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Stability of Linear Time-Variant Functional Differential Equations in a Hilbert Space

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1. Introduction and statement of the main result

It is well-known that quite a few problems of applied mathematics lead to abstract functional differential equations cf. [1], [4], [14], [22], etc. In recent years abstract functional differential and parabolic functional equations have been the object of intensive studies (see [23], [28] and references therein).

Stability of time-invariant parabolic and abstract functional-differential (linear and nonlinear) equations was investigated by Murakami [20], Burton and Zhang [2], Friecske [7], Kuang and Smith [17], Lenhart and Travis [18], Luckhaus [19], Pao and Mahaffy [22], Pao [21], Gil' [11] and other specialists.

At the same time stability of time-variant abstract functional equations was investigated considerably less than the stability of time-invariant equations. Let H be a Hilbert space with a scalar product $(\cdot, \cdot)_H$ and a norm $\|\cdot\|_H$. Let $R(t, s)$ be a left-continuous in s function defined on $[0, \infty) \times [0, \eta]$ with $\eta < \infty$, whose values are bounded linear operators in H . In addition, $R(t, s)$ has the bounded variation in s and is continuous in t . Furthermore, let $A(t)$ be a linear operator in H dependent on $t \geq 0$ with a dense domain $D(A(t))$.

In the present paper, stability conditions are derived for the functional differential equation

$$(1.1) \quad \dot{u} = A(t)u + \int_0^\eta R(t, ds)u(t-s) \quad (\dot{u} = du/dt, t > 0).$$

The integral

$$\int_0^\eta R(t, ds)$$

for each $t \geq 0$ means a limit in the uniform operator topology of the corresponding Riemann-Stieljes sums. Besides,

$$V(t) \equiv \int_0^\eta \|R(t, ds)\|_H$$

is a continuous function. Take the initial condition

$$(1.2) \quad u(t) = \Phi(t) \quad (-\eta \leq t \leq 0),$$

where $\Phi \in C([-\eta, 0], H)$ is a given function. As usually, $C(\omega, H)$ means the space of continuous functions defined on a set ω with values in H and the sup-norm.

Further, suppose the Cauchy problem for the “shortened” equation

$$(1.3) \quad \dot{v} = A(t)v \quad (t \geq 0).$$

is well posed [24]. Then equation (1.3) has the evolution operator (fundamental solution) $U(t, s)$ acting in H and defined by the equality $U(t, s)v(s) = v(t)$, where $v(t)$ is a solution of (1.3). Following Browder’s terminology cf. [26, Chapter 5], let us introduce.

Definition 1.1. Suppose the Cauchy problem for equation (1.3) is well posed. A continuous function $u(t) : [-\eta, \infty) \rightarrow H$ satisfying the equation

$$(1.4) \quad u(t) = U(t, 0)\Phi(0) + \int_0^t U(t, t_1) \int_0^\eta R(t_1, ds)u(t_1 - s) \quad (t > 0)$$

and condition (1.2) will be called the (mild) solution of (1.1) with the initial function Φ .

In the following definitions, the existence of solutions is assumed.

Definition 1.2. Equation (1.1) is said to be stable, if there is a constant N , such that under (1.2) any solution $u(t)$ of (1.1) satisfies the inequality

$$\|u(t)\|_H \leq N \|\Phi\|_{C([-\eta, 0], H)} \quad (t \geq 0).$$

Equation (1.1) is said to be asymptotically stable, if it is stable and, in addition, the relation $\|u(t)\|_H \rightarrow 0$ as $t \rightarrow \infty$ holds for any its solution $u(t)$.

Equation (1.1) is said to be exponentially stable, if there are positive constants N, v , such that under (1.2), any solution $u(t)$ of (1.1) satisfies the inequality

$$\|u(t)\|_H \leq Ne^{-vt} \|\Phi\|_{C([-\eta, 0], H)} \quad (t \geq 0).$$

The aim of the present paper is to prove the following

Theorem 1.3. *Let the Cauchy problem for equation (1.3) be well posed. In addition, assume that*

$$(1.5) \quad \operatorname{Re}(A(t)h, h)_H \leq A(t)(h, h)_H \quad (h \in D(A(t)), t \geq 0)$$

where $\Lambda(t)$ is a continuous negative scalar-valued function. Then for any initial function $\Phi \in C([- \eta, 0], H)$, problem (1.1), (1.2) has on $(0, \infty)$ a unique mild solution $u(t)$, which satisfies the inequality

$$(1.6) \quad \|u(t)\|_H \leq c(\Phi) e^{\int_0^t (\Lambda(r) + Q(r)) dr} \quad (t \geq 0),$$

where

$$Q(r) = \int_0^\eta \|R(r+s, ds)\| e^{-\int_r^{r+s} \Lambda(z) dz}$$

and

$$c(\Phi) = \|\Phi(0)\|_H + \int_0^\eta \|R(t_1, ds)\|_H \int_0^s e^{-\int_0^{t_1} \Lambda(z) dz} \|\Phi(t_1 - s)\|_H dt_1.$$

The proofs of this theorem and of the following corollary are divided into a series of lemmata which are presented in the next two sections.

Corollary 1.4. *Let operator $A(t)$ have a constant dense domain $D(A(t)) \equiv D_A$ ($t \geq 0$) and continuously differentiable on D_A . That is, $A(t)v$ is a strongly continuously differentiable function for any $v \in D_A$. In addition, assume that the inequalities (1.5) and*

$$(1.7) \quad \operatorname{Re}(A^*(t)v, v)_H \leq \Lambda(t)(v, v)_H \quad (v \in D(A^*(t)), t \geq 0),$$

are fulfilled. Then for any initial function $\Phi \in C([- \eta, 0], H)$, problem (1.1), (1.2) has on $(0, \infty)$ a unique mild solution $u(t)$, which satisfies the inequality (1.6).

Put

$$J(t) = \int_0^t (\Lambda(r) + Q(r)) dr.$$

Corollary 1.5. *Let the hypothesis of Theorem 1.3 be fulfilled. Then*

A) *The condition*

$$(1.8) \quad \sup_{t \geq 0} J(t) < \infty$$

provides the stability of equation (1.1).

B) *The conditions (1.8) and $\lim_{t \rightarrow \infty} J(t) = -\infty$ provide the asymptotic stability of equation (1.1).*

C) *The conditions (1.8) and $\overline{\lim}_{t \rightarrow \infty} t^{-1} J(t) < 0$ provide the exponential stability of equation (1.1).*

If $R \equiv 0$, that is, (1.1) is the evolution equation without delay, then Theorem 1.3 gives

$$\|u(t)\|_H \leq \|u(0)\|_H e^{\int_0^t A(r)dr} \quad (t \geq 0).$$

This inequality under some restriction was derived in [10] (see also [13, Section 14.4]). It extends the well-known inequality established by Wazewski in the case of ordinary differential equations (see for instance [15]) and by Winter [27] in the case of equations with bounded operators in a Hilbert space (see also [6]).

Note that in the paper [12], stability conditions for equation (1.1) and the estimate similar to (1.6) were established in the case

$$\int_0^\eta R(t, ds)u(t-s) = B(t)u(h(t)),$$

where $B(t)$ is a bounded operator and $h(t)$ is a differentiable function.

2. Solution existence

Solvability and well-posedness of equations of the type (1.1) with time-varying operators have been considered mainly in the case of scalar kernels and infinite delays cf. [5], [3], [8], etc. In particular, in [5] G. Da Prato and M. Iannelli investigated the case when $A(t)$ for each t is a negative generator. A. Friedman and M. Shinbrot studied in [8] the case when $-A(t)$ for each t generates an analytic semigroup. In this section we consider the simple solvability conditions which supplement these results.

Lemma 2.1. *Let the Cauchy problem for equation (1.3) be well posed. Then for any initial function $\Phi \in C([-\eta, 0], H)$, equation (1.1) has a unique mild solution $u \in C([-\eta, \infty], X)$.*

Proof. Rewrite (1.4) in the form

$$u(t) = U(t, 0)\Phi(0) + \int_0^\eta \int_0^t U(t, t_1)R(t_1, ds)u(t_1-s)dt_1.$$

Hence,

$$(2.1) \quad u(t) = f(t) + \int_0^\eta \int_s^t U(t, t_1)R(t_1, ds)u(t_1-s)dt_1,$$

where

$$\begin{aligned} f(t) &= U(t, 0)\Phi(0) + \int_0^\eta \int_0^s U(t, t_1)R(t_1, ds)u(t_1-s)dt_1 \\ &= U(t, 0)\Phi(0) + \int_0^\eta \int_0^s U(t, t_1)R(t_1, ds)\Phi(t_1-s)dt_1. \end{aligned}$$

We have

$$\begin{aligned}
 & \int_0^\eta \int_s^t U(t, t_1) R(t_1, ds) u(t_1 - s) dt_1 \\
 &= \int_0^\eta \int_0^{t-s} U(t, s+z) R(s+z, ds) u(z) dz \\
 &= \int_0^\eta \int_0^t K_s(t, s+z, z) R(s+z, ds) u(z) dz,
 \end{aligned}$$

where

$$\begin{aligned}
 K_s(t, \tau, z) &= U(t, \tau) \quad \text{for } 0 \leq z \leq t-s \quad \text{and} \\
 K_s(t, \tau, z) &= 0 \quad \text{for } t-s < z \leq t \quad (0 \leq \tau \leq t).
 \end{aligned}$$

So

$$\int_0^\eta \int_0^t K_s(t, s+z, z) R(s+z, ds) u(z) dz = \int_0^t P(t, z) u(z) dz.$$

where

$$P(t, z) = \int_0^\eta K_s(t, s+z, z) R(s+z, ds).$$

That is, equation (2.1) is equivalent to the following one:

$$u(t) = f(t) + \int_0^t P(t, z) u(z) dz.$$

Since kernel $P(t, z)$ is bounded and continuous in t , we arrive at the required result. $\square$

Lemma 2.2. *Let operator $A(t)$ have a constant dense domain D_A and continuously differentiable on it. Then the Cauchy problem for equation (1.3) is well posed.*

Proof. First, note that (1.5) and (1.7) imply that $A(t)$ is maximal dissipative cf. [25, p. 310]. This means that $I - \delta A(t)$ is invertible with a sufficiently small $\delta > 0$. Here I is the unit operator. So simple calculations show that due to (1.5)

$$\|(I - \delta A(t))^{-1}\| \leq 1/(1 + A(t)\delta) \quad (t \geq 0).$$

Hence, it follows that $A(t)$ generates a stable family [24, Section 4.3]. Now the well posedness is due to Corollary to the well-known Theorem 4.4.2 [24, p. 102]. $\square$

Lemmata 2.1 and 2.2 yield.

Corollary 2.3. *Under the hypothesis of Corollary 1.4, problem (1.1), (1.2) has a unique mild solution.*

3. Proof of Theorem 1.3

Lemma 3.1. *Let condition (1.5) hold. Then any solution of the problem (1.1), (1.2) (if it exists) satisfies inequality (1.6).*

Proof. Doing in the “shortened” equation (1.3) the scalar product, we have

$$\operatorname{Re}(\dot{v}(t), v(t))_H = \operatorname{Re}(A(t)v(t), v(t))_H$$

for an arbitrary strong solution v of (1.3). Since

$$2 \operatorname{Re}(\dot{v}(t), v(t))_H = d(v(t), v(t))_H/dt = 2\|v\|_H d\|v\|_H/dt,$$

we arrive at the inequality

$$d\|v(t)\|_H/dt \leq A(t)\|v(t)\|_H.$$

Solving this inequality, we get

$$\|v(t)\|_H \leq e^{\int_s^t A(\tau) d\tau} \|v(s)\|_H \quad (t \geq s \geq 0).$$

This means that the evolution operator $U(t, s)$ of equation (1.3) satisfies the inequality

$$\|U(t, s)\|_H \leq e^{\int_s^t A(t_1) dt_1} \quad (t \geq s \geq 0).$$

Now (1.4) yields

$$\|u(t)\| \leq e^{\int_0^t A(t_1) dt_1} \|\Phi(0)\|_H + \int_0^t e^{\int_{t_1}^t A(\tau) d\tau} \int_0^\eta \|R(t_1, ds)u(t_1 - s)\|_H \quad (t > 0).$$

With the notation

$$(3.1) \quad y(t) = e^{-\int_0^t A(\tau) d\tau} \|u(t)\|_H,$$

we can write out

$$\begin{aligned} y(t) &\leq \|\Phi(0)\|_H + \int_0^\eta \int_0^t \|R(t_1, ds)\|_H \exp \left[\int_{t_1}^{t_1-s} A(z) dz \right] y(t_1 - s) dt_1 \\ &= c(\Phi) + \int_0^\eta \|R(t_1, ds)\|_H \int_s^t e^{\int_{t_1}^{t_1-s} A(z) dz} y(t_1 - s) dt_1. \end{aligned}$$

Or

$$\begin{aligned} y(t) &\leq c(\Phi) + \int_0^\eta \int_0^{t-s} \|R(r+s, ds)\|_H e^{\int_{r+s}^t A(z) dz} y(r) dr \\ &\leq c(\Phi) + \int_0^\eta \int_0^t \|R(r+s, ds)\|_H e^{\int_{r+s}^r A(z) dz} y(r) dr = c(\Phi) + \int_0^t Q(r) y(r) dr. \end{aligned}$$

The Gronwall lemma gives

$$y(t) \leq c(\Phi) \exp \left[\int_0^t Q(r) dr \right].$$

Now (3.1) yields the required result. $\square$

The assertion of Theorem 1.3 follows from the latter lemma and Lemma 2.1.

The assertion of Corollary 1.4 follows from the latter lemma and Corollary 2.3.

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