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Hara, Tadayuki
Miyazaki, Rinko
Morii, Takayasu

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Asymptotic Periodic Solutions for Linear Differential-Difference Equations with N Delays and Hopf Bifurcation

By

Tadayuki HARA,^{1*} Rinko MIYAZAKI² and Takayasu MORII¹

(¹Osaka Prefecture University and ²Shizuoka University, Japan)

1. Introduction

In this paper we consider the linear differential-difference equations with N delays

$$(1.1) \quad x'(t) = -\rho R(\theta) \sum_{k=1}^N x(t - \tau_k)$$

where ρ is a real constant and $R(\theta)$ represents a 2×2 matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

with $|\theta| \leq \pi/2$ and $\{\tau_k\}$ is an arithmetic sequence, that is, $\tau_k = \tau + (k-1)l$ with $\tau \geq 0$ and $l > 0$ for $k = 1, 2, \dots, N$ and $\tau_N > 0$.

Recently, the authors [5] obtained the following result.

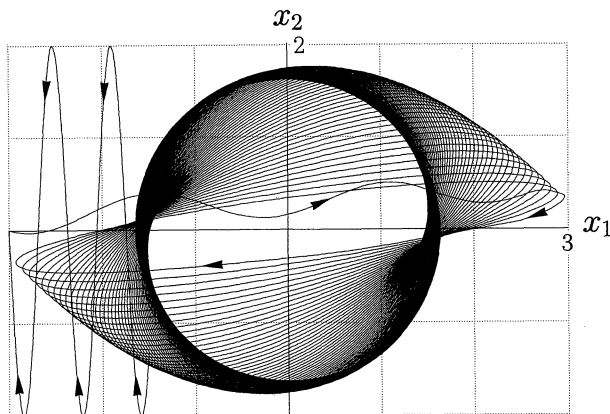
Theorem A. *The zero solution of (1.1) is uniformly asymptotically stable if and only if*

$$\rho > 0 \quad \text{and} \quad \frac{\rho(\tau_1 + \tau_N)}{2} \frac{\sin\left(\frac{Nl}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}{\sin\left(\frac{l}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)} < \frac{\pi}{2} - |\theta|.$$

The purpose of this paper is to deal with the critical case of the above result, that is, $\rho > 0$ and

$$(1.2) \quad \frac{\rho(\tau_1 + \tau_N)}{2} \frac{\sin\left(\frac{Nl}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}{\sin\left(\frac{l}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)} = \frac{\pi}{2} - |\theta|.$$

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$$\rho = \frac{127\pi}{256} \frac{\sin \frac{127\pi}{1024}}{\sin \frac{381\pi}{1024}}, \quad \theta = \frac{\pi}{256}, \quad \tau_1 = 0.5, \quad \tau_2 = 1.0, \quad \tau_3 = 1.5, \quad (t_0, \phi, \psi) = (0, -(t+3), 2 \sin 10t)$$

Fig. 1

When (1.2) holds, by a numerical simulation using the Runge–Kutta Method, we confirmed that (1.1) have asymptotic periodic solutions as the following Figure 1 shows.

In case $N = 1$ ($\tau_1 = 1$), the following result is obtained by Murakami [7] recently.

Theorem B. Suppose that $\rho = \pi/2 - |\theta|$ and $0 < |\theta| < \pi/2$. Then any solution $x(\phi)$ of Equation (1.1) with initial function $\phi \in C([-1, 0], \mathbf{R}^2)$ at $t = 0$ satisfies

$$x_t(\phi) \rightarrow \Phi_t b \quad \text{exponentially as } t \rightarrow \infty,$$

where

$$\Phi(t) = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}, \quad \omega = \begin{cases} \rho, & \theta > 0, \\ -\rho, & \theta < 0, \end{cases}$$

$$b = (I + B)^{-1} \left(\phi(0) - \rho \int_{-1}^0 \Phi^T(\xi + 1) R(\theta) \phi(\xi) d\xi \right),$$

$$B = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

We give the explicit expressions of the periodic orbits of (1.1) and present an application about the characteristic equation of (1.1) having a pair of pure imaginary eigenvalues. In Section 5, we consider the nonlinear equation

$$(1.3) \quad z'(t) = -\rho R(\theta) \sum_{k=1}^N z(t - \tau_k) [1 - |z(t)|^2]$$

and show that a Hopf bifurcation occurs at some value of ρ and obtain the periodic solution explicitly.

2. Preliminaries

In this section we introduce the decomposition theory of the space C given by Hale [2, 3] and Lunel [3]. Suppose $r \geq 0$ is a given real number, $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^n$ is an n -dimensional linear vector space over reals with norm $|\cdot|$, $C([a, b], \mathbf{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbf{R}^n$ with the topology of uniform convergence. If $[a, b] = [-r, 0]$, we let $C = C([-r, 0], \mathbf{R}^n)$ and designate the norm of an element ϕ in C by $\|\phi\| = \sup_{-r \leq \vartheta \leq 0} |\phi(\vartheta)|$. If $T \geq 0$ and $x \in C([-r, T], \mathbf{R}^n)$, then for any $t \in [0, T]$, we let $x_t \in C$ be defined by $x_t(\vartheta) = x(t + \vartheta)$, $-r \leq \vartheta \leq 0$. We consider a linear autonomous delay differential equation

$$(2.1) \quad x'(t) = Lx_t$$

where L is a continuous linear mapping from C into $\mathbf{R}^n$. This hypothesis implies that there exists an $n \times n$ matrix $\eta(\vartheta)$, $-r \leq \vartheta \leq 0$, whose elements are of bounded variation, normalized so that η is continuous from the left of $(-r, 0)$ and $\eta(0) = 0$, such that,

$$L(\phi) = \int_{-r}^0 [d\eta(\vartheta)] \phi(\vartheta), \quad \phi \in C.$$

For any given function $\phi \in C$, we say $x(\phi)$ is a solution of (2.1) with initial function ϕ at $t = 0$ such that $x(\phi)(t)$ satisfies (2.1) for $t \geq 0$ and $x_0(\phi) = \phi$.

If $x(\phi)$ is the unique solution of (2.1) with initial function ϕ at $t = 0$, then the solution operator $T(t) : C \rightarrow C$ is defined by the relation

$$x_t(\phi) = T(t)\phi.$$

Since the family $\{T(t) : t \geq 0\}$ is the strongly continuous semigroup of linear operators on C , the infinitesimal generator A of $T(t)$ can be defined and is given by

$$(2.2) \quad A\phi(\vartheta) = \begin{cases} \frac{d\phi(\vartheta)}{d\vartheta}, & -r \leq \vartheta < 0, \\ \int_{-r}^0 [d\eta(\zeta)]\phi(\zeta), & \vartheta = 0, \end{cases}$$

$$D(A) = \left\{ \phi \in C \mid \frac{d\phi}{d\vartheta} \in C, \frac{d\phi}{d\vartheta}(0) = \int_{-r}^0 [d\eta(\zeta)]\phi(\zeta) \right\},$$

where $D(A)$ is the domain of A . We let $\sigma(A)$ denote the spectrum of A . Also we define $N(A)$ to be the null space of A , the set of all $\phi \in C$ for which $A\phi = 0$, and $R(A)$ to be the range of A . For a given $\lambda \in \sigma(A)$, the generalized eigenspace of λ will be denoted by $M_\lambda(A)$ is defined to be the smallest subspace of C containing all the elements of C that belongs to $N((\lambda I - A)^k)$ for $k = 1, 2, \dots$. The dimension of $M_\lambda(A)$ is called the algebraic multiplicity of λ and the smallest integer k such that $N((\lambda I - A)^k) = N((\lambda I - A)^{k+1})$ is called the ascent of λ .

Lemma 2.1. ([3, p. 198 and p. 207]) *λ is in $\sigma(A)$ if and only if λ satisfies the characteristic equation*

$$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - \int_{-r}^0 e^{\lambda\vartheta} d\eta(\vartheta).$$

The infinitesimal generator A defined by (2.2) has only point spectrum of which elements are called the eigenvalues of A . For $\lambda \in \sigma(A)$, the algebraic multiplicity of the eigenvalue λ equals the order of λ as a zero of $\det \Delta$ and the ascent k of λ equals the order of λ as a pole of Δ^{-1} such that

$$C = N((\lambda I - A)^k) \oplus R((\lambda I - A)^k),$$

where the symbol $\oplus$ means the direct sum.

Let $C^* = C([0, r], \mathbf{R}^{1 \times n})$, and define the bilinear form (α, ϕ) for $\alpha \in C^*$, $\phi \in C$ as

$$(\alpha, \phi) = \alpha(0)\phi(0) - \int_{-r}^0 \int_0^\vartheta \alpha(\zeta - \vartheta) [d\eta(\vartheta)]\phi(\zeta) d\zeta.$$

Then the formal adjoint operator A^* of A relative to the bilinear form is given by

$$A^*\alpha(s) = \begin{cases} -\frac{d\alpha(s)}{ds}, & 0 < s \leq r, \\ \int_{-r}^0 \alpha(-\vartheta) [d\eta(\vartheta)], & s = 0, \end{cases}$$

which satisfies $(\alpha, A\phi) = (A^*\alpha, \phi)$ for $\alpha \in D(A^*)$, $\phi \in D(A)$. The formal adjoint operator A^* has similar properties to those of A . Indeed, λ is in $\sigma(A)$ if and only if λ is in $\sigma(A^*)$, and $\dim M_\lambda(A^*)$ equals to the multiplicity of λ as a zero of $\det \Delta(\lambda)$.

The next two lemmas are very important to obtain the explicit representation of the decomposition given by Lemma 2.1.

Lemma 2.2. ([2, p. 175 and p. 177]) *A function ψ is in $R((\lambda I - A)^k)$ if and only if $(\alpha, \psi) = 0$ for all α in $N((\lambda I - A)^k)$ which coincides with functions ϕ of form*

$$\phi(\vartheta) = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{\vartheta^j}{j!} e^{\lambda \vartheta}, \quad -r \leq \vartheta \leq 0,$$

where $\gamma = \text{col}(\gamma_1, \dots, \gamma_k)$ satisfies $A_k \gamma = 0$ with

$$A_k = \begin{pmatrix} P_1 & P_2 & \dots & P_k \\ 0 & P_1 & \dots & P_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_1 \end{pmatrix},$$

$$P_{j+1} = P_{j+1}(\lambda) = \frac{\Delta^{(j)}(\lambda)}{j!}, \quad \Delta^{(j)}(\lambda) = \frac{d^j \Delta(\lambda)}{d\lambda^j}, \quad j = 0, 1, 2, \dots$$

Also, $N((\lambda I - A^*)^k)$ coincides with functions ψ of form

$$\psi(s) = \sum_{j=1}^k \beta_j \frac{(-s)^{k-j}}{(k-j)!} e^{-\lambda s}, \quad 0 \leq s \leq r,$$

where $\beta = (\beta_1, \dots, \beta_k)$ satisfies $\beta A_k = 0$.

Lemma 2.3. ([2, p. 177]) *For $\lambda \in \sigma(A)$, let $\Psi_\lambda = \text{col}(\psi_1, \dots, \psi_p)$ and $\Phi_\lambda = (\phi_1, \dots, \phi_p)$ be bases for $M_\lambda(A^*)$ and $M_\lambda(A)$, respectively, and let $(\Psi_\lambda, \Phi_\lambda) = ((\psi_i, \phi_j))$, $i, j = 1, 2, \dots, p$. Then $(\Psi_\lambda, \Phi_\lambda)$ is nonsingular and thus may be taken as the identity. The decomposition of C may be written explicitly as*

$$\phi = \phi^{P_\lambda} + \phi^{Q_\lambda}, \quad \phi^{P_\lambda} \in P_\lambda, \quad \phi^{Q_\lambda} \in Q_\lambda,$$

$$P_\lambda = M_\lambda(A) = \{\phi \in C : \phi = \Phi_\lambda b \text{ for some } p\text{-vector } b\},$$

$$Q_\lambda = \{\phi \in C : (\Psi_\lambda, \phi) = 0\},$$

$$\phi^{P_\lambda} = \Phi_\lambda b, \quad b = (\Psi_\lambda, \phi),$$

$$\phi^{Q_\lambda} = \phi - \phi^{P_\lambda}.$$

If $\Lambda = \{\lambda_1, \dots, \lambda_p\}$ is a finite set of eigenvalues of A , we let P_Λ be the linear extension of the $M_{\lambda_j}(A)$, $\lambda_j \in \Lambda$ and refer to this set as the generalized eigenspace of A associated with Λ . In a similar manner, we define P_Λ^* as the generalized eigenspace of A^* associated with Λ .

Lemma 2.4. ([2, p. 170 and p. 178]) *Suppose that Λ is a finite set $\{\lambda_1, \dots, \lambda_p\}$ of eigenvalues of A . If Φ and Ψ are bases for P_Λ and P_Λ^* , respectively, $(\Psi, \Phi) = I$, then for any ϕ in C ,*

$$\phi = \phi^{P_\Lambda} + \phi^{Q_\Lambda},$$

$$P_\Lambda = M_\Lambda(A) = \{\phi \in C : \phi = \Phi b \text{ for some } p\text{-vector } b\},$$

$$Q_\Lambda = \{\phi \in C : (\Psi, \phi) = 0\},$$

$$\phi^{P_\Lambda} = \Phi(\Psi, \phi).$$

Furthermore, the solution $T(t)\Phi b$ with initial value Φb at $t = 0$ may be defined on $(-\infty, \infty)$ by the relation

$$T(t)\Phi b = \Phi e^{Ft}b,$$

$$\Phi(\vartheta) = \Phi(0)e^{F\vartheta}, \quad -r \leq \vartheta \leq 0,$$

where F is defined by $A\Phi = \Phi F$.

For application of the theory of linear systems, we need to have an estimate for the solutions on the complementary subspace Q_Λ , and it is the following theorem.

Lemma 2.5 ([3, p. 214]) *For any real number β , let $\Lambda = \Lambda(\beta) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda > \beta\}$ and suppose C is decomposed by Λ as $C = P_\Lambda \oplus Q_\Lambda$. Then, for any $\gamma > 0$, there exist positive constants K such that*

$$\|T(t)\phi^{P_\Lambda}\| \leq Ke^{(\beta+\gamma)t}\|\phi^{P_\Lambda}\|, \quad t \leq 0,$$

$$\|T(t)\phi^{Q_\Lambda}\| \leq Ke^{(\beta+\gamma)t}\|\phi^{Q_\Lambda}\|, \quad t \geq 0.$$

3. Characteristic equation

In this section we consider the roots of the characteristic equation of (1.1).

Let $C = C([-\tau_N, 0], \mathbf{R}^2)$ and $x(\phi)$ denote the unique solution of (1.1) with the initial function ϕ in C at $t = 0$. For the (1.1), the infinitesimal generator A of the solution operator $T(t)$ is given by

$$A\phi(\sigma) = \begin{cases} \frac{d\phi(\sigma)}{d\sigma}, & -\tau_N \leq \sigma < 0, \\ -\rho R(\theta) \sum_{k=1}^N \phi(-\tau_k), & \sigma = 0. \end{cases}$$

Let $C^* = C([0, \tau_N], \mathbf{R}^{1 \times 2})$, and define the bilinear form (ψ, ϕ) for $\psi \in C^*$, $\phi \in C$ as

$$(3.1) \quad (\psi, \phi) = \psi(0)\phi(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 \psi(\xi + \tau_k) R(\theta) \phi(\xi) d\xi.$$

Then the formal adjoint operator A^* of A relative to the bilinear form is given by

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \leq \tau_N, \\ -\rho \sum_{k=1}^N \psi(\tau_k) R(\theta), & s = 0. \end{cases}$$

which satisfies $(\psi, A\phi) = (A^*\psi, \phi)$.

The characteristic equation of (1.1) is given by

$$(3.2) \quad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I + \rho R(\theta) \sum_{k=1}^N e^{-\lambda \tau_k}.$$

In order to obtain the explicit expression of periodic orbits, we need to show that (3.2) has a pair of pure imaginary eigenvalues and decompose the space C by the the generalized eigenspace associated with those eigenvalues. Indeed, using Theorem 2.1 in [6], we can obtain the following lemma.

Lemma 3.1. *Suppose that the following equality stands:*

$$(3.3) \quad \frac{\rho(\tau_1 + \tau_N)}{2} \sum_{k=1}^N \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right) = \frac{\pi}{2} - |\theta|.$$

If $0 < |\theta| < \pi/2$, (3.2) has a pair of simple roots $\pm i\omega_0$ and the remaining roots have negative real parts, and if $\theta = 0$, (3.2) has a pair of double roots $\pm i\omega_0$ and the remaining roots have negative real parts, where $\omega_0 = (\pi - 2|\theta|)/(\tau_1 + \tau_N)$.

Proof. This is a special case of the example given in [6, Case 1 in Section 4]. And (3.3) is just the same as the condition of Theorem 2.1 (ii) in [6]. Therefore, if $0 < |\theta| < \pi/2$, $p_\theta(\lambda) = 0$ has a simple root $i\omega$ and the remaining roots have negative real parts, where

$$\omega = \begin{cases} \frac{2\theta - \pi}{\tau_1 + \tau_N}, & \theta > 0, \\ -\frac{2\theta + \pi}{\tau_1 + \tau_N}, & \theta < 0, \end{cases}$$

and

$$p_\theta(\lambda) = \lambda + \rho e^{i\theta} \sum_{k=1}^N e^{-\lambda \tau_k}.$$

If $\theta = 0$, $p_\theta(\lambda) = 0$ has a pair of simple roots $\pm i(\pi/(\tau_1 + \tau_N))$ and the remaining roots have negative real parts.

On the other hand, we can easily see that

$$\det \Delta(\lambda) = p_\theta(\lambda) p_{-\theta}(\lambda)$$

and

$$p_{-\theta}(\lambda) = \overline{p_\theta(\bar{\lambda})}.$$

Here $\bar{\lambda}$ is the complex conjugate of any complex number λ . Therefore we can obtain the assertion of Lemma 3.1 immediately. $\square$

Remark 3.1. From the same argument of Lemma 3.1 in [5], we have (3.3) and (1.2) are equivalent.

4. Explicit expressions of asymptotic periodic solutions

We have shown that there exists a pair of pure imaginary eigenvalues in the critical cases. In this section we will give the explicit expressions of periodic orbits of asymptotic periodic solutions by decomposing of the space C .

First, when $0 < |\theta| < \pi/2$, we obtain next theorem.

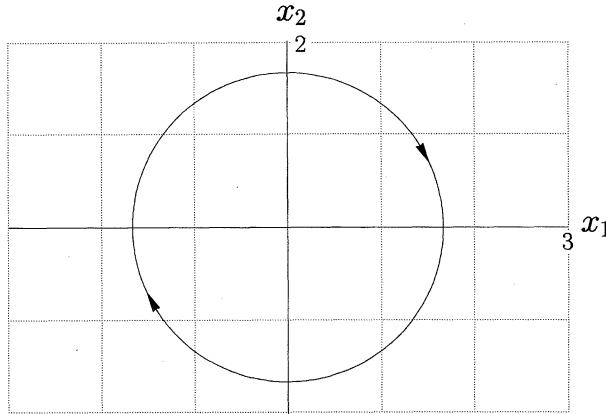
Theorem 4.1. *Suppose that $0 < |\theta| < \pi/2$ and*

$$\frac{\rho(\tau_1 + \tau_N)}{2} \frac{\sin\left(\frac{Nl}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}{\sin\left(\frac{l}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)} = \frac{\pi}{2} - |\theta|.$$

Then, the solution $x(\phi)$ of (1.1) with initial function $\phi \in C$ at $t = 0$ satisfies

$$x_t(\phi) \rightarrow \Phi_{1t} b_1 \quad \text{exponentially as } t \rightarrow \infty,$$

where



$$\Phi_1(t)b_1 = \begin{pmatrix} \cos \frac{127\pi}{256}t & \sin \frac{127\pi}{256}t \\ -\sin \frac{127\pi}{256}t & \cos \frac{127\pi}{256}t \end{pmatrix} \begin{pmatrix} -1.49952 \\ 0.65028 \end{pmatrix}$$

Fig. 2

$$\Phi_1(\sigma) = \begin{pmatrix} \cos \omega\sigma & \sin \omega\sigma \\ -\sin \omega\sigma & \cos \omega\sigma \end{pmatrix}, \quad \omega = \begin{cases} \frac{\pi - 2|\theta|}{\tau_1 + \tau_N}, & \theta > 0, \\ -\frac{\pi - 2|\theta|}{\tau_1 + \tau_N}, & \theta < 0, \end{cases}$$

$$b_1 = (I + B_1)^{-1}(\phi(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 \Phi^T(\xi + \tau) R(\theta) \phi(\xi) d\xi),$$

$$B_1 = \begin{pmatrix} -\rho \sum_{k=1}^N \tau_k \cos(\theta + \omega\tau_k) & \frac{1}{2}(\tau_1 + \tau_N)\omega \\ -\frac{1}{2}(\tau_1 + \tau_N)\omega & -\rho \sum_{k=1}^N \tau_k \cos(\theta + \omega\tau_k) \end{pmatrix}.$$

Figure 2 is the periodic orbit $\Phi_1(t)b_1$ of the asymptotic periodic solution in Figure 1 calculated by Theorem 4.1.

Proof of Theorem 4.1. Notice that we use (3.3) instead of (1.2) in the proof.

Let $\mathcal{A} = \{i\omega, -i\omega\}$. First we will give a basis for the eigenspace of $\mathcal{A}$ associated with $\mathcal{A}$. From Lemma 3.1, we know that $\pm i\omega$ are simple eigenvalues, that is, the order of $\pm i\omega$ as zeros of $\det \mathcal{A}(\lambda)$ is 1. Since

$$\mathcal{A}^{-1}(\lambda) = \frac{1}{\det \mathcal{A}(\lambda)} \begin{pmatrix} \lambda + \rho \cos \theta \sum_{k=1}^N e^{-\lambda\tau_k} & \rho \sin \theta \sum_{k=1}^N e^{-\lambda\tau_k} \\ -\rho \sin \theta \sum_{k=1}^N e^{-\lambda\tau_k} & \lambda + \rho \cos \theta \sum_{k=1}^N e^{-\lambda\tau_k} \end{pmatrix}$$

the order k of λ as a pole of $\mathcal{A}^{-1}(\lambda)$ is less than or equal to 1. Thus, from Lemma 2.1, we compute a basis of $N((\mathcal{A} - \lambda I)^k)$ for $k = 1$ at $\lambda = \pm i\omega$. Noticing that

$$\cos\left(\frac{\pi - 2|\theta|}{2}\right) = \sin|\theta|, \quad \sin\left(\frac{\pi - 2|\theta|}{2}\right) = \cos\theta \quad \text{and} \quad \tau_1 + \tau_N = \tau_k + \tau_{N-k+1}$$

for $k = 1, 2, \dots, N$, we have

$$\begin{aligned} \mathcal{A}(i\omega) &= i\omega I + \rho R(\theta) \sum_{k=1}^N e^{-i\omega\tau_k} \\ &= \rho R(\theta) \sum_{k=1}^N \cos \omega\tau_k + i \left(\omega I - \rho R(\theta) \sum_{k=1}^N \sin \omega\tau_k \right) \\ &= \frac{1}{2} \rho R(\theta) \sum_{k=1}^N (\cos \omega\tau_k + \cos \omega\tau_{N-k+1}) \\ &\quad + i \left(\omega I - \frac{1}{2} \rho R(\theta) \sum_{k=1}^N (\sin \omega\tau_k + \sin \omega\tau_{N-k+1}) \right) \\ &= \frac{1}{2} \rho R(\theta) \sum_{k=1}^N 2 \cos \left(\frac{\omega\tau_k + \omega\tau_{N-k+1}}{2} \right) \cos \left(\frac{\omega\tau_k - \omega\tau_{N-k+1}}{2} \right) \\ &\quad + i \left(\omega I - \frac{1}{2} \rho R(\theta) \sum_{k=1}^N 2 \sin \left(\frac{\omega\tau_k + \omega\tau_{N-k+1}}{2} \right) \cos \left(\frac{\omega\tau_k - \omega\tau_{N-k+1}}{2} \right) \right) \\ &= \rho R(\theta) \cos \left(\frac{\omega(\tau_1 + \tau_N)}{2} \right) \sum_{k=1}^N \cos \left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2} \right) \\ &\quad + i \left(\omega I - \rho R(\theta) \sin \left(\frac{\omega(\tau_1 + \tau_N)}{2} \right) \sum_{k=1}^N \cos \left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2} \right) \right) \\ &= \rho R(\theta) \cos \left(\frac{\pi - 2|\theta|}{2} \right) \sum_{k=1}^N \cos \left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta| \right) \right) \\ &\quad \pm i \left(\frac{\pi - 2|\theta|}{\tau_1 + \tau_N} I - \rho R(\theta) \sin \left(\frac{\pi - 2|\theta|}{2} \right) \sum_{k=1}^N \cos \left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta| \right) \right) \right) \\ &= \frac{\pi - 2|\theta|}{\tau_1 + \tau_N} \sin|\theta| R(\theta) \pm i \frac{\pi - 2|\theta|}{\tau_1 + \tau_N} (I - \cos \theta R(\theta)) \end{aligned}$$

$$\begin{aligned}
&= \omega \left(\sin \theta \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} + i \begin{pmatrix} 1 - \cos^2 \theta & \sin \theta \cos \theta \\ -\sin \theta \cos \theta & 1 - \cos^2 \theta \end{pmatrix} \right) \\
&= \omega \sin \theta \begin{pmatrix} \cos \theta + i \sin \theta & -\sin \theta + i \cos \theta \\ \sin \theta - i \cos \theta & \cos \theta + i \sin \theta \end{pmatrix} \\
&= \omega e^{i\theta} \sin \theta \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.
\end{aligned}$$

Since $\mathcal{A}(\bar{\lambda}) = \overline{\mathcal{A}(\lambda)}$, we also have

$$\mathcal{A}(-i\omega) = \omega e^{-i\theta} \sin \theta \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

From Lemma 2.2, for $\lambda = \pm i\omega$,

$$(4.1) \quad A_1 = P_1 = \omega e^{\pm i\theta} \sin \theta \begin{pmatrix} 1 & \pm i \\ \mp i & 1 \end{pmatrix},$$

and γ satisfying $A_1 \gamma = 0$ are $\gamma = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. Thus, the basis of eigenspace associated with $\mathcal{A}$ is

$$\hat{\phi}_1(\sigma) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\omega\sigma} \quad \text{and} \quad \hat{\phi}_2(\sigma) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\omega\sigma} \quad \text{for } -\tau_N \leq \sigma \leq 0.$$

We define a new basis for eigenspace $P_{\mathcal{A}}$ associated with $\mathcal{A}$ as

$$\begin{aligned}
\phi_1(\sigma) &= \frac{1}{2}(\hat{\phi}_1(\sigma) + \hat{\phi}_2(\sigma)) = \begin{pmatrix} \cos \omega\sigma \\ -\sin \omega\sigma \end{pmatrix}, \\
\phi_2(\sigma) &= \frac{1}{2i}(\hat{\phi}_1(\sigma) - \hat{\phi}_2(\sigma)) = \begin{pmatrix} \sin \omega\sigma \\ \cos \omega\sigma \end{pmatrix}.
\end{aligned}$$

Then we have

$$\Phi_1 = (\phi_1, \phi_2) = \begin{pmatrix} \cos \omega\sigma & \sin \omega\sigma \\ -\sin \omega\sigma & \cos \omega\sigma \end{pmatrix} = R(-\omega\sigma) \quad \text{for } -\tau_N \leq \sigma \leq 0.$$

Also, β satisfying $\beta A_1 = 0$ are $(1, \mp i)$. Thus, the basis of eigenspace of $\mathcal{A}^*$ associated with $\mathcal{A}$ as

$$\hat{\psi}_1^*(s) = (1, i)e^{-i\omega s} \quad \text{and} \quad \hat{\psi}_2^*(s) = (1, -i)e^{i\omega s} \quad \text{for } 0 \leq s \leq \tau_N.$$

We define a new basis for eigenspace $P_{\mathcal{A}}^*$ associated with $\mathcal{A}$ as

$$\begin{aligned}\psi_1^*(s) &= \frac{1}{2}(\hat{\psi}_1^*(s) + \hat{\psi}_2^*(s)) = (\cos \omega s, -\sin \omega s), \\ \psi_2^*(s) &= -\frac{1}{2i}(\hat{\psi}_1^*(s) - \hat{\psi}_2^*(s)) = (\sin \omega s, \cos \omega s).\end{aligned}$$

Then we have

$$\Psi^* = (\psi_1^*, \psi_2^*) = \begin{pmatrix} \cos \omega s & -\sin \omega s \\ \sin \omega s & \cos \omega s \end{pmatrix} = R(\omega s).$$

We will decompose C by eigenspace P_A . From (3.1), we have

$$\begin{aligned}(\Psi^*, \Phi_1) &= \Psi^*(0)\Phi_1(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 \Psi^*(\xi + \tau_k) R(\theta) \Phi_1(\xi) d\xi \\ &= R(0)R(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 R(\omega(\xi + \tau_k)) R(\theta) R(-\omega\xi) d\xi \\ &= I - \rho \sum_{k=1}^N \int_{-\tau_k}^0 R(\omega(\xi + \tau_k) + \theta - \omega\xi) d\xi \\ &= I - \rho \sum_{k=1}^N \int_{-\tau_k}^0 R(\theta + \omega\tau_k) d\xi \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \rho \sum_{k=1}^N \begin{pmatrix} \tau_k \cos(\theta + \omega\tau_k) & -\tau_k \sin(\theta + \omega\tau_k) \\ \tau_k \sin(\theta + \omega\tau_k) & \tau_k \cos(\theta + \omega\tau_k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -S_c & S_s \\ -S_s & -S_c \end{pmatrix},\end{aligned}$$

where

$$S_c = \rho \sum_{k=1}^N \tau_k \cos(\theta + \omega\tau_k) \quad \text{and} \quad S_s = \rho \sum_{k=1}^N \tau_k \sin(\theta + \omega\tau_k).$$

Notice that $\theta + \omega\tau_k = -(\theta + \omega\tau_{N-k+1}) \mp \pi$, then we have

$$(4.2) \quad \sin(\theta + \omega\tau_k) = \sin(\theta + \omega\tau_{N-k+1}).$$

As we can see in the proof of Lemma 3.1, $\lambda = -i\omega$ satisfies $p_\theta(\lambda) = 0$, which yields

$$(4.3) \quad \operatorname{Im} p_\theta(-i\omega) = -\omega + \rho \sum_{k=1}^N \sin(\theta + \omega\tau_k) = 0.$$

Thus,

$$\begin{aligned}
 (4.4) \quad S_s &= \rho \sum_{k=1}^N \tau_k \sin(\theta + \omega \tau_k) \\
 &= \frac{1}{2} \rho \sum_{k=1}^N (\tau_k \sin(\theta + \omega \tau_k) + \tau_{N-k+1} \sin(\theta + \omega \tau_{N-k+1})) \\
 &= \frac{1}{2} \rho \sum_{k=1}^N (\tau_k + \tau_{N-k+1}) \sin(\theta + \omega \tau_k) \\
 &= \frac{1}{2} \rho (\tau_1 + \tau_N) \sum_{k=1}^N \sin(\theta + \omega \tau_k) \\
 &= \frac{1}{2} (\tau_1 + \tau_N) \omega.
 \end{aligned}$$

Hence,

$$(\Psi^*, \Phi_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \rho \sum_{k=1}^N \tau_k \cos(\theta + \omega \tau_k) & -\frac{1}{2} (\tau_1 + \tau_N) \omega \\ \frac{1}{2} (\tau_1 + \tau_N) \omega & \rho \sum_{k=1}^N \tau_k \cos(\theta + \omega \tau_k) \end{pmatrix} = I + B_1.$$

So we define a new basis Ψ for P_A^* by

$$\Psi = (\Psi^*, \Phi_1)^{-1} \Psi^* = (I + B_1)^{-1} \Psi^*,$$

and we have

$$(\Psi, \Phi_1) = ((\Psi^*, \Phi_1)^{-1} \Psi^*, \Phi_1) = (\Psi^*, \Phi_1)^{-1} (\Psi^*, \Phi_1) = I.$$

Therefore, by Lemma 2.3, any $\phi \in C$ can be written as

$$\phi = \phi^{P_A} + \phi^{Q_A},$$

where

$$\phi^{P_A} = \Phi_1 b_1,$$

$$\begin{aligned}
 b_1 &= (\Psi, \phi) = \Psi(0) \phi(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 \Psi(\xi + \tau_k) R(\theta) \phi(\xi) d\xi \\
 &= (I + B_1)^{-1} \left(\phi(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 (I + B_1)^{-1} \Phi_1^T(\xi + \tau_k) R(\theta) \phi(\xi) d\xi \right)
 \end{aligned}$$

$$\phi^{Q_A} = \phi - \phi^{P_A}.$$

When $-\tau_N \leq \sigma < 0$, we have

$$\begin{aligned} A\Phi_1(\sigma) &= \frac{d}{d\sigma} \begin{pmatrix} \cos \omega\sigma & \sin \omega\sigma \\ -\sin \omega\sigma & \cos \omega\sigma \end{pmatrix} = \begin{pmatrix} -\omega \sin \omega\sigma & \omega \cos \omega\sigma \\ -\omega \cos \omega\sigma & -\omega \sin \omega\sigma \end{pmatrix} \\ &= \begin{pmatrix} \cos \omega\sigma & \sin \omega\sigma \\ -\sin \omega\sigma & \cos \omega\sigma \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \Phi_1(\sigma)F \end{aligned}$$

where $F = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$. When $\sigma = 0$, we also have

$$\begin{aligned} A\Phi_1(0) &= -\rho R(\theta) \sum_{k=1}^N \Phi(-\tau_k) = -\rho \sum_{k=1}^N R(\theta + \omega\tau_k) \\ &= \begin{pmatrix} -\rho \sum_{k=1}^N \cos(\theta + \omega\tau_k) & \rho \sum_{k=1}^N \sin(\theta + \omega\tau_k) \\ -\rho \sum_{k=1}^N \sin(\theta + \omega\tau_k) & -\rho \sum_{k=1}^N \cos(\theta + \omega\tau_k) \end{pmatrix} \\ &= \begin{pmatrix} -\cos\left(\pm\frac{\pi}{2}\right) & \sin\left(\pm\frac{\pi}{2}\right) \\ -\sin\left(\pm\frac{\pi}{2}\right) & \cos\left(\pm\frac{\pi}{2}\right) \end{pmatrix} \times \rho \sum_{k=1}^N \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right) \\ &= \begin{pmatrix} 0 & \pm \frac{\pi - 2|\theta|}{\tau_1 + \tau_N} \\ \mp \frac{\pi - 2|\theta|}{\tau_1 + \tau_N} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \Phi_1(0)F \end{aligned}$$

by using (3.3). Hence, from Lemma 2.4, we have

$$\begin{aligned} [T(t)\Phi_1](\sigma) &= \Phi_1(0)e^{Ft}e^{F\tau} = \Phi_1(0)e^{F(t+\sigma)} \\ &= \Phi_1(0) \sum_{k=0}^{\infty} \frac{F^k}{k!} (t+\sigma)^k \\ &= \Phi_1(0) \begin{pmatrix} \sum_{k=0}^{\infty} (-1)^k \frac{(\omega(t+\sigma))^{2k}}{(2k)!} & \sum_{k=0}^{\infty} (-1)^k \frac{(\omega(t+\sigma))^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty} (-1)^k \frac{(\omega(t+\sigma))^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} (-1)^k \frac{(\omega(t+\sigma))^{2k}}{(2k)!} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \omega(t+\sigma) & \sin \omega(t+\sigma) \\ -\sin \omega(t+\sigma) & \cos \omega(t+\sigma) \end{pmatrix} \\ &= \Phi_1(t+\sigma), \quad \text{for } -\tau_N \leq \sigma \leq 0. \end{aligned}$$

This means that $T(t)\Phi_1 = \Phi_{1t}$. Also, from Lemma 2.5, taking $\beta < 0$ sufficiently close to 0, for γ such that $\delta = \beta + \gamma < 0$, there exists a positive constant K such that

$$\|T(t)\phi^{\mathcal{Q}_A}\| \leq Ke^{\delta t} \|\phi^{\mathcal{Q}_A}\|, \quad t \geq 0.$$

Consequently we obtain

$$x_t(\phi) \rightarrow \Phi_{1t}b_1 \quad \text{exponentially as } t \rightarrow \infty$$

since

$$\begin{aligned} \|T(t)\phi^{\mathcal{Q}_A}\| &= \|T(t)(\phi - \phi^{P_A})\| = \|T(t)\phi - T(t)\phi^{P_A}\| \\ &= \|T(t)\phi - T(t)\Phi_{1t}b_1\| \\ &= \|T(t)\phi - \Phi_{1t}b\| \rightarrow 0 \quad \text{exponentially as } t \rightarrow \infty. \quad \square \end{aligned}$$

When $\theta = 0$, we obtain the following theorem.

Theorem 4.2. *Suppose that $\theta = 0$ and*

$$\frac{\rho(\tau_1 + \tau_N)}{2} \frac{\sin\left(\frac{Nl}{\tau_1 + \tau_N} \frac{\pi}{2}\right)}{\sin\left(\frac{l}{\tau_1 + \tau_N} \frac{\pi}{2}\right)} = \frac{\pi}{2}.$$

Then, the solution $x(\phi)$ of (1.1) with initial function $\phi \in C$ at $t = 0$ satisfies

$$x_t(\phi) \rightarrow \Phi_{2t}b_2 \quad \text{exponentially as } t \rightarrow \infty,$$

where

$$\begin{aligned} \Phi_2(\sigma) &= \begin{pmatrix} \cos \omega \sigma & \sin \omega \sigma & 0 & 0 \\ 0 & 0 & \cos \omega \sigma & \sin \omega \sigma \end{pmatrix}, \quad \omega = \frac{\pi}{\tau_1 + \tau_N}, \\ b_2 &= \left(\frac{1}{2}(I + B_2)\right)^{-1} \left(\Phi_2^T(0)\phi(0) - \rho \sum_{k=1}^N \int_{-\tau_k}^0 \Phi_2^T(\xi + \tau_k) \phi(\xi) d\xi \right), \\ B_2 &= \begin{pmatrix} -a & \frac{\pi}{2} & 0 & 0 \\ -\frac{\pi}{2} & -a & 0 & 0 \\ 0 & 0 & -a & \frac{\pi}{2} \\ 0 & 0 & -\frac{\pi}{2} & -a \end{pmatrix}, \quad a = \rho \sum_{k=1}^n \tau_k \cos \omega \tau_k. \end{aligned}$$

In the proof, the only difference from the proof of Theorem 4.1 is the dimension of the generalized eigenspace P_A of A associated with λ is 4 since $\pm i\omega$ are double eigenvalues from Lemma 3.1.

5. Hopf bifurcation

In this section, we consider the following nonlinear equation

$$(5.1) \quad z'(t) = -\rho R(\theta) \sum_{k=1}^N z(t - \tau_k) [1 - |z(t)|^2]$$

where ρ is a real constant, $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $0 < |\theta| < \frac{\pi}{2}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $|z|^2 = z_1^2 + z_2^2$, $\{\tau_k\}$ is an arithmetic sequence, that is, $\tau_k = \tau + (k-1)l$ with $\tau \geq 0$ and $l > 0$ for $k = 1, 2, \dots, N$.

In [4], the author studied the equation (5.1) with $N = 1$ and $\tau_1 = 1$ and gave a conjecture. Related to the conjecture, we consider the equation (5.1).

In the case $\theta = 0$ and $N = 1$ where $\tau_1 = 1$, next two theorems are known [3, pp. 347–348, Theorems 5.1 and 5.2].

Theorem C. (5.1) has a Hopf bifurcation at $\rho = \pi/2$.

Theorem D. If $\rho > \pi/2$, then (5.1) has a nonconstant periodic solution.

In Section 3, we have shown that the characteristic equation of the linearized equation of (5.1) has a pair of pure imaginary eigenvalues when

$$(5.2) \quad \frac{\rho(\tau_1 + \tau_N)}{2} \frac{\sin\left(\frac{Nl}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}{\sin\left(\frac{l}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)} = \frac{\pi}{2} - |\theta|$$

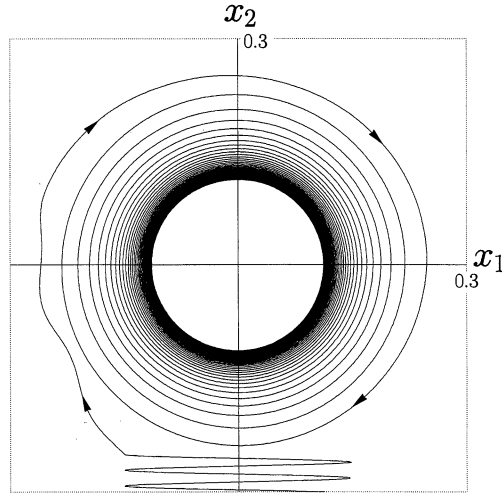
holds. Here, define ρ_0 as ρ satisfying (5.2), that is,

$$\rho_0 = \frac{(\pi - 2|\theta|) \sin\left(\frac{l}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}{(\tau_1 + \tau_2) \sin\left(\frac{Nl}{\tau_1 + \tau_N} \left(\frac{\pi}{2} - |\theta|\right)\right)}.$$

Let us introduced a parameter μ as $\rho = \rho(\mu) = \rho_0 + \mu$ in (5.1), that is,

$$(5.3) \quad z'(t) = -(\rho_0 + \mu)R(\theta) \sum_{k=1}^N z(t - \tau_k) [1 - |z(t)|^2].$$

In this case by numerical simulations, we have confirmed that (5.3) has a periodic solution in the neighborhood of origin for sufficiently small $\mu > 0$ as Figure 3 shows.



$$\rho_0 = \frac{\pi \sin \frac{\pi}{12}}{3 \sin \frac{\pi}{4}}, \quad \mu = 0.005, \quad \theta = \frac{\pi}{6}, \quad \tau_1 = 0.5, \quad \tau_2 = 1, \quad \tau_3 = 1.5,$$

$$(t_0, \phi, \psi) = \left(0, -0.15 \cos 10t, -\left(0.25 - \frac{t}{30}\right)\right)$$

Fig. 3

This indicates that (5.1) has a Hopf bifurcation at $\rho = \rho_0$. In fact, by the same method in [1, p. 299], we can show that the bifurcation is supercritical. We will obtain the periodic solution explicitly.

From Lemma 4.10 and Remark 4.12 in [8, pp. 110–116], the radius r of the orbitally asymptotically stable periodic solution of (5.1) is given by

$$r \approx \sqrt{\frac{-\alpha'(\rho_0)\mu}{\Delta}}$$

where

$$\alpha'(\rho_0) = \operatorname{Re} \left. \frac{d\lambda(\rho)}{d\rho} \right|_{\rho=\rho_0},$$

$$\Delta = \frac{\rho_0(y_1^2 + y_2^2)}{(S_s^2 + (1 - S_c)^2)} \left((1 - S_c) \sum_{k=1}^N \cos(\theta + \omega\tau_k) - S_s \sum_{k=1}^N \sin(\theta + \omega\tau_k) \right).$$

Noticing that

$$\sum_{k=1}^N \cos(\theta + \omega\tau_k) = 0 \quad \text{and} \quad \sum_{k=1}^N \sin(\theta + \omega\tau_k) = \frac{\omega}{\rho_0},$$

we obtain

$$A = \frac{-S_s\omega}{S_s^2 + (1 - S_c)^2}.$$

Also, we calculate the value of $\alpha'(\rho_0)$ by taking a derivative of

$$p_+(\lambda) = \lambda + \rho e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k} = 0$$

at ρ_0 , where λ is a continuous function ρ . From

$$\left(1 - \rho e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k}\right) \frac{d\lambda}{d\rho} + e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k} = 0$$

and

$$e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k} = -\frac{\lambda}{\rho},$$

we have

$$\frac{d\lambda}{d\rho} = \frac{\lambda}{\rho(1 - \rho e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k})}.$$

Since $\lambda(\rho_0) = -i\omega$ is a root of $p_+(\lambda) = 0$,

$$\begin{aligned} \alpha'(\rho) &= \operatorname{Re} \left. \frac{d\lambda}{d\rho} \right|_{\rho=\rho_0} = \operatorname{Re} \left\{ \frac{\lambda}{\rho(1 - \rho e^{i\theta} \sum_{k=1}^N \tau_k e^{-\lambda\tau_k})} \right\} \Big|_{\rho=\rho_0} \\ &= \operatorname{Re} \left\{ \frac{-i\omega}{\rho_0(1 - \rho_0 e^{i\theta} \sum_{k=1}^N \tau_k e^{i\omega\tau_k})} \right\} \\ &= \operatorname{Re} \left\{ \frac{-i\omega}{\rho(1 - (S_c + iS_s))} \right\} \\ &= \operatorname{Re} \left\{ \frac{-i(1 - S_c)\omega + S_s\omega}{\rho_0((1 - S_c)^2 + S_s^2)} \right\} \\ &= \frac{S_s\omega}{\rho_0(S_s^2 + (1 - S_c)^2)}. \end{aligned}$$

Therefore, the periodic orbit of (5.1) is given by

$$\begin{aligned}
 r &= \sqrt{\frac{\alpha'(0)\mu}{\Delta}} \\
 &= \sqrt{\frac{-S_s\omega\mu}{\rho_0(S_s^2 + (1 - S_c)^2)} \frac{S_s^2 + (1 - S_c)^2}{-S_s\omega}} \\
 &= \sqrt{\frac{\mu}{\rho_0}}.
 \end{aligned}$$

Theorem 5.1. (5.1) has a Hopf bifurcation at $\rho = \rho_0$ and a stable periodic orbit. And it is approximated as $p(t) \approx \sqrt{\frac{\mu}{\rho_0}} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Using the Runge–Kutta Method, Figure 4A illustrates a solution of (5.1), which tends to a periodic solution, with the same parameters in Figure 3 and a new initial condition $(t_0, \phi, \psi) = (0, 0.075 + \frac{t}{10}, 0.025 \cos 30t)$. Figure 4B shows the periodic orbit in Figure 4A calculated by Theorem 5.1, which is the

$$\text{orbit } p(t) \approx \sqrt{\frac{0.015 \sin \frac{\pi}{4}}{\pi \sin \frac{\pi}{12}}} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

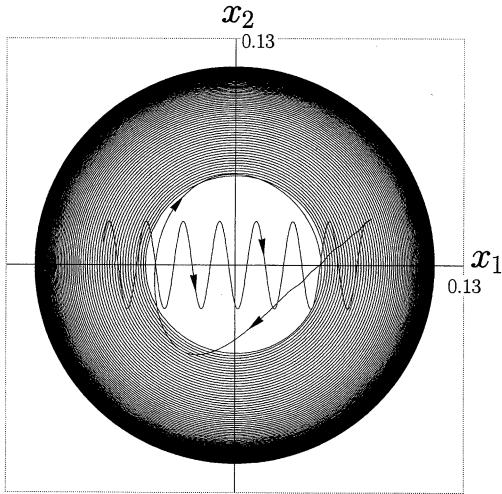


Fig. 4A

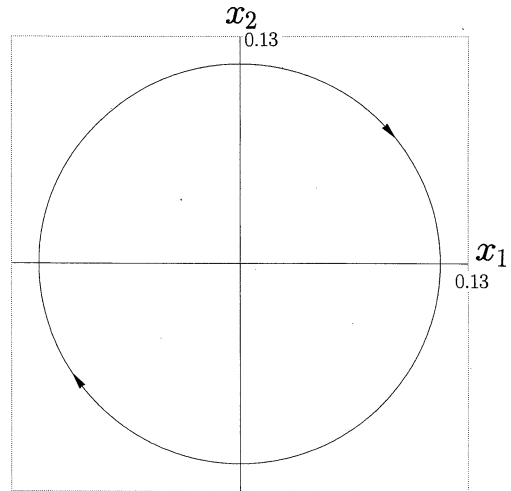


Fig. 4B

Note that, from figures, we can observe the coincidence of the periodic orbits in Figures 3, 4A and 4B.

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Tadayuki Hara
 Department of Mathematical Science
 Osaka Prefecture University
 Sakai, Osaka 599-8531
 Japan
 E-mail: hara@ms.osakafu-u.ac.jp

Rinko Miyazaki
 Department of Systems Engineering
 Sizuoka University
 Hamamatsu, Sizuoka 432-8561
 Japan

Takayasu Morii
 Department of Mathematical Science
 Osaka Prefecture University
 Sakai, Osaka 599-8531
 Japan

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