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The Initial Value Problem for Some Degenerate Hyperbolic Equations of Second Order in Gevrey Classes

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§1. Introduction

In this paper we shall consider the following Cauchy problem for linear weakly hyperbolic equations on $[0, T] \times \mathbf{R}_x^n$

$$(1.1) \quad \begin{cases} L(t, x, \partial_t, \partial_x)u(t, x) = 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \end{cases}$$

where

$$L(t, x, \partial_t, \partial_x) = \partial_t^2 - A_2(t, x, \partial_x) - A_1(t, x, \partial_x),$$

$$A_2(t, x, \partial_x) = \sum_{j,k=1}^n \partial_{x_j} a_{jk}(t, x) \partial_{x_k}, \quad A_1(t, x, \partial_x) = \sum_{j=1}^n a_j(t, x) \partial_{x_j}$$

such that $a_{jk}(t, x) \in C_t^\infty([0, T]; \gamma^{(\kappa)}(\mathbf{R}_x^n))$ satisfy the weak hyperbolicity condition

$$\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \geq 0 \quad \text{for all } (t, x, \xi) \in [0, T] \times \mathbf{R}_{x,\xi}^{2n}$$

and $a_j(t, x) \in C_t^0([0, T]; \gamma^{(\kappa)}(\mathbf{R}_x^n))$. $\gamma^{(\kappa)}(\mathbf{R}^n)$ is the (projective) Gevrey class with exponent κ (>0), that is, the set of all functions $f \in C^\infty(\mathbf{R}^n)$ such that for any $r > 0$ there exists a constant $C_r > 0$ fulfilling

$$\sup_{x \in \mathbf{R}^n} |\partial_x^\alpha f(x)| \leq C_r r^{|\alpha|} |\alpha|!^\kappa$$

for every multi-index $\alpha \in \mathbf{Z}_+^n$. We actually handle some special cases of $a_{jk}(t, x)$ and $a_j(t, x)$ here. As the function spaces to which the solution of the Cauchy problem (1.1) belongs, we define the Gevrey space with exponent κ (>0) and radius ρ (>0) (of convergence)

$$L_{\kappa,\rho}^2(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n); e^{\rho \langle \xi \rangle^{1/\kappa}} \hat{u}(\xi) \in L^2(\mathbf{R}_\xi^n)\}$$

and $L_{\kappa, \infty}^2(\mathbf{R}^n) = \bigcap_{\rho > 0} L_{\kappa, \rho}^2(\mathbf{R}^n)$, where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ and $\hat{u}$ means the Fourier image of u in $L^2(\mathbf{R}^n)$.

First of all, we shall deal with the case that $L(t, x, \partial_t, \partial_x)$ is independent of x , namely

$$L(t, \partial_t, \partial_x) = \partial_t^2 - \sum_{j,k=1}^n a_{jk}(t) \partial_{x_j x_k} - \sum_{j=1}^n a_j(t) \partial_{x_j}$$

with $a_{jk}(t) \in C^\infty([0, T])$ and $a_j(t) \in C^0([0, T])$. Then the weak hyperbolicity condition turns into

$$(1.2) \quad a_2(t, \xi) = \sum_{j,k=1}^n a_{jk}(t) \xi_j \xi_k \geq 0 \quad \text{for all } (t, \xi) \in [0, T] \times \mathbf{R}_\xi^n.$$

For simplicity, put $a_1(t, \xi) = \sum_{j=1}^n a_j(t) \xi_j$ as well. Now we can state our results in this case (see also K. Kajitani [7]).

Theorem 1.1. *Assume that (1.2) and there exist some $\kappa \geq 1$ and $\lambda(t, \xi) \in C_t^\infty([0, T] \times \mathbf{R}_\xi^n)$ such that*

$$(1.3) \quad \lambda(t, \xi) \geq C_0 \langle \xi \rangle^\rho \quad (0 < \rho \leq 2),$$

$$(1.4) \quad \left| \frac{\partial^j}{\partial t^j} \lambda(t, \xi) \right| \leq C_j \langle \xi \rangle^2 \quad (j = 0, 1, \dots),$$

$$(1.5) \quad \int_0^T \frac{|\lambda(t, \xi) - ia_1(t, \xi)|}{\sqrt{a_2(t, \xi) + \lambda(t, \xi)}} dt \leq C(T) \langle \xi \rangle^{1/\kappa} \quad (i = \sqrt{-1})$$

hold for all $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$. Then the Cauchy problem (1.1) is wellposed in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R}^n)$, that is, for any initial data $u_j \in L_{\kappa, \infty}^2(\mathbf{R}^n)$ ($j = 0, 1$) there is a unique solution $u(t, x) \in C^2([0, T]; L_{\kappa, \infty}^2(\mathbf{R}_x^n))$ to the Cauchy problem (1.1).

Further, we give a sufficient condition for the existence of κ and $\lambda(t, \xi)$ in Theorem 1.1 in terms of the symbols $a_2(t, \xi)$ and $a_1(t, \xi)$ only.

Theorem 1.2. *Suppose that for some real numbers ℓ, ν with $0 \leq \nu < \ell - 1$ there exist positive constants δ and C such that*

$$(1.6) \quad a_2(t, \xi) \geq \delta t^{2\ell} |\xi|^2,$$

$$(1.7) \quad |a_1(t, \xi)| \leq C(t^\nu |\xi| + \sqrt{a_2(t, \xi)})$$

are valid for all $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$. Then there are $\lambda(t, \xi) \in C_t^\infty([0, T])$ and $1 \leq \kappa \leq (2\ell - \nu)/(\ell - \nu - 1)$ satisfying the conditions in Theorem 1.1.

Example 1.3.

$$a_2(t, \xi) = t^{2\ell} |\xi|^2 + \sum_{j,k=1}^n b_{jk}(t) \xi_j \xi_k,$$

$$a_1(t, \xi) = t^v \sum_{j=1}^n c_j(t) \xi_j + \sum_{k=1}^n d_k(t) \sum_{j=1}^n b_{jk}(t) \xi_j,$$

where $b_{jk}(t) \in C^\infty([0, T])$, $c_j(t), d_k(t) \in C^0([0, T])$ and $\sum_{j,k=1}^n b_{jk}(t) \xi_j \xi_k \geq 0$ for all $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$.

Here we should make some historical comments. In Example 3.1 of V. Ja. Ivrii [5], he investigated the case of

$$L_0(t, \partial_t, \partial_x) = \partial_t^2 - t^{2\ell} \partial_x^2 - it^v \partial_x$$

with $0 \leq v < \ell - 1$ and $n = 1$. He proved that a necessary and sufficient condition for the wellposedness of the Cauchy problem (1.1) to $L_0(t, \partial_t, \partial_x)$ in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R})$ is $(1 \leq) \kappa \leq (2\ell - v)/(\ell - v - 1)$. Since Theorem 1.2 contains the Ivrii example above, the Gevrey exponent $\kappa = (2\ell - v)/(\ell - v - 1)$ in Theorem 1.2 is optimal for the wellposedness in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R})$. Apart from Ivrii's result, in Remark 3 of F. Colombini-E. Jannelli-S. Spagnolo [2], they showed that if $\sup_{|\xi|=1} \int_0^T |a_1(t, \xi)|/a_2(t, \xi)^\beta dt < \infty$ ($0 \leq \beta < 1/2$), the Cauchy problem (1.1) is wellposed in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R}^n)$ provided $\kappa \leq 1 + 1/(1 - 2\beta)$. So, applying the Ivrii example to this result, we know that the Cauchy problem (1.1) to $L_0(t, \partial_t, \partial_x)$ is wellposed in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R})$ provided $\kappa \leq 1 + 1/(1 - 2\beta) < (2\ell - v - 1)/(\ell - v - 1)$, which is more restricted than Ivrii's result. We shall later observe that Theorem 1.1 is an improvement of the F. Colombini, E. Jannelli and S. Spagnolo result above, namely, $\sup_{|\xi|=1} \int_0^T |a_1(t, \xi)|/a_2(t, \xi)^\beta dt < \infty$ ($0 \leq \beta < 1/2$) implies the conditions in Theorem 1.1 (see §3).

Next we shall treat the case that $L(t, x, \partial_t, \partial_x)$ has the x -dependent coefficients

$$a_{jk}(t, x) = t^{2\ell} b_{jk}(x) + c_{jk}(x),$$

$$a_j(t, x) \text{ satisfying } \left| \sum_{j=1}^n a_j(t, x) \xi_j \right| \leq C_T (t^v \sqrt{b(x, \xi)} + \sqrt{c(x, \xi)}),$$

where we assume that

$$b_{jk}(x) \in \gamma^{(\kappa)}(\mathbf{R}^n), \quad b_{jk}(x) = \overline{b_{kj}(x)}, \quad b(x, \xi) = \sum_{j,k=1}^n b_{jk}(x) \xi_j \xi_k \geq 0,$$

$$c_{jk}(x) \in \gamma^{(\kappa)}(\mathbf{R}^n), \quad c_{jk}(x) = \overline{c_{kj}(x)}, \quad c(x, \xi) = \sum_{j,k=1}^n c_{jk}(x) \xi_j \xi_k \geq 0.$$

Then we obtain the following result.

Theorem 1.4. *If $0 \leq \nu < \ell - 1$, then for every $(1 \leq) \kappa \leq (2\ell - \nu)/(\ell - \nu - 1)$ the Cauchy problem (1.1) is wellposed in the Gevrey space $L_{\kappa, \infty}^2(\mathbf{R}^n)$. Moreover, the energy estimate below is established for any $t \in [0, T]$.*

$$(1.8) \quad \|e^{MA}(t, D_x)u(t, x)\|_{L^2(\mathbf{R}_x^n)} + \|e^{MA}(t, D_x)u_t(t, x)\|_{L^2(\mathbf{R}_x^n)} \\ \leq C(t)(\|e^{MA}(0, D_x)u_0\|_{H^1(\mathbf{R}^n)} + \|e^{MA}(0, D_x)u_1\|_{L^2(\mathbf{R}^n)}),$$

where

$$(1.9) \quad A(t, \xi) = \kappa\sigma\{(t + \langle \xi \rangle_h^{-\sigma})^{-1/(\kappa\sigma)} + (T - t)\langle \xi \rangle_h^{1/\kappa}\}, \quad \sigma = \frac{1}{2\ell - \nu},$$

which was introduced in K. Kajitani-M. Mikami [8], $\langle \xi \rangle_h = \sqrt{h^2 + |\xi|^2}$ and $h, M > 0$ are some large parameters.

Theorem 1.4 presents a generalization of Theorem 1.2 to x -dependent coefficients case in some sense.

Finally we wish to refer to V. Ja. Ivrii [6], T. Kinoshita [10] and K. Yagdjian [11] for other recent topics on the Gevrey wellposedness of the hyperbolic Cauchy problem.

§2. Existence theorem of the solution in Gevrey space

We shall prove Theorem 1.1 in this section. To begin with, we need a lemma of real analysis due to F. Colombini-E. Jannelli-S. Spagnolo [2]. Let k be a nonnegative integer and $0 \leq \alpha \leq 1$. Denote by $C^{k, \alpha}([0, T])$ the Banach space of all k -times continuously differentiable functions on $[0, T]$ whose k -th derivatives are Hölder continuous with exponent α when $\alpha > 0$. The norm in this space is endowed with

$$\|f\|_{C^{k, \alpha}([0, T])} = \begin{cases} \sum_{j=0}^k \sup_{t \in [0, T]} |f^{(j)}(t)| + \sup_{t, s \in [0, T], t \neq s} \frac{|f^{(k)}(t) - f^{(k)}(s)|}{|t - s|^\alpha} & \text{if } \alpha > 0, \\ \sum_{j=0}^k \sup_{t \in [0, T]} |f^{(j)}(t)| & \text{if } \alpha = 0 \end{cases}$$

for $f \in C^{k, \alpha}([0, T])$. Then we can verify the following fact.

Lemma 2.1. *Let $f(t)$ be a real valued function of $C^{k,\alpha}$ on some compact interval $I \subset \mathbf{R}$, with positive integer k and $0 \leq \alpha \leq 1$, and suppose that $f(t) \geq 0$ on I . Then the function $f^{1/(k+\alpha)}(t)$ is absolutely continuous on I and moreover*

$$\|(f^{1/(k+\alpha)})'\|_{L^1(I)}^{k+\alpha} \leq C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}.$$

We refer to Lemma 1 in F. Colombini-E. Jannelli-S. Spagnolo [2] for the proof of Lemma 2.1. Recently, Professor S. Tarama (of Kyoto University) pointed out that the positivity of $f(t)$ in Lemma 2.1 can be removed by considering $\sqrt{f(t)^2 + \varepsilon}$ ($\varepsilon > 0$) instead of $f(t)$.

Now we start the proof of Theorem 1.1. Let us recall the present case that $L(t, x, \partial_t, \partial_x)$ in (1.1) is replaced by $L(t, \partial_t, \partial_x)$. So, operating the Fourier transform in $x \rightarrow \xi$ to (1.1), we get the resulting Cauchy problem for linear ordinary differential equations in $t \in [0, T]$ with parameter $\xi \in \mathbf{R}^n$

$$(2.1) \quad \begin{cases} \partial_t^2 \hat{u}(t, \xi) + a_2(t, \xi) \hat{u}(t, \xi) + ia_1(t, \xi) \hat{u}(t, \xi) = 0, \\ \hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi). \end{cases}$$

Here, assume that there exists a real valued weight function $A(t, \xi) \in C_t^1([0, T]; C^\infty(\mathbf{R}_\xi^n))$ fulfilling

$$(2.2) \quad |A(t, \xi)| \leq C' \langle \xi \rangle^{1/\kappa} \quad \text{for } (t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$$

and set $v(t, \xi) = e^{A(t, \xi)} \hat{u}(t, \xi)$. Then (2.1) is transformed into the Cauchy problem on $[0, T]$

$$(2.3) \quad \begin{cases} (\partial_t - A_t)^2 v(t, \xi) + a_2(t, \xi) v(t, \xi) + ia_1(t, \xi) v(t, \xi) = 0, \\ v(0, \xi) = e^{A(0, \xi)} \hat{u}_0(\xi) = v_0(\xi), \\ (\partial_t - A_t) v(t, \xi)|_{t=0} = e^{A(0, \xi)} \hat{u}_1(\xi) = v_1(\xi). \end{cases}$$

We introduce the energy function associated with (2.3)

$$E(t, \xi) = \frac{1}{2} \{ |(\partial_t - A_t) v(t, \xi)|^2 + (a_2(t, \xi) + \lambda(t, \xi)) |v(t, \xi)|^2 \}.$$

Noting that $E(t, \xi) \leq E(0, \xi)$ if $dE(t, \xi)/dt \leq 0$, we have from (1.4)

$$|(\partial_t - A_t)^j v(t, \xi)| \leq C_1(\langle \xi \rangle |v_0(\xi)| + |v_1(\xi)|) \quad (j = 0, 1),$$

in the sequel, by (1.3) and (2.2)

$$\begin{aligned} |\hat{u}(t, \xi)| &= |e^{-A(t, \xi)} v(t, \xi)| \\ &\leq C'_1 e^{-A(t, \xi)} \langle \xi \rangle^{-(\rho/2)} (\langle \xi \rangle |v_0(\xi)| + |v_1(\xi)|) \\ &= C'_1 e^{A(0, \xi) - A(t, \xi)} \langle \xi \rangle^{-(\rho/2)} (\langle \xi \rangle |\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|) \\ &\leq C'_2 e^{C'' \langle \xi \rangle^{1/\kappa}} (|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|). \end{aligned}$$

Since $u_0, u_1 \in L^2_{\kappa, \infty}(\mathbf{R}^n)$, $e^{\rho \langle \xi \rangle^{1/\kappa}} \hat{u}(t, \xi) \in L^2(\mathbf{R}^n_\xi)$ for any $\rho > 0$ and hence (1.1) admits a unique solution $u(t, x) \in C^2([0, T]; L^2_{\kappa, \infty}(\mathbf{R}^n_x))$ via the inverse Fourier transform of $\hat{u}(t, \xi)$ with respect to $\xi \in \mathbf{R}^n$.

Thus it remains to verify that $dE(t, \xi)/dt \leq 0$ for some $A(t, \xi)$ with (2.2). Hereafter we abbreviate the energy function $E(t, \xi)$ to

$$E(t) = \frac{1}{2} \{ |(\partial_t - A_t)v(t)|^2 + (a_2 + \lambda)|v(t)|^2 \},$$

$a'_2 = da_2/dt$ and so on. Differentiating $E(t)$ and using (2.3), we see that

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2} \{ (a'_2 + \lambda')|v(t)|^2 \} + (a_2 + \lambda) \operatorname{Re}(v'(t) \cdot \overline{v(t)}) \\ &\quad + \operatorname{Re}(((\partial_t - A_t)v(t))' \cdot \overline{(\partial_t - A_t)v(t)}) \\ &= \frac{1}{2} (a'_2 + \lambda')|v(t)|^2 + (a_2 + \lambda) \{ \operatorname{Re}((\partial_t - A_t)v(t) \cdot \overline{v(t)}) + A_t|v(t)|^2 \} \\ &\quad + \operatorname{Re}((\partial_t - A_t)^2 v(t) \cdot \overline{(\partial_t - A_t)v(t)}) + A_t|(\partial_t - A_t)v(t)|^2 \\ &= \frac{1}{2} (a'_2 + \lambda')|v(t)|^2 + (a_2 + \lambda) \{ \operatorname{Re}(v(t) \cdot \overline{(\partial_t - A_t)v(t)}) + A_t|v(t)|^2 \} \\ &\quad - \operatorname{Re}((a_2 + ia_1)v(t) \cdot \overline{(\partial_t - A_t)v(t)}) + A_t|(\partial_t - A_t)v(t)|^2 \\ &= \frac{1}{2} (a'_2 + \lambda')|v(t)|^2 + \operatorname{Re}((\lambda - ia_1)v(t) \cdot \overline{(\partial_t - A_t)v(t)}) \\ &\quad + A_t \{ (a_2 + \lambda)|v(t)|^2 + |(\partial_t - A_t)v(t)|^2 \} \\ &\leq \frac{1}{2} (a'_2 + \lambda')|v(t)|^2 + \frac{|\lambda - ia_1|^2 |v(t)|^2}{2|A_t|} + \frac{1}{2} |A_t| |(\partial_t - A_t)v(t)|^2 \\ &\quad + A_t \{ (a_2 + \lambda)|v(t)|^2 + |(\partial_t - A_t)v(t)|^2 \} \\ &\leq \frac{1}{2} (a'_2 + \lambda')|v(t)|^2 + A_t(a_2 + \lambda)|v(t)|^2 + \frac{|\lambda - ia_1|^2 |v(t)|^2}{2|A_t|} + \frac{1}{2} A_t |(\partial_t - A_t)v(t)|^2 \end{aligned}$$

when $A_t < 0$. Therefore, if we take A like $A_t < 0$ and

$$(a'_2 + \lambda') + 2A_t(a_2 + \lambda) + \frac{|\lambda - ia_1|^2}{|A_t|} \leq 0,$$

then $dE(t)/dt \leq 0$. For this purpose it suffices to establish the inequalities

$$\begin{cases} a'_2 + \lambda' + A_t(a_2 + \lambda) \leq 0, \\ A_t(a_2 + \lambda) + \frac{|\lambda - ia_1|^2}{|A_t|} \leq 0 \end{cases}$$

together with $A_t < 0$. So we may choose

$$A(t, \xi) = - \int_0^t \left(\frac{|a_2'(s, \xi) + \lambda'(s, \xi)|}{a_2(s, \xi) + \lambda(s, \xi)} + \frac{|\lambda(s, \xi) - ia_1(s, \xi)|}{\sqrt{a_2(s, \xi) + \lambda(s, \xi)}} + 1 \right) ds.$$

Up to now, to complete the proof, we must check that this $A(t, \xi)$ satisfies (2.2). Obviously, it holds that

$$\begin{aligned} |A(t, \xi)| &\leq \int_0^t \frac{|a_2'(s, \xi) + \lambda'(s, \xi)|}{a_2(s, \xi) + \lambda(s, \xi)} ds + \int_0^t \frac{|\lambda(s, \xi) - ia_1(s, \xi)|}{\sqrt{a_2(s, \xi) + \lambda(s, \xi)}} ds + T \\ &= I_1(t, \xi) + I_2(t, \xi) + T. \end{aligned}$$

At first, from (1.5) $I_2(t, \xi) \leq C(T) \langle \xi \rangle^{1/\kappa}$ is valid for $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$. Next, pick $\xi = |\xi|\omega$ ($|\omega| = 1$) and $\tilde{\lambda}(t, \xi) = \lambda(t, \xi)|\xi|^{-2}$. Then, because of the homogeneity of a_2 in ξ , I_1 can be rewritten as

$$I_1(t, \xi) = \int_0^t \frac{|(a_2(s, \omega) + \tilde{\lambda}(s, \xi))_s|}{a_2(s, \omega) + \tilde{\lambda}(s, \xi)} ds.$$

Here, taking account of

$$|((a_2(s, \omega) + \tilde{\lambda}(s, \xi))^{1/k})_s| = \frac{1}{k} (a_2(s, \omega) + \tilde{\lambda}(s, \xi))^{1/k} \frac{|(a_2(s, \omega) + \tilde{\lambda}(s, \xi))_s|}{a_2(s, \omega) + \tilde{\lambda}(s, \xi)},$$

we can apply Lemma 2.1 to I_1 and then obtain

$$\begin{aligned} I_1(t, \xi) &= \int_0^t k (a_2(s, \omega) + \tilde{\lambda}(s, \xi))^{-1/k} |((a_2(s, \omega) + \tilde{\lambda}(s, \xi))^{1/k})_s| ds \\ &\leq C(k, T) \langle \xi \rangle^{(2-\rho)/k}. \end{aligned}$$

Indeed, by Lemma 2.1 with $f = a_2 + \tilde{\lambda}$ and $\alpha = 0$

$$\|(a_2(s, \omega) + \tilde{\lambda}(s, \xi))^{1/k}\|_{L^1([0, T])}^k \leq C'(k, T) \|a_2(s, \omega) + \tilde{\lambda}(s, \xi)\|_{C^k([0, T])},$$

where, from (1.4)

$$\begin{aligned} \|a_2(s, \omega)\|_{C^k([0, T])} &\leq \sup_{|\omega|=1} \sum_{j=0}^k \sup_{s \in [0, T]} |a_2^{(j)}(s, \omega)| < \infty, \\ \|\tilde{\lambda}(s, \xi)\|_{C^k([0, T])} &= \frac{1}{|\xi|^2} \sum_{j=0}^k \sup_{s \in [0, T]} |\lambda^{(j)}(s, \xi)| < \infty. \end{aligned}$$

Hence, if we take a natural number k large enough as $(2 - \rho)/k \leq 1/\kappa$, then

$$I_1(t, \xi) \leq C(k, T) \langle \xi \rangle^{1/\kappa}$$

for $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$ and we now finish the proof.

§ 3. Concrete example via the symbols

We shall mention the proof of Theorem 1.2 in this section. Let us first define $\lambda(t, \xi) = \langle \xi \rangle^\rho$, where $\rho \in (0, 2]$ will be determined later. Then our task is to find some $(1 \leq) \kappa \leq (2\ell - \nu)/(\ell - \nu - 1)$ such that

$$(3.1) \quad \int_0^T \frac{|\lambda(t, \xi) - ia_1(t, \xi)|}{\sqrt{a_2(t, \xi) + \lambda(t, \xi)}} dt \leq C(T) \langle \xi \rangle^{1/\kappa}$$

for all $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n$. For this aim, come up with

$$\begin{aligned} \int_0^T \frac{|\lambda - ia_1|}{\sqrt{a_2 + \lambda}} dt &\leq \int_0^T \frac{|\lambda|}{\sqrt{a_2 + \lambda}} dt + \int_0^T \frac{|a_1|}{\sqrt{a_2 + \lambda}} dt \\ &= J_1(T, \xi) + J_2(T, \xi). \end{aligned}$$

As to J_1 , from (1.6) and (1.7)

$$\begin{aligned} J_1(T, \xi) &\leq \int_0^T \frac{\langle \xi \rangle^\rho}{\sqrt{\delta t^{2\ell} |\xi|^2 + \langle \xi \rangle^\rho}} dt = \int_0^T \frac{\langle \xi \rangle^{\rho/2}}{\sqrt{\delta t^{2\ell} |\xi|^2 \langle \xi \rangle^{-\rho} + 1}} dt \\ &= \delta^{-(1/2\ell)} \ell^{-1} |\xi|^{-(1/\ell)} \langle \xi \rangle^{(\rho/2\ell) + (\rho/2)} \int_0^{\sqrt{\delta} T^\ell |\xi| \langle \xi \rangle^{-\rho/2}} \frac{s^{(1/\ell)-1}}{\sqrt{s^2 + 1}} ds \\ &\leq \delta^{-(1/2\ell)} \ell^{-1} |\xi|^{-(1/\ell)} \langle \xi \rangle^{(\rho/2\ell) + (\rho/2)} \int_0^\infty \frac{s^{(1/\ell)-1}}{\sqrt{s^2 + 1}} ds. \end{aligned}$$

So, if $-1/\ell + \rho/(2\ell) + \rho/2 \leq 1/\kappa$, equivalently

$$(3.2) \quad \rho \leq \frac{2(\kappa + \ell)}{(\ell + 1)\kappa},$$

then $J_1(T, \xi) \leq \tilde{C} \langle \xi \rangle^{1/\kappa}$ for $\xi \in \mathbf{R}^n$. Meanwhile, as for J_2 , from (1.7)

$$\begin{aligned} J_2(T, \xi) &\leq \int_0^T \frac{C\{t^\nu |\xi| + \sqrt{a_2(t, \xi)}\}}{\sqrt{a_2(t, \xi) + \langle \xi \rangle^\rho}} dt \\ &= C \int_0^T \frac{t^\nu |\xi|}{\sqrt{a_2 + \langle \xi \rangle^\rho}} dt + C \int_0^T \frac{\sqrt{a_2}}{\sqrt{a_2 + \langle \xi \rangle^\rho}} dt \\ &= CJ_{21}(T, \xi) + CJ_{22}(T, \xi). \end{aligned}$$

It is clear that $J_{22}(T, \xi) \leq T$ for $\xi \in \mathbf{R}^n$. Therefore it is sufficient to evaluate J_{21} only. In view of (1.6)

$$\begin{aligned}
 J_{21}(T, \xi) &\leq \int_0^T \frac{t^v |\xi|}{\sqrt{\delta t^{2\ell} |\xi|^2 + \langle \xi \rangle^\rho}} dt = \int_0^T \frac{t^v |\xi| \langle \xi \rangle^{-(\rho/2)}}{\sqrt{\delta t^{2\ell} |\xi|^2 \langle \xi \rangle^{-\rho} + 1}} dt \\
 &= \delta^{-(1/2\ell) - (v/2\ell)} \ell^{-1} |\xi|^{1 - ((v+1)/\ell)} \langle \xi \rangle^{(\rho(v+1)/2\ell) - (\rho/2)} \int_0^{\sqrt{\delta} T^\ell |\xi| \langle \xi \rangle^{-\rho/2}} \\
 &\quad \times \frac{s^{(1/\ell) - 1 + (v/\ell)}}{\sqrt{s^2 + 1}} ds \\
 &\leq \delta^{-(1/2\ell) - (v/2\ell)} \ell^{-1} |\xi|^{1 - ((v+1)/\ell)} \langle \xi \rangle^{(\rho(v+1)/2\ell) - (\rho/2)} \int_0^\infty \frac{s^{(1/\ell) - 1 + (v/\ell)}}{\sqrt{s^2 + 1}} ds.
 \end{aligned}$$

So, if $1 - (v+1)/\ell + \rho(v+1)/(2\ell) - \rho/2 \leq 1/\kappa$, equivalently

$$(3.3) \quad \rho \geq \frac{2\{(\ell - v - 1)\kappa - \ell\}}{(\ell - v - 1)\kappa},$$

then $J_{21}(T, \xi) \leq \tilde{C} \langle \xi \rangle^{1/\kappa}$ for $\xi \in \mathbf{R}^n$. Thus, for the existence of ρ (>0) satisfying (3.1) with $\lambda(t, \xi) = \langle \xi \rangle^\rho$, from (3.2) and (3.3) the inequality

$$\frac{2\{(\ell - v - 1)\kappa - \ell\}}{(\ell - v - 1)\kappa} \leq \frac{2(\kappa + \ell)}{(\ell + 1)\kappa},$$

that is, $\kappa \leq (2\ell - v)/(\ell - v - 1)$ has to be valid and we complete the proof now. $\square$

In addition, via the same argument as in the proof of Theorem 1.2, we can confirm that if $\sup_{|\xi|=1} \int_0^T |a_1(t, \xi)|/a_2(t, \xi)^\beta dt < \infty$ for some $\beta \in [0, 1/2)$, then (1.5) is fulfilled provided $\kappa \leq 1 + 1/(1 - 2\beta)$. This fact prescribes that Theorem 1.1 is an improvement of a result due to F. Colombini-E. Jannelli-S. Spagnolo [2]. In fact, as in the proof of Theorem 1.2, select $\lambda(t, \xi) = \langle \xi \rangle^\rho$. Then

$$\begin{aligned}
 \int_0^T \frac{|a_1|}{\sqrt{a_2 + \lambda}} dt &= \int_0^T \frac{|a_1|}{(a_2 + \lambda)^\beta (a_2 + \lambda)^{(1/2) - \beta}} dt \\
 &\leq \left(\frac{1}{\lambda}\right)^{(1/2) - \beta} \int_0^T \frac{|a_1(t, \omega)| |\xi|}{a_2(t, \omega)^\beta |\xi|^{2\beta}} dt \\
 &\leq C(T) \left(\frac{1}{\lambda}\right)^{(1/2) - \beta} |\xi|^{1 - 2\beta} = C(T) \langle \xi \rangle^{-\rho((1/2) - \beta) + 1 - 2\beta},
 \end{aligned}$$

where $\xi = |\xi|\omega$ as before, of course. On the other hand,

$$\int_0^T \frac{|\lambda|}{\sqrt{a_2 + \lambda}} dt = \int_0^T \frac{\sqrt{\lambda}\sqrt{\lambda}}{\sqrt{a_2 + \lambda}} dt \leq \int_0^T \sqrt{\lambda} dt = T\langle \xi \rangle^{\rho/2}.$$

These inequalities lead us to the condition

$$1 - \frac{1}{(1 - 2\beta)\kappa} \leq \frac{1}{\kappa},$$

namely $\kappa \leq 1 + 1/(1 - 2\beta)$ for the existence of ρ (> 0). In particular, in the case of Ivrii's example

$$\sup_{|\xi|=1} \int_0^T \frac{|a_1|}{a_2^\beta} dt = \int_0^T \frac{t^\nu}{t^{2\ell\beta}} dt < \infty$$

as long as $\nu - 2\ell\beta > -1$, which is equivalent to

$$1 + \frac{1}{1 - 2\beta} < 1 + \frac{\ell}{\ell - \nu - 1} = \frac{2\ell - \nu - 1}{\ell - \nu - 1}.$$

Hence the range of Gevrey exponent κ in Theorem 1.1 is wider than that in F. Colombini-E. Jannelli-S. Spagnolo [2].

§ 4. Space-variable dependent case

We first prepare a proposition on asymptotic expansions of the symbols of pseudo-differential operators. Stand for by the symbol class $\gamma^\kappa S^m$ ($\kappa > 0, m \in \mathbf{R}$) the set of all functions $p(x, \xi) \in C^\infty(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$ such that for any $r > 0$ there exists a constant $C_r > 0$ satisfying

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leq C_r r^{|\alpha+\beta|} |\alpha + \beta|!^\kappa \langle \xi \rangle_h^{m-|\alpha|}$$

for every multi-indices $\alpha, \beta \in \mathbf{Z}_+^n$. Then we enjoy the following fact.

Lemma 4.1 (see Theorem 6.12 in K. Kajitani-T. Nishitani [9]). *Let $p \in \gamma^\kappa S^m$ and A as in (1.9). Then the symbol $\sigma(e^{MA} p e^{-MA})(t, x, \xi)$ of the pseudo-differential operator $e^{MA}(t, D_x) p(t, x, D_x) e^{-MA}(t, D_x)$ can be expanded in the following forms.*

- (i) $\sigma(e^{MA} p e^{-MA})(t, x, \xi) = p(t, x, \xi) + p^1(t, x, \xi) + r^2(p)(t, x, \xi)$, where $p^1(t, x, \xi) = -iM \sum_{j=1}^n p_{x_j}(t, x, \xi) A_{\xi_j}(t, \xi)$ and $r^2(p) \in \gamma^\kappa S^{m-2(1-1/\kappa)}$.
- (ii) $\sigma(e^{MA} p e^{-MA})(t, x, \xi) = p(t, x, \xi) + p^1(t, x, \xi) + p^2(t, x, \xi) + r^3(p)(t, x, \xi)$, where $p^2(t, x, \xi) = \sum_{j,k=1}^n p_{x_j x_k}(t, x, \xi) (M^2 A_{\xi_j}(t, \xi) A_{\xi_k}(t, \xi) - M A_{\xi_j \xi_k}(t, \xi))$ and $r^3(p) \in \gamma^\kappa S^{m-3(1-1/\kappa)}$.

For the proof of Theorem 1.4 it is enough to derive the energy estimate (1.8). So we shall endeavor to establish (1.8) from now on.

If we transform $u(t, x)$ into $v(t, x)$ by

$$v(t, x) = e^{MA}(t, D_x)u(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi + MA(t, \xi)} \hat{u}(t, \xi) d\xi,$$

then

$$(4.1) \quad u_t(t, x) = e^{-MA}(t, D_x)(\partial_t - MA_t(t, D_x))v(t, x)$$

holds. On account of $u_t(t, x) = e^{-MA}(t, D_x)(\partial_t - MA_t(t, D_x))e^{MA}(t, D_x)u(t, x)$

$$(4.2) \quad u_{tt}(t, x) = e^{-MA}(t, D_x)(\partial_t - MA_t(t, D_x))^2 v(t, x)$$

is also true. Hereforth we indicate $\tilde{\partial}_t = \partial_t - MA_t(t, D_x)$ and $p_{MA}(t, x, D_x) = e^{MA}(t, D_x)p(t, x, D_x)e^{-MA}(t, D_x)$ in brief. Then, from (4.2) the equation (1.1) is changed into

$$(4.3) \quad \tilde{\partial}_t^2 v(t, x) = -A_{2MA}(t, x, D_x)v(t, x) - iA_{1MA}(t, x, D_x)v(t, x),$$

where, by use of Lemma 4.1

$$(4.4) \quad A_{2MA}(t, x, \xi) = A_2(t, x, \xi) + A_2^1(t, x, \xi) + A_2^2(t, x, \xi) + r^3(A_2)(t, x, \xi),$$

$$(4.5) \quad A_{1MA}(t, x, \xi) = A_1(t, x, \xi) + A_1^1(t, x, \xi) + r^2(A_1)(t, x, \xi).$$

Now we define the energy function corresponding to (4.3)

$$E(t) = \frac{1}{2} \{ \|\tilde{\partial}_t v(t, x)\|_{L^2(\mathbb{R}_x^n)}^2 + ((A_2(t, x, D_x) + \lambda(D_x))v(t, x), v(t, x))_{L^2(\mathbb{R}_x^n)} \},$$

where $A_2(t, x, D_x) = \sum_{j,k=1}^n D_{x_j} a_{jk}(t, x) D_{x_k}$ and $\lambda(D_x) = \langle D_x \rangle_h^\rho$ with some $\rho > 0$ which will be taken later. Differentiating $E(t)$, we have

$$(4.6) \quad \begin{aligned} \frac{d}{dt} E(t) &= \operatorname{Re}(\tilde{\partial}_t^2 v(t), \tilde{\partial}_t v(t))_{L^2} + M \operatorname{Re}(A_t \tilde{\partial}_t v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\quad + \frac{1}{2} ((\partial_t A_2)v(t), v(t))_{L^2} + \operatorname{Re}(A_2 v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\quad + M \operatorname{Re}((A_2 + \lambda)A_t v(t), v(t))_{L^2} + \operatorname{Re}(\lambda \tilde{\partial}_t v(t), v(t))_{L^2}. \end{aligned}$$

As well, by means of (4.3)–(4.5)

$$(4.7) \quad \begin{aligned} \operatorname{Re}(\tilde{\partial}_t^2 v(t), \tilde{\partial}_t v(t))_{L^2} &= -\operatorname{Re}(A_2 v(t), \tilde{\partial}_t v(t))_{L^2} - \operatorname{Re}(A_2^1 v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\quad - \operatorname{Re}(A_2^2 v(t), \tilde{\partial}_t v(t))_{L^2} - \operatorname{Re}(iA_1 v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\quad + \operatorname{Im}(A_1^1 v(t), \tilde{\partial}_t v(t))_{L^2} - \operatorname{Re}(r^3(A_2)v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\quad + \operatorname{Im}(r^2(A_1)v(t), \tilde{\partial}_t v(t))_{L^2}. \end{aligned}$$

In the following, according to H. Ishida [4], we shall estimate the respective terms in the right hand side of (4.6) in aid of (4.7). Then we shall adopt the inequality

$$|\partial_\xi^\alpha A(t, \xi)| \leq C r^{|\alpha|} |\alpha|! \langle \xi \rangle_h^{(1/\kappa) - |\alpha|}$$

for every multi-index $\alpha \in \mathbf{Z}_+^n$ without citation. We should note that the positive continuous functions $C(t), C_1(t), \dots, C_6(t)$ on $[0, T]$ appeared below do not depend on the parameters h and M . In advance, since $0 \leq \sigma_0(A_2)(t, x, \xi) = t^{2\ell} b(x, \xi) + c(x, \xi) \in \gamma^\kappa S^2$ ($\sigma_0(A_2)$ means the principal symbol of A_2), the Fefferman-Phong inequality (see C. Fefferman-D. H. Phong [3]) yields

$$\operatorname{Re}(A_2(-A_t)v(t), v(t))_{L^2} \geq -C(t) \| |A_t|^{1/2} v(t) \|_{L^2}^2,$$

so that we retain

$$(4.8) \quad \operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2} + C(t) \| |\lambda A_t|^{1/2} v(t) \|_{L^2}^2 \geq 0.$$

To start with, from (4.4)

$$\begin{aligned} -\operatorname{Re}(A_2^1 v(t), \tilde{\partial}_t v(t))_{L^2} &= -\sum_{j=1}^n \operatorname{Im}(M A_{2x_j} A_{\xi_j} v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\leq \frac{1}{2} n \left(\max_{j=1, \dots, n} \|M |A_t|^{-1/2} A_{2x_j} A_{\xi_j} v(t)\|_{L^2}^2 + \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2 \right). \end{aligned}$$

Here, owing to $|\partial_{x_j} A_2(t, x, \xi)| \leq C \sqrt{A_2(t, x, \xi) + \lambda(\xi)} |\xi|$, we find

$$\begin{aligned} &\|M |A_t|^{-1/2} A_{2x_j} A_{\xi_j} v(t)\|_{L^2}^2 \\ &\leq C_1(t) \{ \operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2} + C(t) \| |\lambda A_t|^{1/2} v(t) \|_{L^2}^2 \} \end{aligned}$$

if $h \geq M$. Thus we get

$$\begin{aligned} (4.9) \quad &-\operatorname{Re}(A_2^1 v(t), \tilde{\partial}_t v(t))_{L^2} - \frac{1}{2} n \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2 \\ &\leq n C_1(t) \{ \operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2} + C(t) \| |\lambda A_t|^{1/2} v(t) \|_{L^2}^2 \}. \end{aligned}$$

Also, similarly to the above

$$\begin{aligned} &-\operatorname{Re}(A_2^2 v(t), \tilde{\partial}_t v(t))_{L^2} \\ &= \sum_{j,k=1}^n \operatorname{Re}(A_{2x_j x_k} (M A_{\xi_j \xi_k} - M^2 A_{\xi_j} A_{\xi_k}) v(t), \tilde{\partial}_t v(t))_{L^2} \\ &\leq \frac{1}{2} n^2 \left\{ \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2 + \max_{j=1, \dots, n} (\|M |A_t|^{-1/2} A_{2x_j x_k} A_{\xi_j \xi_k} v(t)\|_{L^2}^2 \right. \\ &\quad \left. + \|M^2 |A_t|^{-1/2} A_{2x_j x_k} A_{\xi_j} A_{\xi_k} v(t)\|_{L^2}^2) \right\}. \end{aligned}$$

Here, by virtue of $|\partial_{x_j x_k} A_2(t, x, \xi)| \leq C|\xi|^2$ and $\rho \geq 2/\kappa$ (as can be seen soon)

$$\begin{aligned} & \|M|A_t|^{-1/2} A_{2x_j x_k} A_{\xi_j \xi_k} v(t)\|_{L^2}^2 + \|M^2|A_t|^{-1/2} A_{2x_j x_k} A_{\xi_j} A_{\xi_k} v(t)\|_{L^2}^2 \\ & \leq C_2(t) \|\langle D_x \rangle_h^{3/(2\kappa)} v(t)\|_{L^2}^2 \leq C_3(t) \|\lambda A_t|^{1/2} v(t)\|_{L^2}^2 \end{aligned}$$

when $h \geq M$. Therefore we obtain

$$\begin{aligned} (4.10) \quad & -\operatorname{Re}(A_2^2 v(t), \tilde{\partial}_t v(t))_{L^2} - \frac{1}{2} n^2 \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2 \\ & \leq n^2 C_3(t) \|\lambda A_t|^{1/2} v(t)\|_{L^2}^2. \end{aligned}$$

By the way, some simple calculations show that if we choose $\rho = 2/\kappa + 2\sigma$, then there exists a constant $C(T) > 0$ fulfilling

$$\begin{cases} \partial_t A_2(t, x, \xi) + C(T)(A_2(t, x, \xi) + \lambda(\xi))A_t(t, \xi) \leq 0, \\ \frac{|\lambda(\xi) - iA_1(t, x, \xi)|^2}{|A_t(t, \xi)|} + C(T)(A_2(t, x, \xi) + \lambda(\xi))A_t(t, \xi) \leq 0 \end{cases}$$

with $\lambda(\xi) = \langle \xi \rangle_h^\rho$. In consequence, we know

$$\begin{aligned} (4.11) \quad & \frac{1}{2} ((\partial_t A_2)v(t), v(t))_{L^2} - \operatorname{Re}(iA_1 v(t), \tilde{\partial}_t v(t))_{L^2} + \operatorname{Re}(\lambda \tilde{\partial}_t v(t), v(t))_{L^2} \\ & = \frac{1}{2} ((\partial_t A_2)v(t), v(t))_{L^2} + \operatorname{Re}(|A_t|^{-1/2}(\lambda - iA_1)v(t), |A_t|^{1/2} \tilde{\partial}_t v(t))_{L^2} \\ & \leq C_4(t) \{\operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2} + C(t) \|\lambda A_t|^{1/2} v(t)\|_{L^2}^2 \\ & \quad + \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2\}. \end{aligned}$$

Further, thanks to $|\partial_{x_j} A_1(t, x, \xi)| \leq C|\xi|$ and $\rho \geq 2/\kappa$

$$\begin{aligned} (4.12) \quad & \operatorname{Im}(A_1^1 v(t), \tilde{\partial}_t v(t))_{L^2} = -M \sum_{j=1}^n \operatorname{Re}(A_{1x_j} A_{\xi_j} v(t), \tilde{\partial}_t v(t))_{L^2} \\ & \leq \frac{1}{2} nM (C_5(t) \|\lambda^{1/2} v(t)\|_{L^2}^2 + \|\tilde{\partial}_t v(t)\|_{L^2}^2). \end{aligned}$$

Besides, from the boundedness of the symbols $r^3(A_2)(t, x, \xi)$ and $r^2(A_1)(t, x, \xi)$ it is evident that

$$\begin{aligned} (4.13) \quad & -\operatorname{Re}(r^3(A_2)v(t), \tilde{\partial}_t v(t))_{L^2} + \operatorname{Im}(r^2(A_1)v(t), \tilde{\partial}_t v(t))_{L^2} \\ & \leq C_6(t) \|v(t)\|_{L^2}^2 + \|\tilde{\partial}_t v(t)\|_{L^2}^2. \end{aligned}$$

Now, by (4.8) it is reminded that

$$(4.14) \quad M \operatorname{Re}((A_2 + \lambda)A_t v(t), v(t))_{L^2} \\ \leq -\frac{M}{2} \{ \operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2}^2 + C(t) \| |\lambda A_t|^{1/2} v(t) \|_{L^2}^2 \},$$

$$(4.15) \quad M \operatorname{Re}(A_t \tilde{\partial}_t v(t), \tilde{\partial}_t v(t))_{L^2} = -M \| (-A_t)^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2 \\ = -M \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2.$$

Up to now, summing up (4.9)–(4.11) and (4.14)–(4.15), we deduce

$$(4.16) \quad -\operatorname{Re}(A_2^1 v(t), \tilde{\partial}_t v(t))_{L^2} - \operatorname{Re}(A_2^2 v(t), \tilde{\partial}_t v(t))_{L^2} + \frac{1}{2} ((\partial_t A_2) v(t), v(t))_{L^2} \\ - \operatorname{Re}(i A_1 v(t), \tilde{\partial}_t v(t))_{L^2} + \operatorname{Re}(\lambda \tilde{\partial}_t v(t), v(t))_{L^2} \\ + M \operatorname{Re}((A_2 + \lambda)A_t v(t), v(t))_{L^2} + M \operatorname{Re}(A_t \tilde{\partial}_t v(t), \tilde{\partial}_t v(t))_{L^2} \\ \leq \left(-\frac{M}{2} + n C_1(t) + n^2 C_3(t) + C_4(t) \right) \\ \times \{ \operatorname{Re}((A_2 + \lambda)(-A_t)v(t), v(t))_{L^2}^2 + C(t) \| |\lambda A_t|^{1/2} v(t) \|_{L^2}^2 \} \\ + \left(-M + \frac{1}{2} n + \frac{1}{2} n^2 + C_4(t) \right) \| |A_t|^{1/2} \tilde{\partial}_t v(t) \|_{L^2}^2.$$

Hence, if we determine

$$h \geq \max\{M, 1\},$$

$$M \geq 2 \sup_{t \in [0, T]} (n C_1(t) + n + n^2 C_3(t) + n^2 + C_4(t))$$

and $c(t) = nM(C_5(t) + 1) + 2(C_6(t) + 1)$, then it follows from (4.6)–(4.7), (4.12)–(4.13) and (4.16) that

$$\frac{d}{dt} E(t) \leq c(t) E(t)$$

is valid for $t \in [0, T]$, which leads to by Gronwall's inequality

$$(4.17) \quad E(t) \leq e^{\int_0^t c(s) ds} E(0)$$

also holds on $[0, T]$. Here, (4.1) prompts us the relation

$$(4.18) \quad E(t) \geq \frac{1}{2} (\|v(t)\|_{L^2}^2 + \|\tilde{\partial}_t v(t)\|_{L^2}^2) = \frac{1}{2} (\|e^{MA} u(t)\|_{L^2}^2 + \|e^{MA} u_t(t)\|_{L^2}^2).$$

While, because $A_2(t, x, \xi) + \lambda(\xi) \leq C(T) \langle \xi \rangle_h^{\max\{2, \rho\}} = C(T) \langle \xi \rangle_h^2$ (if we may assume $\kappa \geq 2$ without loss of generality due to M. D. Bronshtein [1]),

$$(4.19) \quad E(0) \leq \frac{C(T)}{2} (\|e^{MA} u(0)\|_{H^1}^2 + \|e^{MA} u_t(0)\|_{L^2}^2).$$

Thus we have established the energy estimate (1.8) from (4.17)–(4.19). Finally, it is a standard procedure that the wellposedness of the Cauchy problem (1.1) in $L^2_{\kappa, \infty}(\mathbf{R}^n)$ is derived from (1.8) and so we omit it.

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