get

$$\int_Q \xi (u_t^2 + |\nabla u_t|^2) dQ \leq C \|f\|^2,$$

where C does not depend on N .

The assertion of Lemma 4.2 follows from this inequality. $\square$

Lemma 4.2 gives the estimate $\|u_t\|_{H^1(Q_1)}$ in some vicinity of the surface $t = 0$. On the next step we will prove the similar estimate in $Q \setminus Q_1$.

Lemma 4.3. *Let the Assumptions A1–A8 hold. Then for any $f \in H^1(0, T; L^2(D))$ and $N \in \mathbb{N}$, the approximate solutions $u^N(x, t)$ satisfy the inequality*

$$(4.10) \quad \|u_t^N\|_{H^1(Q_2)} \leq C \|f\|_{H^1(0, T; L^2(D))},$$

where $Q_2 = D \times (T_1/2, T)$; the constant C does not depend on N .

Proof. We define on $[0, T]$ a nonnegative, nondecreasing smooth function $\xi = \xi_1(t)$ as follows:

$$(4.11) \quad \xi_1(t) = \begin{cases} 1, & t \in [T_1, T], \\ 0, & t \in [0, T_1/2]. \end{cases}$$

Differentiating (3.2) with respect to t , that is valid due to Theorem 3.1, multiplying the j -th equation by $e^{-\lambda t} \xi_1(t) g_{jt}(t)$ and summing over $j = 1, \dots, N$; we come, suppressing the index N , to the identity

$$\begin{aligned} (e^{-\lambda t} \xi_1 K u_{tt}, u_{tt})(t) + (e^{-\lambda t} \xi_1 (\alpha + K_t), u_{tt}^2)(t) + (e^{-\lambda t} \xi_1 \nabla u_t, \nabla u_{tt})(t) \\ + \int_\Gamma e^{-\lambda t} \xi_1 \left\{ F_t(t, u_t) + \frac{\partial F}{\partial u_t}(t, u_t) u_{tt} \right\} u_{tt} d\Gamma = (e^{-\lambda t} \xi_1 f_t, u_{tt})(t). \end{aligned}$$

Integrating this over $(0, T)$, we get after standard transformations

$$\begin{aligned}
(4.12) \quad & \int_D e^{-\lambda t} \xi_1(t) \{Ku_u^2 + |\nabla u_t|^2\} dx|_{t=0}^{t=T} + \int_Q e^{-\lambda t} \xi_1(2\alpha + \lambda K + K_t) u_u^2 dQ \\
& + \lambda \int_Q e^{-\lambda t} \xi_1 |\nabla u_t|^2 dQ - \int_Q e^{-\lambda t} \xi_1 |\nabla u_t|^2 dx dt \\
& + 2 \int_0^T \int_\Gamma e^{-\lambda t} \xi_1 \left\{ F_t(t, u_t) u_{tt} + \frac{\partial F}{\partial u_t}(t, u_t) u_{tt}^2 \right\} d\Gamma dt \\
& = 2 \int_Q e^{-\lambda t} \xi_1 f_t u_{tt} dx dt.
\end{aligned}$$

Here λ is an arbitrary positive number.

Choosing $0 < \lambda < \delta \max_{\bar{Q}} |K(x, t)|$ and using (3.9), we have

$$(4.13) \quad 2\alpha + \lambda K + K_t \geq \delta > 0.$$

It follows from Assumptions A7, A8 that the first integral in (4.12) is positive. One can see from (4.11) that $\xi_{1t}(t)$ is not identically zero only on $[T_1/2, T_1]$ where the estimate (4.3) is valid.

Therefore,

$$(4.14) \quad |I_1| = \left| \int_Q e^{-\lambda t} \xi_{1t}(t) |\nabla u_t|^2 dQ \right| \leq C_1 \|u_t\|_{H^1(Q_1)}^2$$

with C_1 independent of N .

The Assumption A5 gives

$$(4.15) \quad \frac{\partial F}{\partial u_t} u_{tt}^2 \geq C_0(1 + |u_t|^\rho) u_{tt}^2.$$

On the other hand, the Assumption A4 implies

$$\begin{aligned}
(4.16) \quad & |F_t(t, u_t) u_{tt}| \leq C(1 + |u_t|^{\rho+1}) |u_{tt}| \\
& \leq \varepsilon(1 + |u_t|^\rho) u_{tt}^2 + C(\varepsilon)(1 + |u_t|^{\rho+2}),
\end{aligned}$$

where ε is an arbitrary positive number.

Taking $0 < 2\varepsilon < C_0$ and using (4.15), (4.16), we have

$$\begin{aligned}
(4.17) \quad I_2 &= 2 \int_0^T \int_\Gamma e^{-\lambda t} \xi_1 \left\{ \frac{\partial F}{\partial u_t} u_{tt}^2 + F_t u_{tt} \right\} d\Gamma dt \\
&\geq C_0 \int_0^T \int_\Gamma e^{-\lambda t} \xi_1 (1 + |u_t|^\rho) u_{tt}^2 d\Gamma dt - C \int_0^T \int_\Gamma (1 + |u_t|^{\rho+2}) d\Gamma dt.
\end{aligned}$$

Choosing in an appropriate way $\lambda > 0$ and taking into account (4.13), (4.17) and properties of $\xi_1(t)$, we reduce (4.12) to the inequality

$$\int_Q \xi_1(t)(u_t^2 + |\nabla u_t|^2) dx dt \leq C \left(\|f_t\|^2 + \|u_t\|_{H^1(Q_t)}^2 + \int_0^T \int_\Gamma (|u_t|^{\rho+2} + 1) d\Gamma dt \right).$$

With the help of (4.1), (4.3) and (4.11), we prove Lemma 4.3. $\square$

Combining Lemmas 4.1–4.3, we come to

Lemma 4.4. *Let Assumptions A1–A8 hold. Then for any $f \in H^1(0, T; L^2(D))$ and $N \in \mathbb{N}$ the approximate solutions $u^N(x, t)$ satisfy the following inequality*

$$(4.18) \quad \|u^N\|_{H^1(Q)} + \|u_t^N\|_{H^1(Q)} + \|u_t^N\|_{L^{\rho+2}(S)} \leq C \|f\|_{H^1(0, T; L^2(D))},$$

where the constant C does not depend on N .

5. Solvability of the problem (1.1)–(1.3)

The estimates of Lemma 4.4 allow us to pass to the limit in (3.2) as $N \rightarrow \infty$. In fact, it follows from (4.18) that there exist a subsequence $\{u^l\}$ of $\{u^N\}$ and such a function $u(x, t)$ that

$$(5.1) \quad \begin{aligned} u^l &\rightharpoonup u \text{ weakly in } H^1(Q) \\ u_t^l &\rightharpoonup u_t \text{ weakly in } H^1(Q) \\ u_t^l &\rightarrow u_t \text{ a.e. on } S. \end{aligned}$$

The Assumption A2 and (4.18) imply

$$F(u_t^N) \in L^{(\rho+2)/(\rho+1)}(S).$$

From here and (5.1)

$$(5.2) \quad F(u_t^N) \rightarrow F(u_t) \text{ weakly in } L^{(\rho+2)/(\rho+1)}(S).$$

Making use of (5.1), (5.2), passing to the limit as $N \rightarrow \infty$ in (3.2); and taking into account density of $\{w_j\}$ in $H^1(D) \cap L^{\rho+2}(\Gamma)$, we obtain for a.e. $t \in (0, T)$:

$$(Ku_t, v)(t) + (\nabla u, \nabla v)(t) + (\alpha u_t, v)(t) + \int_\Gamma F(t, u_t) v d\Gamma = (f, v)(t).$$

Here $v(x)$ is an arbitrary function from $H^1(D) \cap L^{\rho+2}(\Gamma)$.

The boundary condition (1.2) is fulfilled due to the way of construction of $u^N(x, t)$ and passage to the limit in $H^1(Q)$.

Thus, we have proved existence of solutions to the problem (1.1)–(1.3).

Uniqueness can be established by the standard procedure, see [2]. All this can be resumed in the form of

Theorem 5.1. *Let Assumptions A1–A8 hold. Then for any $f \in H^1(0, T; L^2(D))$ there exists a unique function $u(x, t) : u_t \in H^1(Q) \cap L^{p+2}(S)$, which is a solution to the problem (1.1)–(1.3) in the sense of Definition 3.1.*

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