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Some Results of Blow-Up and Formation of Singularities for Non-Strictly Hyperbolic Equations

By

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Introduction

In this paper we give some results of blow-up for non-strictly hyperbolic systems of the form

$$(0.1) \quad \begin{cases} v_t = b(v)v_x + w_x \\ w_t = a(v)v_x + f(v) \end{cases}$$

in $[0, +\infty) \times \mathbf{R}$, which derive from the quasi-linear wave equations

$$(0.2) \quad u_{tt} - a(u_x)u_{xx} = f(u_x)$$

or

$$(0.3) \quad u_{tt} - (a(u)u_x)_x - (b(u)u_x)_t = 0,$$

with, respectively,

$$a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}$$

or

$$b(\eta)^2 + 4a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}.$$

The strictly hyperbolic case was largely investigated by many authors ([L2], [KT], [L], [J1], [TK], [TZK]), under the assumption of “genuine non-linearity” (cf. [L1]).

Assuming only a weak non-linearity condition, the formation of singularities was proved in [KM], [CD1], [CD2], [M2].

Non-strictly hyperbolic systems were instead considered in [KK], [S] and [SV], proving the formation of singularities in some particular cases.

F. John considered in [J1] strictly hyperbolic genuinely nonlinear systems, and proved the formation of singularities for solutions of the corresponding Cauchy problem with small compactly supported initial data.

S. Klainerman and A. Majda in [KM] considered equation (0.2) in the homogeneous ($f \equiv 0$) strictly hyperbolic case, weakening the assumption of

genuinely nonlinearity by requiring that there exists an integer $p \geq 2$ for which $a^{(p)}(0) \neq 0$. They obtained blow-up results for solutions of the corresponding Cauchy problem with small (essentially) periodic initial data. They also gave an estimate of the order of the lifespan of the solution in terms of p . In [CD1] and [CD2] F. Colombini and D. Del Santo generalized the results of [KM] in the case of odd p . In [M2] other generalizations of [KM] were considered, dropping the assumption of small initial data.

In this paper we give up strictly hyperbolicity, genuinely nonlinearity and small initial data. We first consider in §1 (see Theorem 1.1) the Cauchy problem

$$(0.4) \quad \begin{cases} u_{tt} - (a(u)u_x)_x = 0 \\ u(0, x) = \varphi(x) \in C^2(\mathbf{R}) \\ u_t(0, x) = \psi(x) \in C^1(\mathbf{R}) \end{cases}$$

for $a \in C^2(\mathbf{R})$ such that

$$a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, \quad a(0) = 0$$

$$\lim_{u \rightarrow \pm\infty} \int_0^u \sqrt{a(\eta)} d\eta = \pm\infty.$$

We prove that a C^2 solution $u(t, x)$ of (0.4) must develop a singularity if there exist $y_1, y_2 \in \mathbf{R}$ with $y_1 < y_2$ and

$$(0.5) \quad \varphi(y_i) = \psi(y_i) = 0, \quad i = 1, 2$$

$$(0.6) \quad \int_{y_1}^{y_2} \psi(x) dx \neq 0.$$

Note that conditions (0.5) and (0.6) are of course satisfied for initial data with compact support such that $\int_{\mathbf{R}} \psi(x) dx \neq 0$.

We generalize the above results to equations of the form (0.2) with $f \equiv 0$ (see Theorem 1.7), or (0.3) (see Theorem 1.8).

In §2 we start from an idea of R. Saxton in [S], showing the formation of singularities for solutions of the equation

$$(0.7) \quad u_{tt} - a(u_x)u_{xx} = f(u_x)$$

along degenerate characteristic curves $x_1(t, \alpha)$, for $a, f \in C^2(\mathbf{R})$ satisfying

$$a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, \quad a(0) = 0$$

$$a''(0) = \lambda > 0$$

$$f'(0) = 0, \quad f''(0) = \mu \in \mathbf{R},$$

if the initial data $u_{xx}(0, \alpha)$ and $u_{xxl}(0, \alpha)$ satisfy suitable conditions (see Theorem 2.1).

We obtain the same results, moreover, when a and f depend also on t and x , or when equation (0.7) is replaced by (0.3) with $b \equiv 0$ (see §2.1).

In §3 we improve a result of B. L. Keyfitz and H. C. Kranzer, who considered in [KK] some special kinds of weakly hyperbolic systems of the first order. Their results apply, in particular, to system (0.1) for $a(u) \approx u^2$, $b(u) \approx \gamma u$ and $f \equiv 0$.

More precisely, we study the Cauchy problem

$$(0.8) \quad \begin{cases} u_{tt} - a(u_x)u_{xx} = 0 \\ u(0, x) = \varphi(x) \in C^2(\mathbf{R}) \\ u_t(0, x) = \psi(x) \in C^1(\mathbf{R}) \end{cases}$$

for $a \in C^2(\mathbf{R})$ which satisfies for every $R > 0$:

$$a(0) = 0, \quad a(\eta) > 0 \quad \text{if } \eta \neq 0$$

$$a'(\eta) \geq C_R a(\eta)^{5/4} \quad \text{if } \eta \in [0, R]$$

$$a'(\eta) \leq -C_R a(\eta)^{5/4} \quad \text{if } \eta \in [-R, 0].$$

We prove (see Theorem 3.2) that if the initial data are not both identically zero and have compact support, or are periodic but not both constant, then a C^2 solution of (0.8) must develop a singularity in finite time.

The idea that we borrow from [KK] is that along a suitable non-degenerate characteristic curve we have blow-up of some second-order derivative of the solution of (0.8), and that, moreover, if this characteristic curve is non-degenerate at the time $t = 0$, then it will still be non-degenerate for all positive time. If we cannot find such a non-degenerate characteristic curve, we can reduce, however, to the case of §1, so that blow-up occurs in any case.

Finally, in §4 we give some local existence theorems (generalizing [M1]) for those weakly hyperbolic equations for which we gave the blow-up results in the previous sections (see Theorems 4.5 and 4.6).

1. Blow-up for the equation $u_{tt} - (a(u)u_x)_x = 0$

In this section we obtain some blow-up results for quasi-linear equations of the form

$$(1.1) \quad u_{tt} - (a(u)u_x)_x = 0 \quad \text{in } [0, +\infty) \times \mathbf{R}_x.$$

Assuming that (1.1) is weakly hyperbolic, that is $a(\eta) \geq 0$ for every $\eta \in \mathbf{R}$, we shall prove the following:

Theorem 1.1. *Let us consider the following Cauchy problem in $[0, +\infty) \times \mathbf{R}_x$:*

$$(1.2) \quad \begin{cases} u_{tt} - (a(u)u_x)_x = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x), \end{cases}$$

where $\varphi \in C^2(\mathbf{R})$, $\psi \in C^1(\mathbf{R})$ and $a \in C^2(\mathbf{R})$ satisfies:

$$(1.3) \quad a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, \quad a(0) = 0$$

$$(1.4) \quad \lim_{u \rightarrow \pm\infty} \int_0^u \sqrt{a(\eta)} d\eta = \pm\infty.$$

Assume, moreover, that there exist $y_1, y_2 \in \mathbf{R}$ with $y_1 < y_2$ and

$$(1.5) \quad \varphi(y_i) = \psi(y_i) = 0, \quad i = 1, 2$$

$$(1.6) \quad \int_{y_1}^{y_2} \psi(x) dx \neq 0.$$

Then the Cauchy problem (1.2) does not admit a global solution $u \in C^2([0, +\infty) \times [y_1, y_2])$.

Equation (1.1) is equivalent to the following first-order system:

$$(1.7) \quad \begin{cases} u_t - v_x = 0 \\ v_t - a(u)u_x = 0. \end{cases}$$

We can consider the *characteristic roots*

$$(1.8) \quad \lambda_1(u) = -\sqrt{a(u)}, \quad \lambda_2(u) = \sqrt{a(u)}$$

of system (1.7).

Remark 1.2. The characteristic roots $\lambda_1(\eta)$ and $\lambda_2(\eta)$ are locally Lipschitz continuous functions on $\mathbf{R}$, since $a \in C^2(\mathbf{R})$ and $a(\eta) \geq 0$ imply that

$$(1.9) \quad \forall R > 0 \quad \exists c_R > 0 : a'(\eta)^2 \leq c_R a(\eta) \quad \forall \eta \in [-R, R].$$

By Remark 1.2, if $u(t, x)$ is a classical solution of equation (1.1), then the functions

$$x \mapsto \lambda_i(u(t, x)), \quad i = 1, 2$$

are locally Lipschitz continuous on $\mathbf{R}$, and hence the *characteristic curves* $x_1(t, \alpha)$ and $x_2(t, \beta)$ are uniquely determined by the following Cauchy problems:

$$(1.10) \quad \begin{cases} x'_1 = \lambda_1(u(t, x_1)) \\ x_1(0, \alpha) = \alpha \end{cases}$$

and

$$(1.11) \quad \begin{cases} x_2' = \lambda_2(u(t, x_2)) \\ x_2(0, \beta) = \beta. \end{cases}$$

Let us now introduce the *Riemann invariants*

$$(1.12) \quad l(t, x) = l(u(t, x), v(t, x)) = \frac{1}{2} \left(v(t, x) - \int_0^{u(t, x)} \lambda_1(\eta) d\eta \right)$$

$$(1.13) \quad r(t, x) = r(u(t, x), v(t, x)) = \frac{1}{2} \left(v(t, x) + \int_0^{u(t, x)} \lambda_2(\eta) d\eta \right).$$

They satisfy the system

$$\begin{cases} l_t + \lambda_1(u) l_x = 0 \\ r_t + \lambda_2(u) r_x = 0, \end{cases}$$

and clearly

$$l(t, x_1(t, \alpha)) = l(0, \alpha) := l_0(\alpha)$$

$$r(t, x_2(t, \beta)) = r(0, \beta) := r_0(\beta).$$

We now prove the following:

Lemma 1.3. *Let $u(t, x)$ be a C^2 solution of the Cauchy problem*

$$(1.14) \quad \begin{cases} u_{tt} - (a(u)u_x)_x = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

in $[0, +\infty) \times \mathbf{R}_x$, where $\varphi \in C^2(\mathbf{R})$, $\psi \in C^1(\mathbf{R})$ and $a \in C^2(\mathbf{R})$ satisfies (1.3) and (1.4). Then, if there exist $y_1, y_2 \in \mathbf{R}$, with $y_1 < y_2$ satisfying (1.5), the solution $u(t, x)$ is uniformly bounded in the strip $[0, +\infty) \times [y_1, y_2]$.

Proof. Since $a(0) = a'(0) = 0$, the equation in (1.14), which we write as

$$u_{tt} = a(u)u_{xx} + a'(u)u_x^2,$$

together with assumption (1.5) on the initial data, implies that

$$(1.15) \quad u(t, y_1) = u(t, y_2) = 0 \quad \forall t \geq 0.$$

Therefore, for every $i, j = 1, 2$:

$$\lambda_j(u(t, y_i)) = \lambda_j(0) = 0 \quad \forall t \geq 0.$$

From the uniqueness of solutions of the Cauchy problems (1.10) and (1.11) we deduce that if $(t_0, x_j(t_0, \alpha)) \in [0, +\infty) \times [y_1, y_2]$ for some $t_0 \geq 0$, then $(t, x_j(t, \alpha)) \in [0, +\infty) \times [y_1, y_2]$ for every $t \geq 0$, for $j = 1, 2$.

It follows that if we fix $(t_0, x_0) \in (0, +\infty) \times [y_1, y_2]$, then we can find two characteristic curves $x_1(t, \alpha_0)$ and $x_2(t, \beta_0)$, solutions, respectively, of (1.10) with $\alpha = \alpha_0$ and of (1.11) with $\beta = \beta_0$, such that

$$x_1(t_0, \alpha_0) = x_0, \quad x_2(t_0, \beta_0) = x_0,$$

with

$$y_1 \leq \beta_0 \leq \alpha_0 \leq y_2.$$

Moreover, we have:

$$l(t_0, x_0) = l_0(\alpha_0),$$

$$r(t_0, x_0) = r_0(\beta_0).$$

Now, from (1.8), (1.12) and (1.13):

$$l_0(\alpha_0) - r_0(\beta_0) = l(t_0, x_0) - r(t_0, x_0) = \int_0^{u(t_0, x_0)} \sqrt{a(\eta)} d\eta.$$

But we also have that

$$\begin{aligned} l_0(\alpha_0) - r_0(\beta_0) &= \frac{1}{2}(v(0, \alpha_0) - v(0, \beta_0)) + \frac{1}{2} \left(\int_0^{\varphi(\alpha_0)} \sqrt{a(\eta)} d\eta + \int_0^{\varphi(\beta_0)} \sqrt{a(\eta)} d\eta \right) \\ &= \frac{1}{2} \int_{\beta_0}^{\alpha_0} \psi(x) dx + \frac{1}{2} \left(\int_0^{\varphi(\alpha_0)} \sqrt{a(\eta)} d\eta + \int_0^{\varphi(\beta_0)} \sqrt{a(\eta)} d\eta \right), \end{aligned}$$

since $v_x(0, x) = u_t(0, x) = \psi(x)$ by (1.7).

Therefore

$$\left| \int_0^{u(t_0, x_0)} \sqrt{a(\eta)} d\eta \right| \leq \frac{1}{2} \int_{y_1}^{y_2} |\psi(x)| dx + \int_{-\|\varphi\|_{L^\infty(y_1, y_2)}}^{\|\varphi\|_{L^\infty(y_1, y_2)}} \sqrt{a(\eta)} d\eta.$$

From assumption (1.4) we finally deduce that

$$(1.16) \quad |u(t_0, x_0)| \leq M(\varphi, \psi),$$

where $M(\varphi, \psi) < +\infty$ is defined by:

$$M(\varphi, \psi) = \sup \left\{ |u| : \left| \int_0^u \sqrt{a(\eta)} d\eta \right| \leq \frac{1}{2} \int_{y_1}^{y_2} |\psi(x)| dx + \int_{-\|\varphi\|_{L^\infty(y_1, y_2)}}^{\|\varphi\|_{L^\infty(y_1, y_2)}} \sqrt{a(\eta)} d\eta \right\}.$$

By the arbitrariness of $(t_0, x_0) \in (0, +\infty) \times [y_1, y_2]$, the estimate (1.16) proves that $u(t, x)$ is uniformly bounded in $[0, +\infty) \times [y_1, y_2]$. $\square$

Remark 1.4. Observe that condition (1.3) alone is not sufficient to obtain that a C^2 solution $u(t, x)$ of the Cauchy problem (1.14) must be uniformly bounded in $[0, +\infty) \times [y_1, y_2]$. If, for instance, $a(\eta) \equiv 0$, then $u(t, x) = \varphi(x) + t\psi(x)$.

We can now prove Theorem 1.1:

Proof of Theorem 1.1. Let now (1.6) hold true, and argue by contradiction, assuming that there exists a solution $u \in C^2([0, +\infty) \times [y_1, y_2])$ of the Cauchy problem (1.2).

Integrating the equation of (1.2) we obtain that

$$(1.17) \quad \int_{y_1}^{y_2} u_{tt}(t, x) dx = \int_{y_1}^{y_2} (a(u)u_x)_x dx = 0,$$

since $a(u(t, y_i)) = 0$, for $i = 1, 2$, by (1.15).

Therefore

$$\int_{y_1}^{y_2} u_t(t, x) dx = \int_{y_1}^{y_2} u_t(0, x) dx = \int_{y_1}^{y_2} \psi(x) dx \quad \forall t \geq 0,$$

and hence by (1.6)

$$\lim_{t \rightarrow +\infty} \left| \int_{y_1}^{y_2} u(t, x) dx \right| = +\infty.$$

This contradicts the boundness of $u(t, x)$ in $[0, +\infty) \times [y_1, y_2]$, given by Lemma 1.3. $\square$

Remark 1.5. We can also give an estimate of the life-span T of any C^2 solution of the Cauchy problem (1.2) in $[0, +\infty) \times [y_1, y_2]$:

$$T \leq \left| \int_{y_1}^{y_2} \psi(x) dx \right|^{-1} (M(\varphi, \psi)(y_2 - y_1) + \|\varphi\|_{L^1(y_1, y_2)}).$$

As an easy consequence of Theorem 1.1, we have the following:

Corollary 1.6. Let $\varphi \in C_0^2(\mathbf{R})$, $\psi \in C_0^1(\mathbf{R})$ and $a \in C^2(\mathbf{R})$ with

$$a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, \quad a(0) = 0$$

$$\lim_{u \rightarrow \pm\infty} \int_0^u \sqrt{a(\eta)} d\eta = \pm\infty.$$

Then, if $\int_{\mathbf{R}} \psi(x) dx \neq 0$, then Cauchy problem (1.2) does not admit a global solution $u \in C^2([0, +\infty) \times \mathbf{R})$.

1.1. Some generalizations

A) Arguing in the same way as for equation (1.1), we can obtain similar results for the equation

$$(1.18) \quad u_{tt} - a(u_x)u_{xx} = 0.$$

Equation (1.18) reduces, indeed, to the system

$$\begin{cases} v_t = w_x \\ w_t = a(v)v_x \end{cases}$$

by the substitutions $v = u_x$, $w = u_t$.

This system is the same as system (1.7). We can thus prove, as in Theorem 1.1, the following

Theorem 1.7. *Let $u(t, x)$ be a C^2 solution of the equation*

$$(1.19) \quad u_{tt} - a(u_x)u_{xx} = 0$$

in $[0, +\infty) \times \mathbf{R}_x$, where $a \in C^2(\mathbf{R})$ satisfies:

$$a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, \quad a(0) = 0$$

$$(1.20) \quad \lim_{u \rightarrow \pm\infty} \int_0^u \sqrt{a(\eta)} d\eta = \pm\infty.$$

Then, if there exist $y_1, y_2 \in \mathbf{R}$ with $y_1 < y_2$ and

$$(1.21) \quad u_x(0, y_i) = u_{xt}(0, y_i) = 0, \quad i = 1, 2,$$

$u_t(t, x)$ and $u_x(t, x)$ are uniformly bounded in the strip $[0, +\infty) \times [y_1, y_2]$.

It follows that if $u(t, x)$ is a solution of (1.19) which satisfies (1.21) and such that

$$u_t(0, y_1) \neq u_t(0, y_2),$$

then this solution $u(t, x)$ cannot be a C^2 solution on the whole strip $[0, +\infty) \times [y_1, y_2]$.

B) We can extend the preceding results also to weakly hyperbolic equations of the form

$$u_{tt} - (a(u)u_x)_x - (b(u)u_x)_t = 0.$$

This equation reduces to the system

$$\begin{cases} u_t = b(u)u_x + v_x \\ v_t = a(u)u_x. \end{cases}$$

The weakly hyperbolic condition is thus given by

$$b^2(\eta) + 4a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R},$$

and the characteristic roots $\lambda_1(u), \lambda_2(u)$ are

$$\lambda_{1,2}(u) = \frac{-b(u) \pm \sqrt{b^2(u) + 4a(u)}}{2}.$$

Arguing in the same way as for the case $b(u) \equiv 0$, we obtain the following:

Theorem 1.8. *Let $\varphi, a, b \in C^2(\mathbf{R})$, $\psi \in C^1(\mathbf{R})$ and assume that:*

- i) $a(0) = b(0) = 0$
- ii) $b^2(\eta) + 4a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}$
- iii) $\lim_{u \rightarrow \pm\infty} \int_0^u \sqrt{b^2(\eta) + 4a(\eta)} d\eta = \pm\infty$
- iv) *there exists $y_1, y_2 \in \mathbf{R}$ with $y_1 < y_2$ and*

$$(1.22) \quad \varphi(y_i) = \psi(y_i) = 0, \quad i = 1, 2$$

$$(1.23) \quad \int_{y_1}^{y_2} \psi(x) dx \neq 0.$$

Then the Cauchy problem

$$\begin{cases} u_{tt} - (a(u)u_x)_x - (b(u)u_x)_t = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

does not admit a solution $u \in C^2([0, +\infty) \times [y_1, y_2])$.

2. Formation of singularities along degenerate characteristic curves

In this section we show the formation of singularities for solutions of weakly hyperbolic equations of the form

$$(2.1) \quad u_{tt} - a(u_x)u_{xx} = f(u_x) \quad (t, x) \in [0, +\infty) \times \mathbf{R}_x,$$

along degenerate characteristic curves.

We assume here that $a, f \in C^2(\mathbf{R})$ satisfy:

$$(2.2) \quad \begin{cases} a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R}, & a(0) = 0, & a''(0) = \lambda \in \mathbf{R} \\ f'(\eta) = 0, & f''(0) = \mu \in \mathbf{R}. \end{cases}$$

Deriving equation (2.1) with respect to x , by the formal substitution $v = u_x$, we obtain

$$(2.3) \quad v_{tt} - a(v)v_{xx} - a'(v)v_x^2 = f'(v)v_x.$$

This equation is equivalent to the following system:

$$\begin{cases} v_t = w_x \\ w_t = a(v)v_x + f(v), \end{cases}$$

which has locally Lipschitz continuous characteristic roots

$$\lambda_1(v) = -\sqrt{a(v)}, \quad \lambda_2(v) = \sqrt{a(v)}.$$

Then the characteristic curves $x_1(t, \alpha)$ and $x_2(t, \beta)$ are uniquely determined as solutions of the following Cauchy problems:

$$\begin{cases} x_1' = \lambda_1(v(t, x_1)) = \lambda_1(u_x(t, x_1)) \\ x_1(0, \alpha) = \alpha \end{cases}$$

$$\begin{cases} x_2' = \lambda_2(v(t, x_2)) = \lambda_2(u_x(t, x_2)) \\ x_2(0, \beta) = \beta. \end{cases}$$

Let us now assume that there exists a characteristic curve $x_1(t, \alpha)$ such that, for a solution $u(t, x)$ of equation (2.1), we have that

$$u_x(t, x_1(t, \alpha)) = 0 \quad \forall t \geq 0.$$

Then

$$\lambda_1(u_x(t, x_1(t, \alpha))) = \lambda_1(0) = -\sqrt{a(0)} = 0 \quad \forall t \geq 0.$$

In this case the characteristic curve $x_1(t, \alpha) \equiv \alpha$ is a straight-line in the (t, x) -plane orthogonal to the x -axis, and we call it a *degenerate characteristic curve*.

Remark that $u_x(0, \alpha) = u_{xt}(0, \alpha) = 0$ implies that $u_x(t, \alpha) = 0$ for every $t \geq 0$, because of (2.2), (2.3).

Under the assumptions above, we can now prove, following an idea of Saxton (cf. [S]), the formation of singularities for solutions $u(t, x)$ of equation (2.1) in finite time $T \leq t^*$ defined by:

$$(2.4) \quad t^* = \int_{u_{xx}(0, \alpha)}^{+\infty} \left(\frac{\lambda}{2} y^4 + \frac{2}{3} \mu y^3 + 2c \right)^{-1/2} dy,$$

where $\lambda = a''(0)$, $\mu = f''(0)$ and $c = (1/2)u_{xxt}^2(0, \alpha) - (\lambda/4)u_{xx}^4(0, \alpha) - (\mu/3)u_{xx}^3(0, \alpha)$.

We have, indeed, the following:

Theorem 2.1. *Let $u(t, x)$ be a C^4 solution of the equation*

$$(2.5) \quad u_{tt} = a(u_x)u_{xx} + f(u_x),$$

with $a, f \in C^2(\mathbf{R})$ such that (2.2) holds true, with $\lambda > 0$. Assume, moreover, that there exists a degenerate characteristic curve $x_1(t, \alpha)$ such that:

$$\begin{aligned}
(2.6) \quad & u_x(t, x_1(t, \alpha)) = 0 \quad \forall t \geq 0 \\
& u_{xx}(0, \alpha) > -\frac{\mu}{\lambda} \quad \text{and} \quad u_{xxl}(0, \alpha) > 0.
\end{aligned}$$

Then, for t^* defined by (2.4), we have:

$$(2.7) \quad \lim_{t \rightarrow (t^*)^-} u_{xx}(t, x_1(t, \alpha)) = +\infty.$$

Proof. Let us first reduce equation (2.5) to equation (2.3), by substituting v to u_x , as before.

Deriving then equation (2.3) we find:

$$(2.8) \quad v_{ttx} = a(v)v_{xxx} + 3a'(v)v_x v_{xx} + a''(v)v_x^3 + f'(v)v_{xx} + f''(v)v_x^2.$$

Along the degenerate characteristic curve we have that $v(t, \alpha) = u_x(t, x_1(t, \alpha)) = 0$ for all $t \geq 0$, and hence equation (2.8) reduces, by (2.2), to:

$$(2.9) \quad v_{ttx}(t, \alpha) = \lambda v_x^3(t, \alpha) + \mu v_x^2(t, \alpha) \quad \forall t \geq 0.$$

Setting

$$w(t) = v_x(t, \alpha) = u_{xx}(t, \alpha),$$

we thus have to consider the Cauchy problem

$$(2.10) \quad \begin{cases} w''(t) = \lambda w^3(t) + \mu w^2(t) & \text{in } [0, +\infty) \\ w(0) = w_0 = u_{xx}(0, \alpha) > -\frac{\mu}{\lambda} \\ w'(0) = w_1 = u_{xxl}(0, \alpha) > 0. \end{cases}$$

We now prove that a C^2 solution w of (2.10) must blow-up in finite time. To this aim we first show that if $w \in C^2([0, +\infty))$ satisfies (2.10), then $w'(t) > 0$ for every $t \geq 0$. Recalling, indeed, that $w'(0) = w_1 > 0$ we assume by contradiction that there exists $\delta > 0$ such that

$$(2.11) \quad w'(t) > 0 \quad \forall t \in [0, \delta), \quad w'(\delta) = 0.$$

Note that (2.11) implies that $w(t) \geq w_0 > -\mu/\lambda$ for every $t \in [0, \delta]$, and hence $w'' = w^2(\lambda w + \mu) \geq 0$ in $[0, \delta]$. Therefore $w'(t) \geq w_1 > 0$ in $[0, \delta]$, which contradicts $w'(\delta) = 0$. A solution $w \in C^2([0, +\infty))$ of (2.10) must then satisfy:

$$(2.12) \quad w'(t) \geq w_1 > 0 \quad \forall t \geq 0.$$

Multiplying the equation in (2.10) by $w'(t)$ and integrating from 0 to $t > 0$ we obtain:

$$(2.13) \quad \frac{1}{2} (w'(t))^2 = \frac{\lambda}{4} w^4(t) + \frac{\mu}{3} w^3(t) + c,$$

where

$$c = \frac{1}{2}w_1^2 - \frac{\lambda}{4}w_0^4 - \frac{\mu}{3}w_0^3.$$

We deduce that

$$(2.14) \quad w'(t) = \sqrt{\frac{\lambda}{2}w^4(t) + \frac{2}{3}\mu w^3(t) + 2c}$$

because of (2.12).

Condition (2.12) implies also that we can integrate and obtain:

$$t = \int_0^t \frac{w'(s)}{\sqrt{\frac{\lambda}{2}w^4(s) + \frac{2}{3}\mu w^3(s) + 2c}} ds = \int_{w_0}^{w(t)} \frac{dy}{\sqrt{\frac{\lambda}{2}y^4 + \frac{2}{3}\mu y^3 + 2c}}.$$

Setting then

$$t^* = \int_{u_{xx}(0, \alpha)}^{+\infty} \frac{dy}{\sqrt{\frac{\lambda}{2}y^4 + \frac{2}{3}\mu y^3 + 2c}},$$

we finally have that $t^* < +\infty$,

$$\lim_{t \rightarrow (t^*)^-} w(t) = +\infty,$$

and hence (2.7) holds true. $\square$

Remark 2.2. If $a''(0) = \lambda = 0 = \mu = f''(0)$, then from (2.9) we deduce that we may have only linear growth of $v_x(t, \alpha) = u_{xx}(t, \alpha)$.

We can see throughout the proof of Theorem 2.1 that, in fact, we don't need the assumption $a(\eta) \geq 0$, but only $a(0) = a'(0) = 0$. In this case, if $a''(0) = \lambda < 0$ we cannot have blow-up of $u_{xx}(t, x_1(t, \alpha))$ because of (2.13).

Remark 2.3. Note that we can substitute conditions (2.6) by similar ones, upon modification of the estimate (2.4) of the life-span of the solution.

Studying, indeed, the ordinary differential Cauchy problem (2.10), we find that we can assume, instead of $w_0 > -\mu/\lambda$ and $w_1 > 0$, one of the following conditions:

- i) $w_0 > -\frac{\mu}{\lambda}, \quad w_1 > -\sqrt{2} \left(\frac{\lambda}{4}w_0^4 + \frac{\mu}{3}w_0^3 + \frac{1}{12} \frac{\mu^4}{\lambda^3} \right)^{1/2}$
- ii) $w_0 \leq -\frac{\mu}{\lambda}, \quad w_1 > \sqrt{2} \left(\frac{\lambda}{4}w_0^4 + \frac{\mu}{3}w_0^3 + \frac{1}{12} \frac{\mu^4}{\lambda^3} \right)^{1/2}.$

Note also that similar conditions (with $-\mu$ instead of μ) can be obtained if we want to prove that $w(t)$ goes to $-\infty$ (instead of $+\infty$) in finite time.

2.1. Some generalizations

A) Similar results to that obtained for equation (2.1) hold also for the weakly hyperbolic equation

$$(2.15) \quad u_{tt} - (a(u)u_x)_x - (b(u)u_x)_t = 0, \quad (t, x) \in [0, +\infty) \times \mathbf{R}_x,$$

for $a, b \in C^2(\mathbf{R})$ such that

$$b^2(\eta) + 4a(\eta) \geq 0 \quad \forall \eta \in \mathbf{R},$$

$$a(0) = b(0) = 0$$

$$a''(0) = \lambda > 0, \quad b'(0) = \mu.$$

Equation (2.15) is indeed of the form

$$u_{tt} - a(u)u_{xx} - a'(u)u_x^2 - b'(u)u_x u_t - b(u)u_{xt} = 0,$$

where $u(0, \alpha) = u_t(0, \alpha) = 0$ implies $u(t, \alpha) \equiv 0$ for every $t \geq 0$.

Deriving the equation with respect to x , along the degenerate characteristic curve we find

$$u_{xtt} = \lambda u_x^3 + 2\mu u_x u_{xt} \quad \text{for } x = \alpha,$$

because $a'(0) = 0$ and $u_t(t, \alpha) = 0$ for every $t \geq 0$.

Under suitable conditions on λ, μ and the initial data $u_x(0, \alpha)$, $u_{xt}(0, \alpha)$, we thus have blow-up of $u_x(t, \alpha)$.

B) We can obtain the same results as in Theorem 2.1 also for the equation

$$(2.16) \quad u_{tt} - a(t, x, u_x)u_{xx} = f(t, x, u_x),$$

if we substitute to (2.2) the following conditions on $a, f \in C^2(\mathbf{R}^+ \times \mathbf{R}^2)$:

$$(2.17) \quad \begin{cases} D_x^\alpha D_\eta^\beta a(t, x, 0) = 0 & \text{for } \alpha, \beta \in N \text{ with } \alpha + \beta \leq 2 \text{ and } \beta \neq 2 \\ a_{\eta\eta}(t, x, 0) = \lambda > 0 \\ D_x^\alpha D_\eta^\beta f(t, x, 0) = 0 & \text{for } \alpha, \beta \in N \text{ with } \alpha + \beta \leq 2 \text{ and } \beta \neq 2 \\ f_{\eta\eta}(t, x, 0) = \mu \in \mathbf{R}. \end{cases}$$

Deriving, indeed, equation (2.16) with respect to x , we find:

$$\begin{aligned} u_{ttx} - a_x(t, x, u_x)u_{xx} - a_\eta(t, x, u_x)u_{xx}^2 - a(t, x, u_x)u_{xxx} \\ = f_x(t, x, u_x) + f_\eta(t, x, u_x)u_{xx}, \end{aligned}$$

and hence $v = u_x$ satisfies:

$$v_{tt} - a_x(t, x, v)v_x - a_\eta(t, x, v)v_x^2 - a(t, x, v)v_{xx} = f_x(t, x, v) + f_\eta(t, x, v)v_x.$$

Deriving again with respect to x :

$$\begin{aligned} v_{ttx} &= a_{xx}(t, x, v)v_x + a_{x\eta}(t, x, v)v_x^2 + a_x(t, x, v)v_{xx} + a_{\eta x}(t, x, v)v_x^2 + a_{\eta\eta}(t, x, v)v_x^3 \\ &\quad + 2a_\eta(t, x, v)v_xv_{xx} + a_x(t, x, v)v_{xx} + a_\eta(t, x, v)v_xv_{xx} + a(t, x, v)v_{xxx} \\ &\quad + f_{xx}(t, x, v) + f_{x\eta}(t, x, v)v_x + f_{\eta x}(t, x, v)v_x + f_{\eta\eta}(t, x, v)v_x^2. \end{aligned}$$

Since along the degenerate characteristic curve $x_1(t, \alpha) \equiv \alpha$ we have that $v(t, \alpha) = u_x(t, x_1(t, \alpha)) = 0$ for all $t \geq 0$, it follows that

$$v_{ttx}(t, \alpha) = \lambda v_x^3(t, \alpha) + \mu v_x^2(t, \alpha),$$

which is equation (2.9) again. We thus have the following

Theorem 2.4. *Let $u(t, x)$ be a C^4 solution of the equation*

$$(2.18) \quad u_{tt} - a(t, x, u_x)u_{xx} = f(t, x, u_x),$$

with $a, f \in C^2(\mathbf{R}^+ \times \mathbf{R}^2)$ such that (2.17) holds true. Assume, moreover, that there exists a degenerate characteristic curve $x_1(t, \alpha)$ such that

$$u_x(t, x_1(t, \alpha)) = 0 \quad \forall t \geq 0$$

$$u_{xx}(0, \alpha) > -\frac{\mu}{\lambda} \quad \text{and} \quad u_{xxl}(0, \alpha) > 0.$$

Then, for t^* defined by (2.4):

$$\lim_{t \rightarrow (t^*)^-} u_{xx}(t, x_1(t, \alpha)) = +\infty.$$

Remark 2.5. As in Remark 2.2, we note that if $a_{\eta\eta}(t, x, 0) = \lambda = 0$ and $f_{\eta\eta}(t, x, 0) = \mu = 0$, then we may have only linear growth of $u_{xx}(t, \alpha)$.

3. Formation of singularities along non-degenerate characteristic curves

In this section we consider the weakly hyperbolic equation

$$(3.1) \quad u_{tt} - a(u_x)u_{xx} = 0, \quad (t, x) \in [0, +\infty) \times \mathbf{R}_x$$

with $a \in C^2(\mathbf{R})$ such that

$$(3.2) \quad a(0) = 0, \quad a(\eta) > 0 \quad \forall \eta \in \mathbf{R} \setminus \{0\}.$$

We also assume that for every $R > 0$ there exists a constant $C_R > 0$ such that:

$$(3.3) \quad a'(\eta) \geq C_R a(\eta)^{5/4} \quad \text{for } 0 \leq \eta \leq R$$

$$(3.4) \quad a'(\eta) \leq -C_R a(\eta)^{5/4} \quad \text{for } -R \leq \eta \leq 0.$$

We consider the Riemann invariants

$$(3.5) \quad l(t, x) = \frac{1}{2} \left(u_t(t, x) + \int_0^{u_x(t, x)} \sqrt{a(\eta)} d\eta \right)$$

$$(3.6) \quad r(t, x) = \frac{1}{2} \left(u_t(t, x) - \int_0^{u_x(t, x)} \sqrt{a(\eta)} d\eta \right)$$

which satisfy

$$(3.7) \quad \begin{cases} l_t - a(u_x)^{1/2} l_x = 0 \\ r_t + a(u_x)^{1/2} r_x = 0, \end{cases}$$

with initial conditions

$$(3.8) \quad \begin{cases} l(t, x_1(t, \alpha)) = l(0, \alpha) = l_0(\alpha) \\ r(t, x_2(t, \beta)) = r(0, \beta) = r_0(\beta). \end{cases}$$

The assumption (3.2) implies, in particular, that the function

$$y \mapsto g(y) = \int_0^y a(\eta)^{1/2} d\eta$$

is strictly increasing and hence invertible.

By the definition of the Riemann invariants we can thus write

$$u_x = g^{-1}(l - r).$$

Setting then

$$k(\omega) = a(g^{-1}(\omega))^{1/2},$$

we have that $k \in C^0(\mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\})$ and from (3.7) we obtain the system

$$(3.9) \quad \begin{cases} l_t - k(l - r) l_x = 0 \\ r_t + k(l - r) r_x = 0. \end{cases}$$

The characteristic curves $x_1(t, \alpha)$ and $x_2(t, \alpha)$ are thus defined as the solutions of the Cauchy problems

$$(3.10) \quad \begin{cases} x_1' = -a(u_x(t, x_1))^{1/2} = -k(l(t, x_1) - r(t, x_1)) \\ x_1(0, \alpha) = \alpha \end{cases}$$

and

$$\begin{cases} x_2' = a(u_x(t, x_2))^{1/2} = k(l(t, x_2) - r(t, x_2)) \\ x_2(0, \beta) = \beta. \end{cases}$$

Remark 3.1. As in the previous sections, $a(\eta)^{1/2}$ is a locally Lipschitz continuous function and hence the characteristic curves $x_1(t, \alpha)$ and $x_2(t, \beta)$ are well defined.

Under the assumptions above, we shall now prove the following:

Theorem 3.2. *Let $a(\eta) \in C^2(\mathbf{R})$ satisfy:*

- i) $a(0) = 0$, $a(\eta) > 0 \quad \forall \eta \in \mathbf{R} \setminus \{0\}$
- ii) *for every $R > 0$ there exists $C_R > 0$ such that (3.3) and (3.4) hold.*

Let $\varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$ be two functions with compact support and not both identically zero (or two periodic functions not both constant). Then a C^2 solution $u(t, x)$ of the Cauchy problem

$$(3.11) \quad \begin{cases} u_{tt} - a(u_x)u_{xx} = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

must develop a singularity in finite time.

Let $u(t, x)$ be a classical solution of equation (3.1) and assume that there exists a *non-degenerate characteristic curve* $x_1(t, \alpha)$, namely that for some $\alpha \in \mathbf{R}$

$$u_x(t, x_1(t, \alpha)) > 0 \quad \forall t \geq 0,$$

and hence

$$(3.12) \quad a(u_x(t, x_1(t, \alpha))) > 0 \quad \forall t \geq 0.$$

We can then compute the *infinitesimal compression ratio* $(\partial x_1 / \partial \alpha)(t, \alpha)$ in the standard way. Following [M] and [KM] we obtain that

$$(3.13) \quad \frac{\partial x_1}{\partial \alpha}(t, \alpha) = \frac{k(l_0(\alpha) - r(t, x_1(t, \alpha)))^{1/2}}{k(l_0(\alpha) - r_0(\alpha))^{1/2}} \left\{ 1 - l_0'(\alpha) \int_0^t k'(l_0(\alpha) - r(s, x_1(s, \alpha))) \cdot \frac{k(l_0(\alpha) - r_0(\alpha))^{1/2}}{k(l_0(\alpha) - r(s, x_1(s, \alpha)))^{1/2}} ds \right\}.$$

Since $k(\omega) = a(g^{-1}(\omega))^{1/2}$, for $\omega \neq 0$ we have that

$$k'(\omega) = \frac{1}{2} \frac{a'(g^{-1}(\omega))}{a(g^{-1}(\omega))},$$

and hence, substituting in (3.13):

$$(3.14) \quad \frac{\partial x_1}{\partial \alpha}(t, \alpha) = \gamma(t, \alpha) \left(1 - \frac{l'_0(\alpha)}{2} \cdot k(l_0(\alpha) - r_0(\alpha))^{1/2} \right. \\ \left. \cdot \int_0^t \frac{a'(g^{-1}(l_0(\alpha) - r(s, x_1(s, \alpha))))}{a(g^{-1}(l_0(\alpha) - r(s, x_1(s, \alpha))))^{5/4}} ds \right),$$

where

$$\gamma(t, \alpha) = \frac{k(l_0(\alpha) - r(t, x_1(t, \alpha)))^{1/2}}{k(l_0(\alpha) - r_0(\alpha))^{1/2}}.$$

We can then prove the following

Lemma 3.3. *Let $u(t, x)$ be a classical solution of the Cauchy problem*

$$(3.15) \quad \begin{cases} u_{tt} - a(u_x)u_{xx} = 0 & \text{in } [0, +\infty) \times \mathbf{R}_x \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x), \end{cases}$$

where $a, \varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$ satisfy:

- i) $a(0) = 0, a(\eta) > 0 \quad \forall \eta \in \mathbf{R} \setminus \{0\}$
- ii) for every $R > 0$ there exists $C_R > 0$ such that (3.3) and (3.4) hold
- iii) $\|\varphi'\|_{L^\infty(\mathbf{R})} < +\infty, \|\psi\|_{L^\infty(\mathbf{R})} < +\infty.$

Assume, moreover, that there exists $\alpha \in \mathbf{R}$ such that

$$(3.16) \quad l'_0(\alpha) > 0 \quad \text{and} \quad u_x(t, x_1(t, \alpha)) > 0 \quad \forall t \geq 0.$$

Then the solution $u(t, x)$ must develop a singularity in finite time.

Proof. Let $M > 0$ such that

$$(3.17) \quad \|\varphi'\|_{L^\infty(\mathbf{R})} + \|\psi\|_{L^\infty(\mathbf{R})} \leq M.$$

Assumptions i) and ii) imply, in particular, that

$$\lim_{y \rightarrow +\infty} g(y) = \lim_{y \rightarrow +\infty} \int_0^y a(\eta)^{1/2} d\eta = +\infty,$$

and hence there exists a constant $R > 0$ such that

$$0 < u_x(t, x_1(t, \alpha)) = g^{-1}(l_0(\alpha) - r(t, x_1(t, \alpha))) \leq R \quad \forall t \geq 0,$$

since (3.17) implies that also $l_0(\alpha)$ and $r(t, x_1(t, \alpha)) = r(t, x_2(t, \bar{\alpha})) = r_0(\bar{\alpha})$ (for some $\bar{\alpha} \in \mathbf{R}$) are bounded. By (3.3) we can thus find a positive constant C_R such that

$$\frac{a'(g^{-1}(l_0(\alpha) - r(t, x_1(t, \alpha))))}{a(g^{-1}(l_0(\alpha) - r(t, x_1(t, \alpha))))^{5/4}} \geq C_R > 0 \quad \forall t \geq 0.$$

Then from (3.14):

$$\frac{\partial x_1}{\partial \alpha}(t, \alpha) \leq \gamma(t, \alpha) \left(1 - \frac{l'_0(\alpha)}{2} k(l_0(\alpha) - r_0(\alpha))^{1/2} C_R t \right).$$

It follows that there exists $T > 0$ such that $(\partial x_1 / \partial \alpha)(t, \alpha) > 0$ in $[0, T)$ and $(\partial x_1 / \partial \alpha)(T, \alpha) = 0$, since

$$l'_0(\alpha) k(l_0(\alpha) - r_0(\alpha))^{1/2} > 0$$

by (3.16).

Therefore, having

$$(3.18) \quad l_x(t, x_1(t, \alpha)) \cdot \frac{\partial x_1}{\partial \alpha}(t, \alpha) = l'_0(\alpha) > 0$$

by (3.8), we can affirm that

$$\limsup_{t \rightarrow T^-} l_x(t, x_1(t, \alpha)) = +\infty,$$

from which we deduce that the solution $u(t, x)$ must develop some singularity for $t \leq T$ along the characteristic curve $x_1(t, \alpha)$. $\square$

By the same arguments we can prove also the following lemma, where we substitute to condition (3.16) (of Lemma 3.3) similar ones:

Lemma 3.4. *Assume that conditions i), ii), and iii) of Lemma 3.3 hold, and that one of the following conditions is satisfied:*

$$(3.19) \quad \text{a) } l'_0(\alpha) < 0 \quad \text{and} \quad u_x(t, x_1(t, \alpha)) < 0 \quad \forall t \geq 0$$

$$(3.20) \quad \text{b) } r'_0(\beta) > 0 \quad \text{and} \quad u_x(t, x_2(t, \beta)) > 0 \quad \forall t \geq 0$$

$$(3.21) \quad \text{c) } r'_0(\beta) < 0 \quad \text{and} \quad u_x(t, x_2(t, \beta)) < 0 \quad \forall t \geq 0.$$

Then a classical solution $u(t, x)$ of the Cauchy problem (3.15) must develop a singularity in finite time.

Proof. If condition a) is satisfied we can follow the same proof as in Lemma 3.3 using condition (3.4) instead of (3.3).

If condition b) or c) holds, we can again follow the same proof employing the infinitesimal compression ratio

$$\frac{\partial x_2}{\partial \beta}(t, \beta) = \tilde{\gamma}(t, \beta) \left(1 - r'_0(\beta) \int_0^t \frac{k'(l(s, x_2(s, \beta)) - r_0(\beta))}{\tilde{\gamma}(s, \beta)} ds \right)$$

for

$$\tilde{\gamma}(t, \beta) = \frac{k(l(t, x_2(t, \beta)) - r_0(\beta))^{1/2}}{k(l_0(\beta) - r_0(\beta))^{1/2}}. \quad \square$$

Let us now prove, following an idea of [KK], that if $l'_0(\alpha) > 0$ or $r'_0(\alpha) > 0$ (resp.: $l'_0(\alpha) < 0$ or $r'_0(\alpha) < 0$) and $u_x(0, \alpha) > 0$ (resp.: $u_x(0, \alpha) < 0$), then the characteristic curve is non-degenerate until $u(t, x)$ is a regular solution of the Cauchy problem (3.15). It will follow that conditions (3.16), (3.19), (3.20) and (3.21) can be weakened by requiring the condition on the sign of u_x only at the initial time $t = 0$.

Lemma 3.5. *Let $u(t, x)$ be a classical solution of the equation (3.1). Then conditions*

$$(3.22) \quad l'_0(\alpha) > 0, \quad u_x(0, \alpha) > 0$$

imply

$$(3.23) \quad u_x(t, x_1(t, \alpha)) > 0 \quad \forall t \geq 0.$$

Similar results hold also for the other cases considered in Lemma 3.4.

Proof. Let (3.22) be valid, assume by contradiction that there exists $\tau > 0$ such that

$$(3.24) \quad u_x(\tau, x_1(\tau, \alpha)) = 0, \quad u_x(t, x_1(t, \alpha)) > 0 \quad \forall t \in [0, \tau),$$

and let us give an estimate of $(d/dt)u_x(t, x_1(t, \alpha))$ in a left neighbourhood of τ in $[0, \tau)$.

Along the characteristic curve $x_1(t, \alpha)$, by (3.5) and (3.8) we have, for $0 \leq t < \tau$ (where $u_x(t, x_1(t, \alpha)) > 0$ and hence $(\partial x_1 / \partial \alpha)(t, \alpha)$ is well defined):

$$(3.25) \quad \begin{aligned} 2l'_0(\alpha) &= \frac{\partial}{\partial \alpha} \left(u_t(t, x_1(t, \alpha)) + \int_0^{u_x(t, x_1(t, \alpha))} a(\eta)^{1/2} d\eta \right) \\ &= u_{tx}(t, x_1(t, \alpha)) \cdot \frac{\partial x_1}{\partial \alpha}(t, \alpha) \\ &\quad + u_{xx}(t, x_1(t, \alpha)) \cdot \frac{\partial x_1}{\partial \alpha}(t, \alpha) \cdot a(u_x(t, x_1(t, \alpha)))^{1/2}, \end{aligned}$$

and, moreover, by (3.10):

$$(3.26) \quad \begin{aligned} \frac{d}{dt} u_x(t, x_1(t, \alpha)) &= u_{xt}(t, x_1(t, \alpha)) + u_{xx}(t, x_1(t, \alpha)) \cdot \frac{dx_1}{dt}(t, \alpha) \\ &= u_{xt}(t, x_1(t, \alpha)) - u_{xx}(t, x_1(t, \alpha)) \cdot a(u_x(t, x_1(t, \alpha)))^{1/2}. \end{aligned}$$

Substituting (3.26) in (3.25):

$$(3.27) \quad 2l'_0(\alpha) = \frac{\partial x_1}{\partial \alpha}(t, \alpha) \cdot \frac{d}{dt} u_x(t, x_1(t, \alpha)) \\ + 2a(u_x(t, x_1(t, \alpha)))^{1/2} \cdot u_{xx}(t, x_1(t, \alpha)) \cdot \frac{\partial x_1}{\partial \alpha}(t, \alpha).$$

Let us now remark that $(\partial x_1 / \partial \alpha)(t, \alpha) > 0$ for every $t \in [0, \tau]$ and, moreover, if $u(t, x)$ is a classical solution of equation (3.1), then there exists a constant $c_0 > 0$ such that

$$(3.28) \quad |u_x(t, x_1(t, \alpha))| \leq c_0 \quad \forall t \in [0, \tau]$$

and

$$(3.29) \quad |u_{xx}(t, x_1(t, \alpha))| \leq c_0 \quad \forall t \in [0, \tau].$$

It follows that there is a constant $c_1 > 0$ such that

$$(3.30) \quad \left| \frac{\partial x_1}{\partial \alpha}(t, \alpha) \right| \leq c_1 \quad \forall t \in [0, \tau].$$

Indeed, from (3.10) we have that

$$(3.31) \quad \frac{d}{dt} \frac{\partial x_1}{\partial \alpha}(t, \alpha) = -\frac{\partial}{\partial \alpha} a(u_x(t, x_1(t, \alpha)))^{1/2} \\ = -\frac{1}{2} \frac{a'(u_x(t, x_1(t, \alpha)))}{a(u_x(t, x_1(t, \alpha)))^{1/2}} \cdot u_{xx}(t, x_1(t, \alpha)) \cdot \frac{\partial x_1}{\partial \alpha}(t, \alpha).$$

From (3.28) we can estimate

$$\left| \frac{a'(u_x(t, x_1(t, \alpha)))}{2\sqrt{a(u_x(t, x_1(t, \alpha)))}} \right| \leq L \quad \forall t \in [0, \tau],$$

where L is the Lipschitz constant of $\sqrt{a(\eta)}$ for $|\eta| \leq c_0$.

Then from (3.29) and (3.31) we have that

$$\left| \frac{d}{dt} \frac{\partial x_1}{\partial \alpha}(t, \alpha) \right| \leq Lc_0 \left| \frac{\partial x_1}{\partial \alpha}(t, \alpha) \right| \quad \forall t \in [0, \tau],$$

from which (3.30) follows.

From (3.27) we can thus find $\delta > 0$ such that

$$(3.32) \quad \frac{d}{dt} u_x(t, x_1(t, \alpha)) > 0 \quad \forall t \in [\tau - \delta, \tau],$$

since $l'_0(\alpha) > 0$, $0 \leq (\partial x_1 / \partial \alpha)(t, \alpha) \leq c_1$ and $a(u_x(t, x_1(t, \alpha))) \rightarrow 0$ as $t \nearrow \tau$.

But (3.32) contradicts (3.24), so that (3.23) holds true.

We can argue in the same way for the other cases considered in Lemma

3.4. $\square$

From Lemmas 3.3, 3.4 and 3.5 we can finally write:

Corollary 3.6. *Let $u(t, x)$ be a classical solution of the Cauchy problem*

$$\begin{cases} u_{tt} - a(u_x)u_{xx} = 0 & \text{in } [0, +\infty) \times \mathbf{R}_x \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x), \end{cases}$$

where $a, \varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$ satisfy:

- i) $a(0) = 0$, $a(\eta) > 0 \ \forall \eta \in \mathbf{R} \setminus \{0\}$
- ii) for every $R > 0$ there exists $C_R > 0$ such that

$$a'(\eta) \geq C_R a(\eta)^{5/4} \quad \text{for } 0 \leq \eta \leq R$$

$$a'(\eta) \leq -C_R a(\eta)^{5/4} \quad \text{for } -R \leq \eta \leq 0$$

- iii) $\|\varphi'\|_{L^\infty(\mathbf{R})} < +\infty$, $\|\psi\|_{L^\infty(\mathbf{R})} < +\infty$.

Assume, moreover, that there exists $\alpha \in \mathbf{R}$ such that at least one of the following conditions is satisfied:

$$(3.33) \quad \text{A) } l'_0(\alpha) > 0 \quad \text{or} \quad r'_0(\alpha) > 0, \quad \text{and} \quad u_x(0, \alpha) > 0$$

$$(3.34) \quad \text{B) } l'_0(\alpha) < 0 \quad \text{or} \quad r'_0(\alpha) < 0, \quad \text{and} \quad u_x(0, \alpha) < 0.$$

Then the solution $u(t, x)$ must develop a singularity in finite time.

We are now able to prove Theorem 3.2:

Proof of Theorem 3.2. If one of conditions (3.33) or (3.34) holds true, then the thesis follows from Corollary 3.6. Otherwise, the following two conditions must be satisfied:

$$(3.35) \quad \begin{cases} l'_0(\alpha) \leq 0 & \text{if } u_x(0, \alpha) > 0 \\ l'_0(\alpha) \geq 0 & \text{if } u_x(0, \alpha) < 0 \end{cases}$$

and

$$\begin{cases} r'_0(\beta) \leq 0 & \text{if } u_x(0, \beta) > 0 \\ r'_0(\beta) \geq 0 & \text{if } u_x(0, \beta) < 0. \end{cases}$$

We now use the assumption that at least one of the two initial data φ or ψ is not identically zero and has compact support (or is non-constant and

periodic). This means that at least one of the two Riemann invariants $l_0(\alpha)$ or $r_0(\alpha)$ is non-constant and attains its maximum and minimum over $\mathbf{R}$.

Let us then assume, for instance, that $l_0(\alpha)$ is a non-constant function, and prove that

$$(3.36) \quad \max_{\mathbf{R}} l_0(\alpha) = \max_{\Sigma} l_0(\alpha)$$

and

$$(3.37) \quad \min_{\mathbf{R}} l_0(\alpha) = \min_{\Sigma} l_0(\alpha),$$

where

$$\Sigma = \{x \in \mathbf{R} : u_x(0, x) = 0\}.$$

Let $\alpha_0 \in \mathbf{R}$ with

$$l_0(\alpha_0) = \max_{\alpha \in \mathbf{R}} l_0(\alpha).$$

Assume that

$$\alpha_0 \in \Sigma^+ = \{x \in \mathbf{R} : u_x(0, x) > 0\}.$$

There exists then a maximal interval $(\alpha', \alpha'') \subset \Sigma^+$ with $\alpha', \alpha'' \in \Sigma$ and $\alpha_0 \in (\alpha', \alpha'')$. From (3.35) we deduce that

$$l'_0(\alpha) \leq 0 \quad \forall \alpha \in (\alpha', \alpha'')$$

and hence

$$l_0(\alpha') \geq l_0(\alpha_0) = \max_{\mathbf{R}} l_0(\alpha),$$

i.e. $\alpha' \in \Sigma$ is a maximum point for $l_0(\alpha)$ on $\mathbf{R}$.

We can argue in a similar way if

$$\alpha_0 \in \Sigma^- = \{x \in \mathbf{R} : u_x(0, x) < 0\}.$$

Therefore (3.36) holds true, and similarly also (3.37) holds true.

We can thus find $\alpha_1, \alpha_2 \in \mathbf{R}$ such that

$$(3.38) \quad u_x(0, \alpha_1) = u_x(0, \alpha_2) = 0$$

and

$$\min_{\mathbf{R}} l_0(\alpha) = l_0(\alpha_1) < l_0(\alpha_2) = \max_{\mathbf{R}} l_0(\alpha).$$

It follows that $l'_0(\alpha_1) = l'_0(\alpha_2) = 0$ and hence, for each $i = 1, 2$:

$$\begin{aligned}
0 &= l'_0(\alpha_i) = \frac{1}{2} \frac{\partial}{\partial \alpha} \left(u_t(0, \alpha) + \int_0^{u_x(0, \alpha)} \sqrt{a(\eta)} d\eta \right) \Big|_{\alpha=\alpha_i} \\
&= \frac{1}{2} (u_{tx}(0, \alpha_i) + u_{xx}(0, \alpha_i) a(u_x(0, \alpha_i))^{1/2}) \\
&= \frac{1}{2} u_{tx}(0, \alpha_i)
\end{aligned}$$

because of (3.38).

We have thus found $\alpha_1, \alpha_2 \in \mathbf{R}$ such that:

$$\begin{cases} u_x(0, \alpha_i) = 0, & i = 1, 2 \\ u_{xt}(0, \alpha_i) = 0, & i = 1, 2 \\ u_t(0, \alpha_2) - u_t(0, \alpha_1) = 2(l_0(\alpha_2) - l_0(\alpha_1)) > 0. \end{cases}$$

Since condition (1.20) of §1 is automatically satisfied because of assumptions i) and ii), by Theorem 1.7 we obtain that the solution $u(t, x)$ of the Cauchy problem (3.11) cannot be a C^2 solution on the whole strip $[0, +\infty) \times [\alpha_1, \alpha_2]$.

The same conclusion holds if the non-constant Riemann invariant is $r_0(\alpha)$ instead of $l_0(\alpha)$. This completes the proof. $\square$

4. Local existence of smooth solutions

In this section we give some local existence results for weakly hyperbolic equations of the same kind as those considered in the previous sections.

To this aim we recall the Theorem of Nash and Moser, referring to [H] for more details:

Theorem 4.1. (NASH-MOSER) *Let $\mathcal{F}, \mathcal{G}$ be tame spaces, W an open subset of $\mathcal{F}$, and $P : W \rightarrow \mathcal{G}$ a smooth tame map. Assume that the equation $DP(u)h = k$ has a unique solution $h = VP(u)k$ for all $u \in W$ and $k \in \mathcal{G}$, and that this map $VP : W \times \mathcal{G} \rightarrow \mathcal{F}$ is smooth tame. Then P is locally invertible, and the local inverse map is smooth tame.*

4.1. Local existence for the equation $u_{tt} - (a(u)u_x)_x = 0$

In this subsection we prove, following [M1] and [M3], the existence of a unique local solution $u(t, x) \in C^\infty([0, T) \times \mathbf{R}_x)$ of the Cauchy problem

$$(4.1) \quad \begin{cases} u_{tt} - (a(u)u_x)_x = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x), \end{cases}$$

for smooth initial data φ, ψ with compact support, and for a real analytic non-negative function $a(\eta)$.

Remark that a real analytic function $a \in \mathcal{A}(\mathbf{R})$ has only finite order zeros, and hence only a finite number of zeros in any compact subset of $\mathbf{R}$. We can thus prove, (see Proposition 2.2 of [M3]), that it is not restrictive to assume $a(\eta)$ of the form

$$a(\eta) = \eta^{2q} a_0(\eta) \quad \forall \eta \in \mathbf{R},$$

with $q \in \mathbf{N} \setminus \{0\}$ and $a_0 \in C^\infty(\mathbf{R})$ such that

$$(4.2) \quad a_0(\eta) > 0 \quad \forall \eta \in \mathbf{R}.$$

The idea of the motivation of such a reduction is that we can “localize” the Cauchy problem in a neighbourhood of each zero of the real analytic function $a(\eta)$.

Remark 4.2. It can be showed, indeed, that we can assume $a \in C^\infty(\mathbf{R})$ with only zeros of finite order, instead of $a \in \mathcal{A}(\mathbf{R})$, for proving the local existence of a unique solution of the Cauchy problem (4.1) (see Theorem 4.5 in the following).

We also note that since we are looking for local solutions, and since the initial data $\varphi, \psi \in C_0^\infty(\mathbf{R})$ are bounded, the local solution will be also bounded and hence it is not restrictive to assume that there are two positive constants λ_0, A_0 such that

$$0 < \lambda_0 \leq a_0(\eta) \leq A_0 \quad \forall \eta \in \mathbf{R},$$

instead of (4.2).

For the sake of simplicity we shall study here only the case $q = 1$, since the generic case $q \in \mathbf{N} \setminus \{0\}$ can be treated by the same arguments.

Theorem 4.3. *Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$ and consider*

$$a(\eta) = \eta^2 a_0(\eta), \quad \eta \in \mathbf{R},$$

for $a_0 \in C^\infty(\mathbf{R})$ which satisfies, for some positive constants λ_0, A_0 :

$$0 < \lambda_0 \leq a_0(\eta) \leq A_0 \quad \forall \eta \in \mathbf{R}.$$

Then there exists $T > 0$ such that the Cauchy problem

$$(4.3) \quad \begin{cases} u_{tt} - a(u)u_{xx} = a'(u)u_x^2 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

has a unique local solution $u \in C^\infty([0, T] \times \mathbf{R})$.

Proof. Existence

We first remark that for every $t \geq 0$

$$(4.4) \quad \text{supp } u(t, \cdot) \subset (\text{supp } \varphi \cup \text{supp } \psi).$$

Indeed, if $x_0 \in (\mathcal{C} \text{supp } \varphi) \cap (\mathcal{C} \text{supp } \psi)$, then $v(t) = u(t, x_0)$ satisfies:

$$(4.5) \quad \begin{cases} v''(t) = F(t, v) \\ v(0) = \varphi(x_0) = 0 \\ v'(0) = \psi(x_0) = 0, \end{cases}$$

where the C^∞ function $F(t, v)$ is given by

$$F(t, v) = a(v)u_{xx}(t, x_0) + a'(v)u_x^2(t, x_0).$$

From the classical uniqueness theorem for the Cauchy problem (4.5) we obtain that $v \equiv 0$ since $F(t, 0) = 0$, and hence $x_0 \in \mathcal{C} \text{supp } u(t, \cdot)$, and (4.4) is proved.

We are thus led to search a solution $u(t, x)$ of the Cauchy problem (4.3) of the form

$$(4.6) \quad u(t, x) = \varphi(x)g(t, x) + \psi(x)h(t, x),$$

with smooth functions g, h which satisfy the initial conditions

$$\begin{aligned} g(0, x) &= 1, & g_t(0, x) &= 0 \\ h(0, x) &= 0, & h_t(0, x) &= 1. \end{aligned}$$

Condition (4.4) allows also to consider our Cauchy problem as a periodic Cauchy problem in $\mathbf{R}_x$, that is we can assume φ, ψ to be two smooth 2π -periodic functions and search a C^∞ solution $u(t, x)$, 2π -periodic with respect to x . This will be useful in the following.

For simplicity we write here $a_0 = a_0(\varphi g + \psi h)$ and

$$H = \varphi(x)g(t, x) + \psi(x)h(t, x).$$

Substituting (4.6) in our equation

$$u_{tt} - u^2 a_0 u_{xx} = 2u a_0 u_x^2 + u^2 a_0' u_x^2$$

we thus obtain:

$$\begin{aligned} \varphi g_{tt} + \psi h_{tt} - H^2 a_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)_x &= 2H a_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 \\ &+ H^2 a_0' (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2, \end{aligned}$$

i.e.

$$\begin{aligned}
& \varphi g_{tt} + \psi h_{tt} - \varphi H^2 a_0 g_{xx} - \psi H^2 a_0 h_{xx} \\
& \quad - \varphi g H a_0 (\varphi_{xx} g + 2\varphi_x g_x) - \varphi g H a_0 (\psi_{xx} h + 2\psi_x h_x) \\
& \quad - \psi h H a_0 (\varphi_{xx} g + 2\varphi_x g_x) - \psi h H a_0 (\psi_{xx} h + 2\psi_x h_x) \\
& = 2H a_0 (\varphi_x g + \psi_x h)^2 + 4H a_0 (\varphi_x g + \psi_x h) (\varphi g_x + \psi h_x) \\
& \quad + 2H a_0 (\varphi g_x + \psi h_x)^2 + \varphi g H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 \\
& \quad + \psi h H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 \\
& = 2a_0 \varphi g (\varphi_x g + \psi_x h)^2 + 2a_0 \psi h (\varphi_x g + \psi_x h)^2 \\
& \quad + 4H a_0 \varphi g_x (\varphi_x g + \psi_x h) + 4H a_0 \psi h_x (\varphi_x g + \psi_x h) \\
& \quad + 2H a_0 \varphi (\varphi g_x^2 + \psi g_x h_x) + 2H a_0 \psi (\psi h_x^2 + \varphi g_x h_x) \\
& \quad + \varphi g H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 + \psi h H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2.
\end{aligned}$$

It is then sufficient to solve, in a neighbourhood of $\text{supp } \varphi \cup \text{supp } \psi$, the following system:

$$(4.7) \quad \begin{cases} g_{tt} - H^2 a_0 g_{xx} - g H a_0 (\varphi_{xx} g + 2\varphi_x g_x) - g H a_0 (\psi_{xx} h + 2\psi_x h_x) \\ \quad = 2a_0 g (\varphi_x g + \psi_x h)^2 + 4H a_0 g_x (\varphi_x g + \psi_x h) \\ \quad \quad + 2H a_0 g_x (\varphi g_x + \psi h_x) + g H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 \\ h_{tt} - H^2 a_0 h_{xx} - h H a_0 (\varphi_{xx} g + 2\varphi_x g_x) - h H a_0 (\psi_{xx} h + 2\psi_x h_x) \\ \quad = 2a_0 h (\varphi_x g + \psi_x h)^2 + 4H a_0 h_x (\varphi_x g + \psi_x h) \\ \quad \quad + 2H a_0 h_x (\psi h_x + \varphi g_x) + h H a'_0 (\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2. \end{cases}$$

Setting $U = \begin{pmatrix} g \\ h \end{pmatrix}$, system (4.7) is equivalent to the following:

$$(4.8) \quad \begin{cases} U_{tt} - H^2 a_0 U_{xx} + H A U_x + B U = 0 \\ U(0, x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U_t(0, x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{cases}$$

where $A = A(x, U, U_x)$ and $B = B(x, U)$ are 2×2 smooth matrices; more precisely:

$$A = A(\partial^\alpha \varphi, \partial^\beta \psi, U, U_x) \quad \text{with } 0 \leq \alpha, \beta \leq 1$$

$$B = B(\partial^\alpha \varphi, \partial^\beta \psi, U) \quad \text{with } 0 \leq \alpha, \beta \leq 2.$$

Note that system (4.8) is weakly hyperbolic and the Levi condition is satisfied because

$$\|D_V A(x, U, V)V\|^2 \leq ca_0,$$

for some positive constant c , when g, h and their first-order derivatives are bounded in $\mathbf{R}^+ \times \mathbf{R}$. Therefore, we can argue as in [M1] and prove its local solvability using the Nash-Moser theorem (Theorem 4.1).

To this aim we set $\mathcal{T} = \mathbf{R}/2\pi\mathbf{Z}$ for the 1-dimensional torus, and consider the tame spaces (cf. [H], [M1]):

$$\begin{aligned} \mathcal{F} &= \{v \in C^2([0, T]; C^\infty(\mathcal{T})) : v_t(0, x) = 0\} \\ &\quad \times \{v \in C^2([0, T]; C^\infty(\mathcal{T})) : v(0, x) = 0\} \end{aligned}$$

and

$$\mathcal{G} = (C^0([0, T]; C^\infty(\mathcal{T})))^2 \times (C^\infty(\mathcal{T}))^2,$$

with $T > 0$ to be chosen in the following.

We then consider the non-linear map

$$P : \mathcal{F} \rightarrow \mathcal{G}$$

defined by

$$P(g, h) = (P_1(g, h), P_2(g, h), g(0, x) - 1, h_t(0, x) - 1),$$

with $P_1, P_2 : \mathcal{F} \rightarrow C^0([0, T]; C^\infty(\mathcal{T}))$ given by:

$$\begin{aligned} P_1(g, h) &= g_{tt} - H^2 a_0 g_{xx} - gHa_0(\varphi_{xx}g + 2\varphi_x g_x) \\ &\quad - gHa_0(\psi_{xx}h + 2\psi_x h_x) - 2a_0g(\varphi_x g + \psi_x h)^2 \\ &\quad - 4Ha_0g_x(\varphi_x g + \psi_x h) - 2Ha_0g_x(\varphi g_x + \psi h_x) \\ &\quad - gHa'_0(\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2 \end{aligned}$$

and

$$\begin{aligned} P_2(g, h) &= h_{tt} - H^2 a_0 h_{xx} - hHa_0(\varphi_{xx}g + 2\varphi_x g_x) \\ &\quad - hHa_0(\psi_{xx}h + 2\psi_x h_x) - 2a_0h(\varphi_x g + \psi_x h)^2 \\ &\quad - 4Ha_0h_x(\varphi_x g + \psi_x h) - 2Ha_0h_x(\varphi g_x + \psi h_x) \\ &\quad - hHa'_0(\varphi_x g + \varphi g_x + \psi_x h + \psi h_x)^2. \end{aligned}$$

The map P defined above is smooth tame, being composition of differential operators, which are all smooth tame (cf. [H]).

We define now $(g_0, h_0) \in \mathcal{F}$ by

$$g_0(t, x) = 1 + \frac{t^2}{2} \{ \varphi \varphi_{xx} a_0(\varphi) + 2\varphi_x^2 a_0(\varphi) + \varphi \varphi_x^2 a'_0(\varphi) \}$$

and

$$h_0(t, x) = t,$$

and set

$$P(g_0, h_0) = (P_1(g_0, h_0), P_2(g_0, h_0), 0, 0) := (z_1(t, x), z_2(t, x), 0, 0).$$

Note that

$$z_1(0, x) = z_2(0, x) = 0 \quad \forall x \in \mathcal{T}.$$

To prove the existence of a local solution $U(t, x)$ of (4.8) it is enough to prove that the image of P contains a neighbourhood of $P(g_0, h_0)$ in $\mathcal{G}$. As a matter of fact, every neighbourhood of $(z_1(t, x), z_2(t, x), 0, 0)$ in $\mathcal{G}$ contains elements of the form $(\tilde{z}_1(t, x), \tilde{z}_2(t, x), 0, 0)$ with

$$\tilde{z}_1(t, x) = \tilde{z}_2(t, x) = 0 \quad \forall (t, x) \in [0, \delta] \times \mathcal{T},$$

for some $\delta > 0$ small enough. On the other hand, if $P(g, h) = (\tilde{z}_1(t, x), \tilde{z}_2(t, x), 0, 0)$, then $U = \begin{pmatrix} g \\ h \end{pmatrix}$ is a local solution of (4.8) in $[0, \delta] \times \mathcal{T}$. (We shall choose $T = \delta$).

In order to show that the image of P contains a neighbourhood of $P(g_0, h_0)$ in $\mathcal{G}$, we now use the Nash-Moser theorem and prove that $P : \mathcal{F} \rightarrow \mathcal{G}$ is locally invertible in a neighbourhood of (g_0, h_0) in $\mathcal{F}$.

To this aim we first compute the Fréchet derivative $DP(g, h)$ of P in (g, h) :

$$\begin{aligned} DP(g, h)\{k, l\} &= \lim_{\varepsilon \rightarrow 0} \frac{P(g + \varepsilon k, h + \varepsilon l) - P(g, h)}{\varepsilon} \\ &= (DP_1(g, h)\{k, l\}, DP_2(g, h)\{k, l\}, k(0, x), l_t(0, x)) \end{aligned}$$

with

$$DP_1(g, h)\{k, l\} = k_{tt} - H^2 a_0 k_{xx} + H \tilde{A} k_x + H \tilde{B} l_x + \tilde{M} k + \tilde{N} l$$

and

$$DP_2(g, h)\{k, l\} = l_{tt} - H^2 a_0 l_{xx} + H \hat{A} k_x + H \hat{B} l_x + \hat{M} k + \hat{N} l,$$

where $\tilde{A}, \tilde{B}, \tilde{M}, \tilde{N}, \hat{A}, \hat{B}, \hat{M}, \hat{N}$ are polynomial functions of a_0, a'_0 and of g, h, φ, ψ together with their partial derivatives of order less or equal than 2.

Equation

$$(4.9) \quad DP(g, h)\{k, l\} = q$$

is then a weakly hyperbolic equation which satisfies the Levi condition on the coefficients of the first order terms.

We can thus prove, as in [M1], that equation (4.9) has a unique solution (k, l) for every (g, h) in a suitable neighbourhood W of (g_0, h_0) in $\mathcal{F}$ and for every $q \in \mathcal{G}$.

We can also prove, as in [M1], that the map

$$\begin{aligned} VP : W \times \mathcal{G} &\rightarrow \mathcal{F} \\ ((g, h), q) &\mapsto (k, l) \end{aligned}$$

is smooth tame.

Then, by the Nash-Moser theorem, $P : \mathcal{F} \rightarrow \mathcal{G}$ is locally invertible and hence the Cauchy problem (4.8) has a local solution $U = \begin{pmatrix} g \\ h \end{pmatrix}$.

This proves the existence of a local solution $u(t, x)$ of (4.3) of the form

$$u(t, x) = \varphi(x)g(t, x) + \psi(x)h(t, x).$$

Uniqueness

We shall finally prove that this is the unique smooth solution of the Cauchy problem (4.3) in $[0, T) \times \mathbf{R}$.

Assume by contradiction that $v(t, x)$ is another smooth solution of (4.3) in $[0, T) \times \mathbf{R}$. Then $u(t, x)$ and $v(t, x)$ satisfy the system:

$$(4.10) \quad \begin{cases} u_{tt} - a(u)u_{xx} = a'(u)u_x^2 \\ v_{tt} - a(v)v_{xx} = a'(v)v_x^2 \end{cases}$$

Setting $w = u - v$, and writing

$$a(u(t, x)) - a(v(t, x)) = (u(t, x) - v(t, x)) \cdot \tilde{a}(t, x)$$

and

$$a'(u(t, x)) - a'(v(t, x)) = (u(t, x) - v(t, x)) \cdot \tilde{b}(t, x)$$

for two smooth functions $\tilde{a}(t, x)$ and $\tilde{b}(t, x)$, from (4.10) we obtain:

$$w_{tt} - a(u)w_{xx} - (a(u) - a(v))v_{xx} = a'(u)(u_x^2 - v_x^2) + (a'(u) - a'(v))v_x^2.$$

Therefore

$$w_{tt} - a(u)w_{xx} - w\tilde{a}(t, x)v_{xx} = a'(u)w_x(u_x + v_x) + w\tilde{b}(t, x)v_x^2,$$

i.e.

$$(4.11) \quad w_{tt} - a(u)w_{xx} - a'(u)\tilde{c}(t, x)w_x - \tilde{d}(t, x)w = 0,$$

for two smooth bounded functions $\tilde{c}(t, x)$ and $\tilde{d}(t, x)$.

Recalling that $a(u) = u^2 a_0(u)$ and $u = \varphi g + \psi h = H$, equation (4.11) is of the form:

$$w_{tt} - H^2 a_0 w_{xx} - H \hat{c}(t, x) w_x - \tilde{d}(t, x) w = 0,$$

for a smooth bounded function $\hat{c}(t, x)$.

This equation has a unique solution $w(t, x) \in C^\infty([0, T] \times \mathbf{R})$ (see [M1], and in particular Theorem 2.1).

Since $w(0, x) = 0$ and $w_t(0, x) = 0$, we finally have that $w(t, x) \equiv 0$, i.e. $v(t, x) = u(t, x) = \varphi(x)g(t, x) + \psi(x)h(t, x)$. $\square$

Remark 4.4. Following the proof of Theorem 4.3 we also see that the solution $u(t, x)$ of the Cauchy problem (4.3) has the special form:

$$u(t, x) = \varphi(x)g(t, x) + \psi(x)h(t, x)$$

for two smooth functions g, h .

As we already noticed at the beginning of this subsection, Theorem 4.3 can be generalized to the following:

Theorem 4.5. *Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$ and $a \in C^\infty(\mathbf{R})$ with only zeros of finite order and such that $a(\eta) \geq 0$ for all $\eta \in \mathbf{R}$. Then there exists $T > 0$ such that the Cauchy problem*

$$\begin{cases} u_{tt} - (a(u)u_x)_x = 0 \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

has a unique local solution $u \in C^\infty([0, T] \times \mathbf{R})$.

4.2. Local existence for the equation $u_{tt} - a(t, x, u_x)u_{xx} = f(t, x, u_x)$

In this subsection we consider the Cauchy problem

$$(4.12) \quad \begin{cases} u_{tt} - a(t, x, u_x)u_{xx} = f(t, x, u_x) \\ u(0, x) = \varphi(x) \in C_0^\infty(\mathbf{R}) \\ u_t(0, x) = \psi(x) \in C_0^\infty(\mathbf{R}) \end{cases}$$

and state the existence of a unique local smooth solution under some suitable assumptions on the coefficients a, f (see Theorem 4.6 below).

The proof will be omitted since it is very similar to that of Theorem 4.3. We also refer to [M1] (see Remark 1.2).

Theorem 4.6. Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$ and $a, f \in C^\infty(\mathbf{R}^+ \times \mathbf{R}^2)$ such that:

i) $a(t, x, \eta) = a_1(t, x, \eta)a_2(\eta)$ with $a_2(\eta) \geq 0$ for every $\eta \in \mathbf{R}$ and

$$0 < \lambda_0 \leq a_1(t, x, \eta) \leq A_0 \quad \text{on bounded subsets of } \mathbf{R}^+ \times \mathbf{R}^2;$$

ii) the function $a_2(\eta)$ has only zeros of finite order;

iii) $f_x(t, x, \eta) = 0$ if $a_2(\eta) = 0$, $f(t, x, 0) = 0$;

iv) the following Levi condition is satisfied:

$$|f_\eta(t, x, \eta)|^2 \leq ca(t, x, \eta) \quad \forall (t, x) \in \mathbf{R}^+ \times \mathbf{R}, \eta \text{ bounded}$$

for some constant $c \geq 0$.

Then there exists $T > 0$ such that the Cauchy problem (4.12) has a unique local solution $u \in C^\infty([0, T) \times \mathbf{R})$.

As a particular case, we have:

Corollary 4.7. Let $\varphi, \psi \in C_0^\infty(\mathbf{R})$ and $a, f \in C^\infty(\mathbf{R}^+ \times \mathbf{R}^2)$ of the form:

$$a(t, x, \eta) = \eta^{2q}a_0(t, x, \eta) \quad \forall (t, x, \eta) \in \mathbf{R}^+ \times \mathbf{R}^2$$

$$f(t, x, \eta) = \eta^N f_0(t, x, \eta) \quad \forall (t, x, \eta) \in \mathbf{R}^+ \times \mathbf{R}^2,$$

for some $q, N \in \mathbf{N} \setminus \{0\}$ with $N \geq q + 1$, where a_0 and f_0 are smooth functions which satisfy, for some positive constants λ_0, A_0 :

$$0 < \lambda_0 \leq a_0(t, x, \eta) \leq A_0 \quad \forall (t, x, \eta) \in \mathbf{R}^+ \times \mathbf{R}^2$$

$$|f_0(t, x, \eta)| \leq A_0 \quad \forall (t, x, \eta) \in \mathbf{R}^+ \times \mathbf{R}^2.$$

Then there exists $T > 0$ such that the Cauchy problem

$$\begin{cases} u_{tt} - a(t, x, u_x)u_{xx} = f(t, x, u_x) \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

has a unique local solution $u \in C^\infty([0, T) \times \mathbf{R})$.

Remark 4.8. Corollary 4.7 may apply, in particular, to equation (1.19) of § 1, to equations (2.5) and (2.18) of § 2 (with $q = 1$), and to equation (3.11) of § 3.

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