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Intersection Numbers for 1-Forms Associated with Confluent Hypergeometric Functions

By

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1. Introduction

There are four confluent hypergeometric functions of the Gauss hypergeometric function, which are called the Kummer function, the Bessel function, the Hermite function and the Airy function. They admit the following type of integral representation

$$\int_\gamma u(t)\varphi(t), \tag{1}$$

where $u(t)$ is a multi-valued function such that the logarithmic derivative $\omega := d\log(u(t))$ is a rational 1-form on the projective line $\boldsymbol{P}$ with the polar divisor $D \subset \{0, 1, \infty\}$, $\varphi(t)$ is a rational 1-form on $\boldsymbol{P}$ with poles only on D and γ is a curve such that the function $u(t)$ vanishes at the terminals of γ. By the Stokes theorem

$$\int_{\partial\rho} u(t)\eta(t) = \int_\rho d(u(t)\eta(t)) = \int_\rho u(t)(d\eta(t) + \omega \wedge \eta(t)),$$

we have

$$\int_\gamma u(t)\nabla_\omega f(t) = \int_{\partial\gamma} u(t)f(t) = 0, \tag{2}$$

where ∇_ω is the derivation $d + \omega\wedge$ and $f(t)$ is a rational function on $\boldsymbol{P}$ with poles only on D. The equality (2) implies that the integral (1) depends only on the class of the form $\varphi(t) \in \Gamma(\boldsymbol{P}, \Omega^1(*D))$ modulo $\nabla_\omega\Gamma(\boldsymbol{P}, \mathcal{O}(*D))$. In this way, we are naturally led to the first rational twisted de-Rham cohomology group

$$H^1(\Gamma(\Omega^\bullet(*D), \nabla_\omega)) = \Gamma(\boldsymbol{P}, \Omega^1(*D))/\nabla_\omega\Gamma(\boldsymbol{P}, \mathcal{O}(*D))$$

with respect to ω. This group is important for the study of confluent hypergeometric functions and the differential equations annihilating these.

In this paper, we introduce a pairing between $H^1(\Gamma(\Omega^\bullet(*D), \nabla_\omega))$ and $H^1(\Gamma(\Omega^\bullet(*D), \nabla_{-\omega}))$ for general effective divisors D on $\boldsymbol{P}$, following the idea in

[2], and present intersection matrices with respect to some bases of these groups. It is shown in [2], [5], [8] and [10] that intersection matrices of the first twisted cohomology groups with respect to $\pm\omega$ with at most simple poles play an important role in studying hypergeometric functions with regular singularities.

Our intersection matrices appear in [7] when considering the limit (confluence) of the intersection matrices associated with Lauricella's hypergeometric functions F_D's in [2].

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2. Isomorphisms

Let $n_1, \dots, n_m$ be natural numbers such that

$$n_1 \geq n_2 \geq \cdots \geq n_m, \qquad n = \sum_{i=1}^{m} n_i \geq 3$$

and let $x_1, \dots, x_m$ be distinct m points on the complex projective line $\boldsymbol{P}$. Put

$$X = \boldsymbol{P} \backslash \{x_1, \dots, x_m\},$$

$$\omega = \sum_{i=1}^{m} \left(\frac{\alpha_{i;1}}{t - x_i} + \frac{\alpha_{i;2}}{(t - x_i)^2} + \cdots + \frac{\alpha_{i;n_i - 1}}{(t - x_i)^{n_i - 1}} + \frac{\alpha_{i;n_i}}{(t - x_i)^{n_i}} \right) dt,$$

where $\alpha_{i;k} \in \boldsymbol{C}$, $\alpha_{i;n_i} \neq 0$ and $\alpha_{i;1}$'s satisfy that $\alpha_{i;1} \notin \boldsymbol{Z}$ if $n_i = 1$ and that

$$\sum_{i=1}^{m} \alpha_{i;1} = 0.$$

For the connection $\nabla_\omega = d + \omega \wedge$ with respect to ω and the polar divisor

$$D = \sum_{i=1}^{m} n_i x_i$$

of the rational 1-form ω on $\boldsymbol{P}$, we have a complex:

$$(\Omega^\bullet(*D), \nabla_\omega) : \cdots 0 \to \mathcal{O}(*D) \xrightarrow{\nabla_\omega} \Omega^1(*D) \to 0 \cdots,$$

where $\mathcal{O}(*D)$ and $\Omega^1(*D)$ are the sheave of meromorphic functions and 1-forms with poles only on $\{x_1, \dots, x_m\}$, respectively. The group

$$H^1(\Gamma(\Omega^\bullet(*D), \nabla_\omega)) = \Gamma(\boldsymbol{P}, \Omega^1(*D)) / \nabla_\omega \Gamma(\boldsymbol{P}, \mathcal{O}(*D))$$

is called the first rational twisted de-Rham cohomology group with respect to ∇_ω.

Proposition 2.1. *We have the following isomorphism:*

$$\jmath_\omega : H^1(\Gamma(\Omega^\bullet(*D), \nabla_\omega)) \to \Gamma(\boldsymbol{P}, \Omega^1(D))/\boldsymbol{C} \cdot \omega,$$

where $\Omega^1(D)$ *is the sheaf of meromorphic* 1*-forms* ψ *such that*

$$(\psi) + D \geq 0.$$

The rank of $H^1(\Gamma(\Omega^\bullet(*D), \nabla_\omega))$ *is* $n-2$.

Proof. We can eliminate poles of any element of $\Gamma(\boldsymbol{P}, \Omega^1(*D))$ higher than D by ∇_ω-images; refer to [4] for details. The rank of $\Gamma(\boldsymbol{P}, \Omega^1(D))/\boldsymbol{C} \cdot \omega$ is

$$(\deg(D) - 1) - 1 = \left(\sum_{i=1}^{m} n_i\right) - 2 = n - 2. \quad \square$$

We have two complexes:

$$F_1(\Omega^\bullet(*D), \nabla_\omega) : \cdots 0 \to \mathcal{O} \xrightarrow{\nabla_\omega} \Omega^1(D) \to 0 \cdots,$$

$$F_0(\Omega^\bullet(*D), \nabla_\omega) : \cdots 0 \to \mathcal{O}(-D) \xrightarrow{\nabla_\omega} \Omega^1 \to 0 \cdots,$$

where $\mathcal{O}$ is the sheaf of holomorphic functions on $\boldsymbol{P}$, Ω^1 the sheaf of holomorphic 1-forms on $\boldsymbol{P}$, and $\mathcal{O}(-D)$ the sheaf of holomorphic functions f such that

$$(f) - D \geq 0.$$

Proposition 2.2. *We have two isomorphisms*

$$\mathbf{H}^1(\boldsymbol{P}, F_1(\Omega^\bullet(*D), \nabla_\omega)) \simeq \Gamma(\boldsymbol{P}, \Omega^1(D))/\boldsymbol{C} \cdot \omega,$$

$$\mathbf{H}^1(\boldsymbol{P}, F_0(\Omega^\bullet(*D), \nabla_\omega)) \simeq \ker(\nabla_\omega : H^1(\boldsymbol{P}, \mathcal{O}(-D)) \to H^1(\boldsymbol{P}, \Omega^1)).$$

Proof. Since the spectral sequence

$$E_1^{0q} \simeq H^q(\boldsymbol{P}, \mathcal{O}), \qquad E_1^{1q} \simeq H^q(\boldsymbol{P}, \Omega^1(D))$$

abuts to $\mathbf{H}^{p+q}(\boldsymbol{P}, F_1(\Omega^\bullet(*D), \nabla_\omega))$ and $E_1^{pq} = 0$ for $q > 0$, we have an isomorphism

$$\mathbf{H}^1(\boldsymbol{P}, F_1(\Omega^\bullet(*D), \nabla_\omega)) \simeq \Gamma(\boldsymbol{P}, \Omega(D))/\boldsymbol{C} \cdot \omega.$$

Since the spectral sequence

$$E_1^{0q} \simeq H^q(\boldsymbol{P}, \mathcal{O}(-D)), \qquad E_1^{1q} \simeq H^q(\boldsymbol{P}, \Omega^1)$$

abuts to $\mathbf{H}^{p+q}(\boldsymbol{P}, F_0(\Omega^\bullet(*D), \nabla_\omega))$ and $E_1^{pq} = 0$ for $q = 0$, we have an isomorphism

$$\mathbf{H}^1(\boldsymbol{P}, F_0(\Omega^\bullet(*D), \nabla_\omega)) \simeq \ker(\nabla_\omega : H^1(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-D)) \to H^1(\boldsymbol{P}, \Omega^1_{\boldsymbol{P}})). \quad \square$$

Consider the following exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\iota} & \mathcal{O} & \longrightarrow & \bigoplus_{i=1}^{n} \left(\sum_{k=1}^{n_i} b_{i;k}(t-x_i)^{k-1} \right)_{x_i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla_\omega} & & \downarrow{\scriptstyle \nabla_\omega} & & \downarrow{\scriptstyle \nabla_\omega} & & \\
0 & \longrightarrow & \Omega^1 & \xrightarrow{\iota} & \Omega^1(D) & \xrightarrow{\varpi} & \bigoplus_{i=1}^{n} \left(\sum_{k=1}^{n_i} \frac{c_{i;k}\, dt}{(t-x_i)^k} \right)_{x_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\left(\sum_{k=1}^{n_i} b_{i;k}(t-x_i)^{k-1} \right)_{x_i}$ is the germ of polynomials of degree $(n_i - 1)$ at x_i, $\left(\sum_{k=1}^{n_i} \frac{c_{i;k}\, dt}{(t-x_i)^k} \right)_{x_i}$ is the germ of the n_i-th principal parts of the Laurent expansion of meromorphic forms at x_i, and ϖ is defined by taking the principal part of a meromorphic form at x_i.

Proposition 2.3. *The inclusion ι induces the isomorphism between two hypercohomology groups*

$$\iota : \mathbf{H}^\bullet(\boldsymbol{P}, F_0(\Omega^\bullet(*D), \nabla_\omega)) \to \mathbf{H}^\bullet(\boldsymbol{P}, F_1(\Omega^\bullet(*D), \nabla_\omega)).$$

Proof. Note that

$$\nabla_\omega : \sum_{k=1}^{n_i} b_{i;k}(t-x_i)^{k-1} \to \sum_{k=1}^{n_i} \frac{c_{i;k}\, dt}{(t-x_i)^k}$$

is expressed as

$$(c_{i;n_i}, \ldots, c_{i;1}) = (b_{i;1}, \ldots, b_{i;n_i}) A_i$$

$$A_i = \begin{pmatrix}
\alpha_{i;n_i} & \alpha_{i;n_i-1} & \cdots & \alpha_{i;2} & \alpha_{i;1} \\
 & \alpha_{i;n_i} & \cdots & \alpha_{i;3} & \alpha_{i;2} \\
 & & \ddots & \vdots & \vdots \\
 & \mathbf{0} & & \alpha_{i;n_i} & \alpha_{i;n_i-1} \\
 & & & & \alpha_{i;n_i}
\end{pmatrix},$$

Since $\det(A_i) = \alpha_{i;n_i}^{n_i} \neq 0$ for any i, ι induces the isomorphism between the hypercohomology groups. □

By Propositions 2.2 and 2.3, we have the isomorphism

$$\ker(\nabla_\omega : H^1(\boldsymbol{P}, \mathcal{O}(-D)) \to H^1(\boldsymbol{P}, \Omega^1)) \to \Gamma(\boldsymbol{P}, \Omega(D))/\boldsymbol{C} \cdot \omega.$$

In order to give the inverse of the above explicitly, we prepare two propositions.

Proposition 2.4. *We have an isomophism*

$$\mathbf{H}^1(\boldsymbol{P}, F_0(\Omega^\bullet(*D), \nabla_\omega)) \simeq \frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11}))}{\nabla_\omega \Gamma(\mathscr{E}^{00}(-D))},$$

where $\mathscr{E}^{pq}$ stands for the sheaf of smooth (p,q)-forms on $\boldsymbol{P}$ and $\mathscr{E}^{pq}(-D)$ the sheaf of smooth (p,q)-forms ζ on $\boldsymbol{P}$ such that $\zeta/(t-x_i)^{n_i}$ is smooth around x_i.

Proof. A fine resolution of $F_0(\Omega^\bullet(*D), \nabla_\omega)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla_\omega} & \Omega^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}^{00}(-D) & \xrightarrow{\partial+\omega} & \mathscr{E}^{10} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial} & & \downarrow{\scriptstyle \bar\partial} & & \\
0 & \longrightarrow & \mathscr{E}^{01}(-D) & \xrightarrow{\partial+\omega} & \mathscr{E}^{11} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

yields the associated single complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla_\omega} & \Omega^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}^{00}(-D) & \xrightarrow{\nabla_\omega} & \mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10} & \xrightarrow{\nabla_\omega} & \mathscr{E}^{11} & \longrightarrow & 0
\end{array}$$

which implies this proposition. □

Note that the Dolbeault theorem implies

$$H^1(\boldsymbol{P}, \mathcal{O}(-D)) \simeq \frac{\Gamma(\mathscr{E}^{01}(-D))}{\bar\partial \Gamma(\mathscr{E}^{00}(-D))}, \qquad H^1(\boldsymbol{P}, \Omega^1) \simeq \frac{\Gamma(\mathscr{E}^{11})}{\bar\partial \Gamma(\mathscr{E}^{10})}.$$

The connection ∇_ω annihilates an element

$$\eta \in \frac{\Gamma(\mathscr{E}^{01}(-D))}{\bar{\partial}\Gamma(\mathscr{E}^{00}(-D))} \simeq H^1(\boldsymbol{P}, \mathscr{O}(-D))$$

means that $\nabla_\omega \eta$ is zero in

$$H^1(\boldsymbol{P}, \Omega^1) \simeq \frac{\Gamma(\mathscr{E}^{11})}{\bar{\partial}\Gamma(\mathscr{E}^{10})},$$

i.e. there exists $\xi(\eta) \in \Gamma(\mathscr{E}^{10})$ such that

$$\nabla_\omega \eta - \bar{\partial}\xi(\eta) = 0.$$

This gives an explicit expression of the isomorphism

$$\ker(\nabla_\omega : H^1(\boldsymbol{P}, \mathscr{O}(-D)) \to H^1(\boldsymbol{P}, \Omega^1)) \ni \eta$$
$$\mapsto \eta - \xi(\eta) \in \frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11}))}{\nabla_\omega \Gamma(\mathscr{E}^{00}(-D))}.$$

Proposition 2.5. *We have an isomorphism*

$$\mathbf{H}^1(\boldsymbol{P}, F_1(\Omega^\bullet(*D), \nabla_\omega)) \simeq \frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01} \oplus \mathscr{E}^{10}(D)) \to \Gamma(\mathscr{E}^{11}(D)))}{\nabla_\omega \Gamma(\mathscr{E}^{00})},$$

where $\mathscr{E}^{pq}(D)$ stands for the sheaf of smooth (p,q)-forms ζ on X such that $(t - x_i)^{n_i}\zeta$ is smooth around x_i.

Proof. A fine resolution of $F_1(\Omega^\bullet(*D), \nabla_\omega)$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{O} & \xrightarrow{\nabla_\omega} & \Omega^1(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}^{00} & \xrightarrow{\partial+\omega} & \mathscr{E}^{10}(D) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} & & \\
0 & \longrightarrow & \mathscr{E}^{01} & \xrightarrow{\partial+\omega} & \mathscr{E}^{11}(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

yields the associated single complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\nabla_\omega} & \Omega^1(D) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E}^{00} & \xrightarrow{\nabla_\omega} & \mathscr{E}^{01} \oplus \mathscr{E}^{10}(D) & \xrightarrow{\nabla_\omega} & \mathscr{E}^{11}(D) & \longrightarrow & 0 \end{array}$$

which implies this proposition. □

Note that an explicit expression of the map

$$\Gamma(\boldsymbol{P}, \Omega(D))/\boldsymbol{C} \cdot \omega \to \frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01} \oplus \mathscr{E}^{10}(D)) \to \Gamma(\mathscr{E}^{11}(D)))}{\nabla_\omega \Gamma(\mathscr{E}^{00})}$$

is given by

$$\psi \mapsto 0 \oplus \psi.$$

The isomorphism ι in Proposition 2.3 is expressed as the natural map

$$\frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11}))}{\nabla_\omega \Gamma(\mathscr{E}^{00}(-D))}$$

$$\to \frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01} \oplus \mathscr{E}^{10}(D)) \to \Gamma(\mathscr{E}^{11}(D)))}{\Gamma(\nabla_\omega \mathscr{E}^{00})}.$$

Let us express the inverse of $0 \oplus \psi$ $(\psi \in \Gamma(\boldsymbol{P}, \Omega(D))/\boldsymbol{C} \cdot \omega)$ of the above isomorphism. Let

$$\sum_{k=1}^{n_i} \frac{c_{i;k}\, dt}{(t - x_i)^k}$$

be the principal part of the Laurent expression of ψ at x_i. Put

$$\eta_i = \sum_{k=1}^{n_i} b_{i;k}(t - x_i)^{k-1},$$

where

$$(b_{i,1}, \ldots, b_{i;n_i}) = (c_{i;n_i}, \ldots, c_{i;1}) A_i^{-1}.$$

A smooth function on $\boldsymbol{P}$

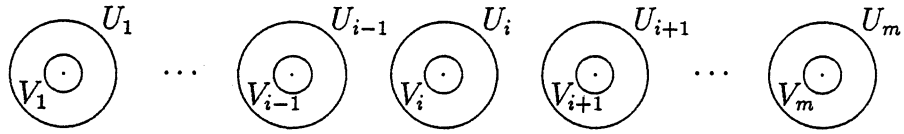
$$\eta = \sum_{i=1}^{m} h_i \eta_i$$

satisfies

$$\nabla_\omega \eta = \sum_{i=1}^{m} (dh_i \eta_i + h_i \nabla_\omega \eta_i),$$

where h_i is a smooth function such that

$$\begin{aligned} h_i(t) &= 1 \qquad t \in V_i, \\ 0 \le h_i(t) &\le 1 \qquad t \in U_i \backslash V_i, \\ h_i(t) &= 0 \qquad t \in U_i^c. \end{aligned}$$



Since $\nabla_\omega \eta_i$ is the principal part of the Laurent expression of ψ at x_i and

$$dh_i(t) = 0, \qquad h_i(t) = 1$$

for $t \in V_i$, we have

$$\psi - \nabla_\omega \eta \in \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10}),$$

which represents $\iota^{-1}(\psi)$ in

$$\frac{\ker(\nabla_\omega : \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11}))}{\nabla_\omega \Gamma(\mathscr{E}^{00}(-D))}.$$

We conclude this result in the following Proposition.

Proposition 2.6. *For any $\psi \in \Gamma(\boldsymbol{P}, \Omega(D))$, construct a smooth function*

$$\eta = \sum_{i=1}^{m} h_i \eta_i,$$

where h_i is a smooth function with small support to be identically 1 on the small neighborhood V_i of x_i, and η_i is the polynomial such that $\nabla_\omega \eta_i$ is equal to the principal part of the Laurent expression of ψ at x_i. Then the isomorphism

$$\iota_\omega : \Gamma(\boldsymbol{P}, \Omega(D))/\boldsymbol{C} \cdot \omega \to \ker(\nabla_\omega : H^1(\boldsymbol{P}, \mathscr{O}(-D)) \to H^1(\boldsymbol{P}, \Omega^1))$$

is given by

$$\psi \mapsto \iota_\omega(\psi) = \psi - \nabla_\omega \eta,$$

of which (01)-part is $-\sum_{i=1}^{m} (\eta_i dh_i)^{(01)}$, where $\zeta^{(01)}$ stands for the (01)-part of smooth 1-form ζ.

3. Duality

By Propositions in the previous section, we have the isomorphisms

$$H^1(\Gamma(\Omega^\bullet(*D),\nabla_\omega)) \xrightarrow{\jmath_\omega} \Gamma(\boldsymbol{P},\Omega^1(D))/\boldsymbol{C}\cdot\omega$$
$$\xrightarrow{\imath_\omega} \ker(\nabla_\omega : H^1(\boldsymbol{P},\mathcal{O}(-D)) \to H^1(\boldsymbol{P},\Omega^1)),$$
$$H^1(\Gamma(\Omega^\bullet(*D),\nabla_{-\omega})) \xrightarrow{\jmath_{-\omega}} \Gamma(\boldsymbol{P},\Omega^1(D))/\boldsymbol{C}\cdot(-\omega).$$

We define the intersection number $\langle\psi^+,\psi^-\rangle_\omega$ of $\psi^+ \in H^1(\Gamma(\Omega^\bullet(*D),\nabla_\omega))$ and $\psi^- \in H^1(\Gamma(\Omega^\bullet(*D),\nabla_{-\omega}))$ as

$$\langle\psi^+,\psi^-\rangle_\omega = \int_{\boldsymbol{P}} \imath_\omega\circ\jmath_\omega(\psi^+)\wedge\jmath_{-\omega}(\psi^-).$$

Since $\imath_\omega\circ\jmath_\omega(\psi^+)$ and $\jmath_{-\omega}(\psi^-)$ are represented by elements of $\Gamma(\mathscr{E}^{01}(-D))$ and $\Gamma(\mathscr{E}^{10}(D))$, the above integral converges. Since we have

$$\int_{\boldsymbol{P}} \imath_\omega\circ\jmath_\omega(\psi^+)\wedge\omega = \int_{\boldsymbol{P}}\{\nabla_\omega(\imath_\omega\circ\jmath_\omega(\psi^+)) - d(\imath_\omega\circ\jmath_\omega(\psi^+))\}$$
$$= \int_{\boldsymbol{P}} \bar\partial\xi(\imath_\omega\circ\jmath_\omega(\psi^+)) = \int_{\boldsymbol{P}} d\xi(\imath_\omega\circ\jmath_\omega(\psi^+)) = 0,$$

where $\xi(\imath_\omega\circ\jmath_\omega(\psi^+))$ is an element $\Gamma(\mathscr{E}^{10})$ such that

$$\nabla_\omega(\imath_\omega\circ\jmath_\omega(\psi^+)) - \bar\partial\xi(\imath_\omega\circ\jmath_\omega(\psi^+)) = 0,$$

and

$$\int_{\boldsymbol{P}} (\nabla_\omega(\zeta^{01}))^{(01)}\wedge\jmath_{-\omega}(\psi^-) = \int_{\boldsymbol{P}} (\bar\partial\zeta)\wedge\jmath_{-\omega}(\psi^-) = \int_{\boldsymbol{P}} d(\zeta\wedge\jmath_{-\omega}(\psi^-)) = 0,$$

for $\zeta\in\Gamma(\mathscr{E}^{00}(-D))$, the intersection number $\langle\psi^+,\psi^-\rangle_\omega$ is well-defined. Note that this pairing is naturally induced by the Serre duality for $H^1(\boldsymbol{P},\mathcal{O}(-D))$ and $\Gamma(\boldsymbol{P},\Omega^1(D))$.

4. Intersection matrices

To state the main theorem, we recall notations: $n_1,\dots,n_m$ are natural numbers such that

$$n_1 \geq n_2 \geq \cdots \geq n_m, \qquad n = \sum_{i=1}^m n_i \geq 3$$

and $x_i, \dots, x_m$ are distinct m points on $\boldsymbol{P}$. For

$$\omega = \sum_{i=1}^{m} \left(\frac{\alpha_{i;1}}{t - x_i} + \frac{\alpha_{i;2}}{(t - x_i)^2} + \cdots + \frac{\alpha_{i;n_i-1}}{(t - x_i)^{n_i-1}} + \frac{\alpha_{i;n_i}}{(t - x_i)^{n_i}} \right) dt,$$

where $\alpha_{i;k} \in \boldsymbol{C}$, $\alpha_{i;n_i} \neq 0$ and $\alpha_{i;1}$'s satisfy that $\alpha_{i;1} \notin \boldsymbol{Z}$ if $n_i = 1$ and that

$$\sum_{i=1}^{m} \alpha_{i;1} = 0,$$

we put $\nabla_\omega = d + \omega \wedge$, the polar divisor

$$D = \sum_{i=1}^{n} n_i x_i$$

of the meromorphic 1-form ω on $\boldsymbol{P}$ and (n_i, n_i)-matrix

$$A_i = \begin{pmatrix} \alpha_{i;n_i} & \alpha_{i;n_i-1} & \cdots & \alpha_{i;2} & \alpha_{i;1} \\ & \alpha_{i;n_i} & \cdots & \alpha_{i;3} & \alpha_{i;2} \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & \alpha_{i;n_i} & \alpha_{i;n_i-1} \\ & & & & \alpha_{i;n_i} \end{pmatrix}.$$

Note that

$$A_i^{-1} = \begin{pmatrix} \lambda_{n_i}(\alpha_i) & \lambda_{n_i-1}(\alpha_i) & \cdots & \lambda_2(\alpha_i) & \lambda_1(\alpha_i) \\ & \lambda_{n_i}(\alpha_i) & \cdots & \lambda_3(\alpha_i) & \lambda_2(\alpha_i) \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & \lambda_{n_i}(\alpha_i) & \lambda_{n_i-1}(\alpha_i) \\ & & & & \lambda_{n_i}(\alpha_i) \end{pmatrix},$$

where $\alpha_i = (\alpha_{i;n_i}, \dots, \alpha_{i;1})$ and

$$\lambda_{n_i}(\alpha_i) = \frac{1}{\alpha_{i;n_i}}, \qquad \lambda_{n_i-1}(\alpha_i) = -\frac{\alpha_{i;n_i-1}}{\alpha_{i;n_i}^2}, \qquad \lambda_{n_i-2}(\alpha_i) = -\frac{\alpha_{i;n_i}\alpha_{i;n_i-2} - \alpha_{i;n_i-1}^2}{\alpha_{i;n_i}^3},$$

$$\lambda_{n_i-3}(\alpha_i) = -\frac{\alpha_{i;n_i}^2\alpha_{i;n_i-3} - 2\alpha_{i;n_i}\alpha_{i;n_i-1}\alpha_{i;n_i-2} + \alpha_{i;n_i-1}^3}{\alpha_{i;n_i}^4}, \dots.$$

Take a base

$$\overbrace{\varphi_{1;1}, \dots, \varphi_{1;n_1-1}}^{n_1-1}, \quad \overbrace{\varphi_{2;1}, \dots, \varphi_{2;n_2-1}}^{n_2-1}, \dots, \quad \overbrace{\varphi_{m;1}, \dots, \varphi_{m;n_m-1}}^{n_m-1},$$

$$\overbrace{\varphi_{1;n_1}, \varphi_{2;n_2}, \dots, \varphi_{m-1;n_{m-1}}}^{m-1}$$

of $(n-1)$-dimensional vector space of $\Gamma(\boldsymbol{P}, \Omega^1(D))$ as

$$\varphi_{i;k} = \begin{cases} \dfrac{(t-x_i)^{k-1}\,dt}{(t-x_i)^{n_i}}, & 1 \le i \le m,\ 1 \le k \le n_i - 1, \\ \dfrac{dt}{(t-x_i)} - \dfrac{dt}{(t-x_{i+1})}, & 1 \le i \le m-1,\ k = n_i. \end{cases}$$

Let $\varphi^{\pm}_{i;k} \in \Gamma(\boldsymbol{P}, \Omega^1(D))/\boldsymbol{C}\cdot(\pm\omega)$ be the natural projections of $\varphi_{i;k}$.

Theorem 4.1. *The intersection matrix* $\langle \varphi^{+}_{i;k}, \varphi^{-}_{j;l} \rangle_{\omega}$ *is*

$$2\pi\sqrt{-1}\begin{pmatrix} \Lambda_1 & & & & E_1 \\ & \Lambda_2 & & & E_2 \\ & & \ddots & & \vdots \\ & & & \Lambda_p & E_p \\ {}^tE_1 & {}^tE_2 & \cdots & {}^tE_p & H \end{pmatrix},$$

of which components are given as follows: *the* $(n_i - 1, n_i - 1)$*-matrix* Λ_i *is*

$$\begin{pmatrix} \lambda_1(\alpha_i) & \lambda_2(\alpha_i) & \lambda_3(\alpha_i) & \cdots & \lambda_{n_i-2}(\alpha_i) & \lambda_{n_i-1}(\alpha_i) \\ \lambda_2(\alpha_i) & \lambda_3(\alpha_i) & \lambda_4(\alpha_i) & \cdots & \lambda_{n_i-1}(\alpha_i) & \lambda_{n_i}(\alpha_i) \\ \lambda_3(\alpha_i) & \lambda_4(\alpha_i) & \lambda_5(\alpha_i) & \cdots & \lambda_{n_i}(\alpha_i) & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \lambda_{n_i-2}(\alpha_i) & \lambda_{n_i-1}(\alpha_i) & \lambda_{n_i}(\alpha_i) & \cdots & 0 & 0 \\ \lambda_{n_i-1}(\alpha_i) & \lambda_{n_i}(\alpha_i) & 0 & \cdots & 0 & 0 \end{pmatrix},$$

the $(n_i - 1, m-1)$*-matrix* E_i *is*

$$\begin{pmatrix} & & \overset{(i-1)\text{-th}}{} & \overset{i\text{-th}}{} & & \\ 0 & \cdots & -\lambda_{n_i}(\alpha_i) & \lambda_{n_i}(\alpha_i) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and the $(m-1, m-1)$*-matrix* H *is*

$$H = \begin{matrix} \\ p-1 \\ m-p \end{matrix}\overset{\begin{matrix} p-1 & m-p \end{matrix}}{\begin{pmatrix} 0 & 0 \\ 0 & H' \end{pmatrix}},$$

$$H' = \begin{pmatrix} \frac{1}{\alpha_{p+1}} & -\frac{1}{\alpha_{p+1}} & 0 & \cdots & 0 & 0 \\ -\frac{1}{\alpha_{p+1}} & \frac{1}{\alpha_{p+1}}+\frac{1}{\alpha_{p+2}} & -\frac{1}{\alpha_{p+2}} & \ddots & 0 & 0 \\ 0 & -\frac{1}{\alpha_{p+2}} & \frac{1}{\alpha_{p+2}}+\frac{1}{\alpha_{p+3}} & \ddots & 0 & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & \cdots & & & \frac{1}{\alpha_{m-2}}+\frac{1}{\alpha_{m-1}} & -\frac{1}{\alpha_{m-1}} \\ 0 & \cdots & & & -\frac{1}{\alpha_{m-1}} & \frac{1}{\alpha_{m-1}}+\frac{1}{\alpha_m} \end{pmatrix},$$

where

$$p = \#\{j \mid n_j > 1\}$$

and we regard α_i *as* $\alpha_{i;1}$ *for* $i \geq p+1$. *When* $p = 1$, *we regard* H *as* H', *and when* $p = 0$, *we regard* H *as the submatrix of* H' *obtained by deleting the first row and column.*

Proof. By Proposition 2.6, the (01)-part $\iota_\omega(\varphi_{i;k})$ is

$$\{-\eta_i(\varphi_{i;k})\, dh_i + \delta(n_i, k)\eta_{i+1}(\varphi_{i;k})\, dh_{i+1}\}^{(01)},$$

where $\delta(n_i, k)$ is Kronecker's symbol and $\eta_i(\varphi_{i;k})$ and $\eta_{i+1}(\varphi_{i;k})$ are polynomials such that the principal parts the Laurent expressions of $\nabla_\omega \eta_i(\varphi_{i;k})$ at x_i and $\nabla_\omega \eta_{i+1}(\varphi_{i;k})$ at x_{i+1} coincide with those of $\varphi_{i;k}$ at x_i and x_{i+1}, respectively. Note that the coefficients of polynomial $\eta_i(\varphi_{i;k})$ are obtained from

$$(0, \ldots, 0, \overset{k\text{-th}}{1}, 0, \ldots, 0)A^{-1},$$

which is the k-th row vector of A^{-1}, and that

$$\eta_{i+1}(\varphi_{i;n_i}) = \lambda_{n_{i+1}}(\alpha_{i+1})t^{n_{i+1}-1}.$$

By the Stokes theorem and the residue theorem, for $\varphi \in \Gamma(\boldsymbol{P}, \Omega^1(D))$, we have

$$\begin{aligned} \langle \varphi^+_{i;k}, \varphi^- \rangle_\omega &= \int_{\boldsymbol{P}} (-\eta_i(\varphi_{i;k})\, dh_i + \delta(n_i, k)\eta_{i+1}(\varphi_{i;k})\, dh_{i+1}) \wedge \varphi \\ &= \int_{U_i + U_{i+1}} (-\eta_i(\varphi_{i;k})\, dh_i + \delta(n_i, k)\eta_{i+1}(\varphi_{i;k})\, dh_{i+1}) \wedge \varphi \\ &= \int_{\partial U_i - \partial V_i} -h_i \eta_i(\varphi_{i;k})\varphi + \delta(n_i, k) \int_{\partial U_{i+1} - \partial V_{i+1}} h_{i+1}\eta_{i+1}(\varphi_{i;k})\varphi \end{aligned}$$

$$= \int_{\partial V_i} \eta_i(\varphi_{i;k})\varphi - \delta(n_i, k) \int_{\partial V_{i+1}} \eta_{i+1}(\varphi_{i;k})\varphi$$
$$= 2\pi\sqrt{-1}\{(\mathrm{Res}_{t=x_i} \eta_i(\varphi_{i;k})\varphi) - \delta(n_i, k)(\mathrm{Res}_{t=x_{i+1}} \eta_{i+1}(\varphi_{i;k})\varphi)\}.$$

Note that

$$\mathrm{Res}_{t=x_i} \eta_i(\varphi_{i;k})\varphi_{j;l} = \begin{cases} \lambda_{k+l-1}(\alpha_i), & i = j,\ k + l \leq n_i + 1, \\ -\lambda_{n_i}(\alpha_i), & j = i - 1,\ k = 1,\ l = n_{i-1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathrm{Res}_{t=x_{i+1}} \eta_{i+1}(\varphi_{i;n_i})\varphi_{j;l} = \begin{cases} \dfrac{\delta(1, n_{i+1})}{\alpha_{i+1}}, & i = j,\ l = n_i, \\ -\lambda_{n_{i+1}}(\alpha_{i+1}), & j = i + 1,\ l = 1, \\ 0, & \text{otherwise}, \end{cases}$$

which imply our results. □

5. Examples

1. Intersection matrix associated with the gamma function $\Gamma(\alpha)$. We put

$$\omega = -dt + \alpha \frac{dt}{t} (\alpha \neq 0), \qquad x_1 = \infty,\ x_2 = 0.$$

Note that the gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-1} \exp\left(\int_1^t \omega\right) \frac{dt}{t} = \int_0^\infty e^{-t} t^\alpha \frac{dt}{t}$$

for $\mathrm{Re}(\alpha) > 0$. Since ω is expressed as

$$\omega = \frac{ds}{s^2} - \alpha \frac{ds}{s}$$

around $t = \infty$ by the local coordinate $s = 1/t$, the polar divisor D of ω is $2x_1 + x_2$ and A_i's are

$$A_1 = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \alpha.$$

The vector space $\Gamma(\boldsymbol{P}, \Omega^1(D))$ is 2-dimensional; we give its base as

$$\varphi_{1,1} = \frac{ds}{s^2} = -dt, \qquad \varphi_{1,2} = \frac{ds}{s} = \frac{-dt}{t}.$$

Since

$$A_1^{-1} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad A_2^{-1} = \frac{1}{\alpha},$$

the intersection matrix with respect to this base is

$$2\pi\sqrt{-1}\begin{pmatrix} \alpha & 1 \\ 1 & 1/\alpha \end{pmatrix}$$

of which determinant is 0.

2. Intersection matrix associated with the integral $\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt$. We put

$$\omega = -t\,dt, \qquad x_1 = \infty.$$

Note that

$$\int_{-\infty}^{+\infty} e^{-t^2/2}\,dt = \int_{-\infty}^{+\infty} \exp\left(\int_0^t \omega\right) dt.$$

Since ω is expressed as

$$\omega = \frac{ds}{s^3}$$

around $t = \infty$ by the local coordinate $s = 1/t$, the polar divisor D of ω is $3x_1$ and A_1 is

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vector space $\Gamma(\boldsymbol{P}, \Omega^1(D))$ is 2-dimensional; we give its base as

$$\varphi_{1,1} = \frac{ds}{s^3} = -t\,dt, \qquad \varphi_{1,2} = \frac{ds}{s^2} = -dt.$$

The intersection matrix with respect to this base is

$$2\pi\sqrt{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $\varphi_{1,1}^{+} = 0$ as an element of the quotient the space $\Gamma(\boldsymbol{P}, \Omega^1(D))/\mathbf{C}\cdot\omega$.

3. The case $n = 4$, intersection matrices associated with confluent hypergeometric functions.

There are four confluent hypergeometric functions of the Gauss hypergeometric function

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{\alpha}(1-t)^{c-a}(1-zt)^{-b}\left(\frac{dt}{t} - \frac{dt}{t-1}\right) :$$

the Kummer function

$$K(a,c;z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^a(1-t)^{c-a}\left(\frac{dt}{t}-\frac{dt}{t-1}\right),$$

the Bessel function

$$J(a;z)=\frac{1}{2\pi\sqrt{-1}}\int_\gamma \exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)t^{-a}\frac{dt}{t},$$

the Hermite function

$$He(a,z)=\frac{1}{2\pi\sqrt{-1}}\int_0^\infty \exp\left(-\frac{1}{2}t^2+zt\right)t^{-a}\frac{dt}{t},$$

and the Airy function

$$Ai(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\ell_0-\ell_1}\exp\left(-\frac{t^3}{3}+zt\right)dt,$$

where γ is a path starting from $-\infty$, turning around 0 counterclockwise and going to $-\infty$, and ℓ_j is a ray starting from 0 with argument $2j\pi/3$. Intersection matrices associated with above five functions are obtained as follows:

(i) the Gauss hypergeometric function

$$\omega=a\frac{dt}{t}+(c-a)\frac{dt}{t-1}-b\frac{dt}{t-1/z},$$

$$D=\sum_{i=1}^4 x_i,\quad x_1=\infty,\quad x_2=0,\quad x_3=1,\quad x_4=1/z,$$

$$A_1=b-c,\quad A_2=a,\quad A_3=c-a,\quad A_4=-b,$$

$$\varphi_{1;1}=\frac{-dt}{t},\quad \varphi_{2;1}=\frac{dt}{t}-\frac{dt}{t-1},\quad \varphi_{3;1}=\frac{dt}{t-1}-\frac{dt}{t-1/z},$$

$$\langle\varphi_{i;k},\varphi_{j;l}\rangle_\omega=2\pi\sqrt{-1}\begin{pmatrix}\frac{a+b-c}{(b-c)a} & \frac{-1}{a} & 0\\ \frac{-1}{a} & \frac{c}{(c-a)a} & \frac{-1}{c-a}\\ 0 & \frac{-1}{c-a} & \frac{-a-b+c}{(c-a)b}\end{pmatrix}.$$

(ii) the Kummer function

$$\omega = -z\,dt + a\frac{dt}{t} + (c-a)\frac{dt}{t-1},$$

$$D = 2x_1 + x_2 + x_3, \quad x_1 = \infty, \quad x_2 = 0, \quad x_3 = 1,$$

$$A_1 = \begin{pmatrix} z & -c \\ 0 & z \end{pmatrix}, \quad A_2 = a, \quad A_3 = c - a,$$

$$\varphi_{1;1} = -dt = \frac{ds}{s^2}, \quad \varphi_{1;2} = \frac{-dt}{t}, \quad \varphi_{2;1} = \frac{dt}{t} - \frac{dt}{t-1},$$

$$\langle \varphi_{i;k}, \varphi_{j;l} \rangle_\omega = 2\pi\sqrt{-1} \begin{pmatrix} \frac{c}{z} & \frac{1}{z} & 0 \\ \frac{1}{z} & \frac{1}{a} & \frac{-1}{a} \\ 0 & \frac{-1}{a} & \frac{c}{a(c-a)} \end{pmatrix}.$$

(iii) the Bessel function

$$\omega = \frac{z}{2}\,dt - a\frac{dt}{t} + \frac{z}{2}\frac{dt}{t^2},$$

$$D = 2x_1 + 2x_2, \quad x_1 = \infty, \quad x_2 = 0,$$

$$A_1 = \begin{pmatrix} -\frac{z}{2} & a \\ 0 & -\frac{z}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{z}{2} & -a \\ 0 & \frac{z}{2} \end{pmatrix},$$

$$\varphi_{1;1} = -dt = \frac{ds}{s^2}, \quad \varphi_{1;2} = \frac{-dt}{t}, \quad \varphi_{2;1} = \frac{dt}{t^2},$$

$$\langle \varphi_{i;k}, \varphi_{j;l} \rangle_\omega = 2\pi\sqrt{-1} \begin{pmatrix} \frac{-4a}{z^2} & 0 & \frac{-2}{z} \\ 0 & \frac{4a}{z^2} & \frac{-2}{z} \\ \frac{-2}{z} & \frac{-2}{z} & 0 \end{pmatrix}.$$

(iv) the Hermite function

$$\omega = -t\,dt + z\,dt - a\frac{dt}{t},$$

$$D = 3x_1 + x_2, \quad x_1 = \infty, \quad x_2 = 0,$$

$$A_1 = \begin{pmatrix} 1 & -z & a \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = -a,$$

$$\varphi_{1;1} = -t\,dt = \frac{ds}{s^3}, \quad \varphi_{1;2} = -dt = \frac{ds}{s^2}, \quad \varphi_{1;3} = \frac{-dt}{t},$$

$$\langle \varphi_{i;k}, \varphi_{j;l} \rangle_\omega = 2\pi\sqrt{-1} \begin{pmatrix} z^2 - a & z & 1 \\ z & 1 & 0 \\ 1 & 0 & \dfrac{-1}{a} \end{pmatrix}.$$

(v) the Airy function

$$\omega = -t^2\,dt + z\,dt,$$

$$D = 4x_1, \quad x_1 = \infty.$$

$$A_1 = \begin{pmatrix} 1 & 0 & -z & 0 \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\varphi_{1;1} = -t^2\,dt = \frac{ds}{s^4}, \quad \varphi_{1;2} = -t\,dt = \frac{ds}{s^3}, \quad \varphi_{1;3} = -dt = \frac{ds}{s^2},$$

$$\langle \varphi_{i;k}, \varphi_{j;l} \rangle_\omega = 2\pi\sqrt{-1} \begin{pmatrix} 0 & z & 0 \\ z & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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