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Multiplicity Results for a Semilinear Elliptic Equation with Indefinite Critical Nonlinearity

By

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1. Introduction

Consider the problem

$$(1) \quad \begin{aligned} -\Delta u &= \lambda g(x)u + h(x)|u|^{2^*-2}u \quad \text{in } \mathbf{R}^N \\ u &> 0 \quad \text{on } \mathbf{R}^N, \end{aligned}$$

where $N \geq 4$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, and $u \in D^{1,2}(\mathbf{R}^N)$. In this paper we investigate the influence of $h(x)$ on the existence of solutions to (1) for $\lambda > 0$ small. Throughout the paper, we assume the following hold:

(G1) $g(x) \in L^{N/2}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ is indefinite in sign but $g^+ \not\equiv 0$,

(H1) $h(x) \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ is nonconstant and indefinite in sign.

Further conditions supplement these restrictions. The requirements on $g(x)$ are used in [6] to analyse the associated linear eigenvalue problem. The integrability requirements provide a compactness condition.

Problems such as (1) have captured interest since the seminal work of Brézis and Nirenberg [4]. In particular, the difficulties associated with overcoming a critical Sobolev exponent are discussed there. The concentration-compactness principle (CCP) of Lions [17] and a CCP at infinity [2] are the dominant tools used in overcoming such difficulties.

Several papers have considered small perturbations and correlated the number of peaks of a weighted nonlinearity with the number of solutions. Examples in bounded domains are [8], [20], [10], and for unbounded domains [16], [9], [26].

Without a perturbation term $\lambda \int g u^2$, an early analysis of problem (1) is contained in [13]. Certainly, the scenario of nonexistence of positive solutions is discussed. The case when $h(x)$ possesses a group symmetry is analysed in [2]. Reference [14] has deduced the existence of a least energy solution for λ below (or slightly above) the first resonance.

Suppose $h(x)$ assumes k isolated global peaks, and assume that $g(x)$ is

positive and $h(x)$ ‘flat’ at these locations. Huang [16] has recently derived that there are at least k solutions to (1), which concentrate as the perturbation coefficient λ vanishes. Using a closely related technique, we remove the constraint that maxima be isolated. As the coefficient $\lambda \rightarrow 0$, solutions concentrate somewhere in the set where $h(x)$ assumes its maximum. These results further improve on existing publications by allowing $g(x)$ and $h(x)$ to be indefinite in sign, and allowing sharper estimates on the shape of $h(x)$ near its peaks. Contrasting with [16] the most important properties are deduced on a manifold associated with $\lambda = 0$. These results are then projected to a manifold associated with small $\lambda > 0$.

Let $M = \{x \in \mathbf{R}^N : h(x) = \|h\|_\infty\}$. For the subcritical problem, on an unbounded domain, Musina [19] has located multiple solutions in $H^1(\mathbf{R}^N)$ depending on $\text{cat}_{M_\delta}(M)$. In Section 5 we show that a similar result extends to the critical exponent.

2. Preliminaries

We seek weak solutions to (1) in the space $D^{1,2}(\mathbf{R}^N)$, defined as the completion of C_0^∞ with respect to norm $\|\nabla u\|^2 \equiv \int_{\mathbf{R}^N} |\nabla u|^2 dx$ and inner product $\langle u, v \rangle = \int_{\mathbf{R}^N} \nabla u \nabla v dx$. Denote its dual as $D^{-1,2}(\mathbf{R}^N)$. By S we refer to the best Sobolev constant for the continuous imbedding: $D^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2^*}(\mathbf{R}^N)$:

$$S = \inf_{u \in D^{1,2}, \|u\|_{2^*}=1} \int_{\mathbf{R}^N} |\nabla u|^2 dx$$

The minimiser is achieved by a unique family of functions (Talenti functions) [23]. We express the L^{2^*} -normalised Talenti extremal function with scale factor r as $T_r(x)$, that is

$$(2) \quad T_r(x) = \frac{(r + |x|^2)^{(2-N)/2}}{\|(r + |x|^2)^{(2-N)/2}\|_{2^*}}.$$

Two particular cut-off functions defined here will be frequently utilised. Let $\phi^R \in C^\infty(\mathbf{R}^N, \mathbf{R})$ satisfy

$$(3) \quad 0 \leq \phi^R(x) \leq 1, \quad \phi^R(x) = 0 \text{ if } |x| < R, \quad \phi(x) = 1 \text{ if } |x| > R + 1$$

Let $\phi_\delta(x_0) \in C^\infty(\mathbf{R}^N, \mathbf{R})$ satisfy

$$(4) \quad \begin{aligned} 0 \leq \phi_\delta(x_0)(x) \leq 1, & \quad |D\phi_\delta(x_0)(x)| \leq 4/\delta, \\ \phi_\delta(x_0)(x) = 0 & \text{ if } |x - x_0| > \delta, \quad \phi_\delta(x_0)(x) = 1 \text{ if } |x - x_0| < \delta/2. \end{aligned}$$

If x_0 is clearly understood, then it is not explicitly included.

Weak convergence shall be denoted “ $\rightharpoonup$ ” while strong convergence is represented by “ $\rightarrow$ ”. The space of finite measures on $\mathbf{R}^N$ is denoted $\mathcal{M}(\mathbf{R}^N)$. Let $\mathbf{R}^N \cup \{\infty\}$ be the one-point compactification of $\mathbf{R}^N$, and $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$ the associated space of finite measures.

For a nonnegative measurable function $l(x)$, define the weighted Lebesgue space $L_l^p(\mathbf{R}^N)$ by all measurable functions u which satisfy $\int_{\mathbf{R}^N} l(x)|u|^p dx < \infty$.

Critical points of the C^1 functional

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\mathbf{R}^N} g(x) u^2 dx - \frac{1}{2^*} \int_{\mathbf{R}^N} h(x) |u|^{2^*} dx$$

correspond with weak solutions to the Euler equations (1). In the sequel, we may omit the notation for the domain of integration which shall by default be $\mathbf{R}^N$.

If u is a critical point of I_λ , then so too is $|u|$. By familiar applications of a Harnack-type inequality ([14], [21]), nontrivial critical points of I_λ may be taken to be positive solutions.

Assume (G1) and consider the eigenvalue problem

$$(5) \quad -\Delta u = \lambda g(x)u, \quad x \in \mathbf{R}^N, \quad \lim_{|x| \rightarrow \infty} u(x) = 0$$

Define a linear form on $D^{1,2}(\mathbf{R}^N)$ by

$$\beta(u, v) = \int_{\mathbf{R}^N} g(x) uv dx.$$

By the Riesz representation theorem, there is a bounded linear operator L such that

$$\beta(u, v) = \langle Lu, v \rangle \quad \forall u, v \in D^{1,2}(\mathbf{R}^N).$$

Lemma 2.1. *L is compact.*

This is proved in [6] and consequently the existence of a principal positive eigenvalue results. We set

$$(6) \quad 0 < \lambda_1 = \left[\sup_{v \in D^{1,2} \setminus \{0\}} \frac{\int g(x) v^2}{\int |\nabla v|^2} \right]^{-1}.$$

Then λ_1 is the principal positive eigenvalue of the eigenvalue problem (5) and the associated eigenfunction ϕ_1 is strictly positive by construction in [5]. Tertikas [24] has gained eigenvalue results similar to these with relaxed conditions on $g(x)$, however, we maintain the condition (G1) to take advantage of compactness properties. If $g^- \not\equiv 0$, then a first negative eigenvalue also results [6]. We remark that $g^+ \in L^{N/2}(\mathbf{R}^N)$ suffices to attain the positive eigenvalue [7].

Drábek and Huang [14] provide impetus for the following convergence result:

Lemma 2.2. *Let $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbf{R}^N)$ and $\phi \in L^{2^*}(\mathbf{R}^N)$. Let $h(x) \in L^\infty(\mathbf{R}^N)$. Then*

$$\int h(x)|u_n|^{2^*-2}u_n\phi \rightarrow \int h(x)|u_0|^{2^*-2}u_0\phi$$

Proof. For $\phi \in C_0^\infty(\mathbf{R}^N)$, the proof is contained in [14]. We extend this result to account for $\phi \in L^{2^*}$. By density arguments, there exists $\tilde{\phi} \in C_0^\infty(\mathbf{R}^N)$ with $\|\phi - \tilde{\phi}\|_{2^*} \leq \varepsilon$ for any given $\varepsilon > 0$ arbitrarily small. Now,

$$\begin{aligned} & \left| \int h(x)|u_m|^{2^*-2}u_m\phi - \int h(x)|u_0|^{2^*-2}u_0\phi \right| \\ &= \left| \int h(x)|u_m|^{2^*-2}u_m(\phi - \tilde{\phi}) + \int (h(x)|u_m|^{2^*-2}u_m - h(x)|u_0|^{2^*-2}u_0)\tilde{\phi} \right. \\ & \quad \left. + \int h(x)|u_0|^{2^*-2}u_0(\tilde{\phi} - \phi) \right| \\ &\leq \|h\|_\infty (\|u_m\|_{2^*}^{(N+2)/2N} + \|u_0\|_{2^*}^{(N+2)/2N}) \|\phi - \tilde{\phi}\|_{2^*} \\ & \quad + \left| \int h(x)(|u_m|^{2^*-2}u_m - |u_0|^{2^*-2}u_0)\tilde{\phi} \right| \end{aligned}$$

As $m \rightarrow \infty$, the final term in this expression vanishes, and the first term can be made arbitrarily small with improving approximations $\tilde{\phi}$ for ϕ . ■

3. Palais-smale condition

Let $\{u_m\}$ be a sequence in $D^{1,2}(\mathbf{R}^N)$ satisfying

$$I_\lambda(u_m) \rightarrow c \quad \text{and} \quad I'_\lambda(u_m) \rightarrow 0 \quad \text{in } D^{-1,2}(\mathbf{R}^N) \quad \text{as } m \rightarrow \infty.$$

Such a sequence is called a Palais-Smale sequence at level c , abbreviated as a $(\text{PS})_c$ -sequence. The functional I_λ satisfies the Palais-Smale condition at level c (the $(\text{PS})_c$ -condition) if every $(\text{PS})_c$ -sequence is relatively compact.

Due to the critical exponent, we require the concentration-compactness principle of P. L. Lions [17] to establish the (PS) -condition.

Proposition 3.1. *Let $\{u_m\}$ be a sequence in $D^{1,2}(\mathbf{R}^N)$, weakly convergent to u in $D^{1,2}(\mathbf{R}^N)$ and such that*

- (i) $|\nabla u_m|^2 \rightharpoonup \mu$ in $\mathcal{M}(\mathbf{R}^N)$, a non-negative measure
- (ii) $|u_m|^{2^*} \rightharpoonup \nu$ in $\mathcal{M}(\mathbf{R}^N)$, a non-negative measure

then there exists an at most countable index set J and sequences $\{x_j\} \subset \mathbf{R}^N$, $\{\mu_j\}, \{v_j\} \subset (0, \infty)$, $j \in J$ such that

- (a) $v = |u|^{2^*} + \sum_{j \in J} v_j \delta_{x_j}$,
- (b) $\mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$, and
- (c) $S v_j^{2/2^*} \leq \mu_j$,

where δ_{x_j} is the Dirac measure assigned to point x_j .

The notion of concentration at infinity, introduced in [2] and enhanced in [11], is used to provide estimates for the components of the sequence which escape to infinity. Later, this behaviour may be captured by the notation of $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$, but this formulation expresses the result explicitly:

Proposition 3.2. Let $\{u_m\} \subset D^{1,2}(\mathbf{R}^N)$ be a sequence such that $u_m \rightharpoonup u$ in $D^{1,2}(\mathbf{R}^N)$ and define

- (i) $v_\infty = \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{|x| > R} |u_m|^{2^*} dx$,
- (ii) $\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{|x| > R} |\nabla u_m|^2 dx$.

The quantities v_∞ and μ_∞ are well defined and satisfy

- (a) $\limsup_{m \rightarrow \infty} \int_{\mathbf{R}^N} |u_m|^{2^*} dx = \int_{\mathbf{R}^N} dv + v_\infty$,
- (b) $\limsup_{m \rightarrow \infty} \int_{\mathbf{R}^N} |\nabla u_m|^2 dx = \int_{\mathbf{R}^N} d\mu + \mu_\infty$, and
- (c) $S v_\infty^{2/2^*} \leq \mu_\infty$.

A further result from [25] (Lemma 1.40) or [17] (Lemma I.1 (iv)) indicates that with particular energy, only a single concentration can occur:

Proposition 3.3. Suppose $u_m \rightarrow 0$ in $D^{1,2}(\mathbf{R}^N)$ and

$$|\nabla u_m|^2 \rightharpoonup \mu \text{ in } \mathcal{M}(\mathbf{R}^N); \quad |u_m|^{2^*} \rightharpoonup v \text{ in } \mathcal{M}(\mathbf{R}^N)$$

with $S\|v\|^{2/2^*} = \|\mu\|$. Then μ and v concentrate at a single point.

Due to the elegance of concentration at infinity, we wish to prove boundedness and compactness of (PS)-sequences using these tools. However, some notational obstructions must be overcome to avoid imposition of the stringent condition that $\lim_{|x| \rightarrow \infty} h(x)$ exists, as in [11], [18] and [19].

From Propositions 3.1 and 3.2 any bounded sequence $\{u_m\} \subset D^{1,2}(\mathbf{R}^N)$ must satisfy

$$|\nabla u_m|^2 \rightharpoonup |\nabla u|^2 + \sum \mu_j \delta_{x_j} + \mu_\infty; \quad |u_m|^{2^*} \rightharpoonup |u|^{2^*} + \sum v_j \delta_{x_j} + v_\infty$$

where weak convergence is in the sense of $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$. Define ϕ^R in (3). Proposition 3.2(i) defines v_∞ . Taking a subsequence if necessary, we may assume $v_\infty = \lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbf{R}^N} \phi^R |u_m|^{2^*} dx$. Consequently, for a given sequence $\{u_m\}$ we define $h(\infty)$ by the following relation:

$$\lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \int h(x) \phi^R |u_m|^{2^*} dx = h(\infty) v_\infty.$$

Thus, $h(\infty)$ is a property of the sequence $\{u_m\}$ and represents an ‘average’ of $h(x)$ at infinity, weighted according to the way $\{u_m\}$ concentrates to infinity. Clearly, $|h(\infty)| \leq \|h\|_\infty$ and if $\lim_{|x| \rightarrow \infty} h(x)$ exists then it corresponds with $h(\infty)$.

Lemma 3.4. *Let $0 \leq \lambda < \lambda_1$. Suppose that $\{u_m\}$ is a (PS)-sequence. Then $\{u_m\}$ is bounded in $D^{1,2}(\mathbf{R}^N)$.*

Proof. Let $\{u_m\}$ be a $(PS)_c$ -sequence. Then there is a sequence of real numbers $\varepsilon_m \rightarrow 0$ such that

$$\begin{aligned} \|\nabla u_m\|^2 - \lambda \int g u_m^2 - \int h |u_m|^{2^*} &\geq -\varepsilon_m \|\nabla u_m\|, \\ \frac{1}{2} \|\nabla u_m\|^2 - \frac{\lambda}{2} \int g u_m^2 - \frac{1}{2^*} \int h |u_m|^{2^*} &= c + o(1). \end{aligned}$$

Subtracting and using the Rayleigh characterisation,

$$\frac{1}{N} \left(1 - \frac{\lambda}{\lambda_1}\right) \|\nabla u_m\|^2 \leq \frac{1}{N} \left(\|\nabla u_m\|^2 - \lambda \int g u_m^2\right) \leq \frac{\varepsilon_m}{2^*} \|\nabla u_m\| + O(1).$$

Consequently, u_m must be bounded in $D^{1,2}(\mathbf{R}^N)$. ■

The Palais-Smale condition can be achieved for I_λ , provided that λ lies within a range of values related to the first eigenvalue.

Chabrowski [11] provides a proof which is only valid for $h \geq 0$ on $\mathbf{R}^N$. Although Drábek and Huang [14] achieve similar results, they neglect to provide full analysis to eliminate the possibility of a weakly convergent sequence concentrating at infinity.

Lemma 3.5. *Define $c^* = (1/N)S^{N/2}\|h\|_\infty^{-(N-2)/2}$. Suppose $0 \leq \lambda < \lambda_1$. Then I_λ satisfies the $(PS)_c$ -condition for all $c < c^*$.*

Proof. For the appropriate conditions on $h(x)$, let λ lie in the range specified. Let $\{u_m\}$ be a $(PS)_c$ -sequence with $c < c^*$. Lemma 3.4 ensures that u_m is bounded in $D^{1,2}(\mathbf{R}^N)$, and consequently for a relabelled subsequence $u_m \rightharpoonup u_0$ in $D^{1,2}(\mathbf{R}^N)$.

We remark that u_0 forms a weak solution to (1) by an application of Lemma 2.2. For $\phi \in C_0^1(\mathbf{R}^N)$,

$$\begin{aligned} \langle I'_\lambda(u_m), \phi \rangle &= \int \nabla u_m \nabla \phi - \lambda \int g u_m \phi - \int h |u_m|^{2^*-2} u_m \phi \\ &\rightarrow \int \nabla u_0 \nabla \phi - \lambda \int g u_0 \phi - \int h |u_0|^{2^*-2} u_0 \phi = 0. \end{aligned}$$

By Proposition 3.1 there exist measures μ and ν and an at most countable set J , sequences of positive constants $\{\mu_j\}, \{\nu_j\}, j \in J$ and a sequence $\{x_j\} \subset \mathbf{R}^N, j \in J$ such that relations (a), (b) and (c) are satisfied. Let x_k be a singular point of measures μ and ν and let $\phi_\delta = \phi_\delta(x_k) \in C_0^1(\mathbf{R}^N)$ satisfy (4). Since $\langle I'_\lambda(u_m), u_m \phi_\delta \rangle \rightarrow 0$ as $m \rightarrow \infty$, we get

$$\begin{aligned} & \int_{\mathbf{R}^N} \phi_\delta d\mu - \lambda \int g u_m^2 \phi_\delta dx - \int h \phi_\delta d\nu \\ & \leq \limsup_{m \rightarrow \infty} \int_{\mathbf{R}^N} |\nabla u_m| |u_m| |\nabla \phi_\delta| dx \leq \tilde{C} \left(\int_{\mathbf{R}^N} u_0^2 |\nabla \phi_\delta|^2 dx \right)^{1/2} \end{aligned}$$

where $\tilde{C} = \sup_{m \geq 1} \|\nabla u_m\|$. By the Hölder inequality,

$$\begin{aligned} (7) \quad & \int_{\mathbf{R}^N} u_0^2 |\nabla \phi_\delta|^2 dx \leq \left(\int_{|x-x_k| < \delta} |u_0|^{2^*} dx \right)^{2/2^*} \left(\int_{|x-x_k| < \delta} |\nabla \phi_\delta|^N dx \right)^{2/N} \\ & \leq 16 \omega_N^{2/N} \left(\int_{|x-x_k| < \delta} |u_0|^{2^*} dx \right)^{2/2^*} \end{aligned}$$

where ω_N is the volume of a unit ball in $\mathbf{R}^N$. Letting $\delta \rightarrow 0$ in (7) we deduce that $h(x_k) \nu_k \geq \mu_k$.

If $h(x) \leq 0$ almost everywhere, then concentration is impossible: $\nu_k = \mu_k = 0$. For concentration to remain a possibility, $h(x_k)$ must be positive, and combined with (c) of Proposition 3.1 we deduce that either (i) $\nu_k = 0$ or (ii) $\nu_k \geq (S/h(x_k))^{N/2}$. There must only be a finite number of local concentration points for otherwise $\sum \mu_j$ is unbounded contradicting the fact that $\|\nabla u_m\| < \infty$. Hence, a sufficiently large ball contains all of the singular points of measure.

We now mollify at infinity to investigate the possibility of concentration at infinity. Define $\phi^R \in C^1(\mathbf{R}^N)$ by (3), and use Lemma 2.1 to reveal

$$(8) \quad \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{\mathbf{R}^N} \phi^R h(x) |u_m|^{2^*} dx \leq h(\infty) \nu_\infty,$$

$$(9) \quad \lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbf{R}^N} \phi^R g(x) |u_m|^2 dx = 0.$$

Suppose that $h(\infty) \leq 0$. Then $h(\infty) \nu_\infty \leq 0$ and there is no more to do, so assume the contrary is true. Since $\langle I'_\lambda(u_m), u_m \phi^R \rangle \rightarrow 0$ as $m \rightarrow \infty$, equations (8) and (9) imply that either (iii) $\nu_\infty = 0$ or (iv) $\nu_\infty \geq (S/h(\infty))^{N/2}$. We now show that $\nu_\infty = \nu_k = 0$ for each k .

Suppose $\nu_k > 0$ or $\nu_\infty > 0$. Then for arbitrarily small $\sigma > 0$ and m large enough,

$$\begin{aligned}
c + \sigma > I_\lambda(u_m) &= \frac{1}{2} \left(\|\nabla u_m\|^2 - \lambda \int g u_m^2 - \int h |u_m|^{2^*} \right) + \frac{1}{N} \int h |u_m|^{2^*} \\
&= \frac{1}{N} \int h |u_m|^{2^*} + o(1) \\
&= \frac{1}{N} \int h |u_0|^{2^*} + \frac{1}{N} \sum_j h(x_j) v_j + \frac{1}{N} h(\infty) v_\infty + o(1) \\
&\geq \frac{1}{N} \int h |u_0|^{2^*} + \sum_j \frac{1}{N} \frac{S^{N/2}}{h(x_j)^{(N-2)/2}} + \frac{1}{N} \frac{S^{N/2}}{h(\infty)^{(N-2)/2}} + o(1) \\
&\geq \frac{1}{N} \int h |u_0|^{2^*} + \frac{1}{N} \frac{S^{N/2}}{\|h\|_\infty^{(N-2)/2}} + o(1)
\end{aligned}$$

which reveals that $\int h |u_0|^{2^*} < 0$ since $c < c^*$.

However, u_0 forms a solution to (1), and so

$$\int |\nabla u_0|^2 - \lambda \int g u_0^2 = \int h |u_0|^{2^*} \geq 0$$

where nonnegativity results for $\lambda \in [0, \lambda_1]$ from the Rayleigh characterisation of the eigenvalues.

This contradiction can only mean that $v_k = v_\infty = 0$, concentration is impossible and it follows simply that $u_m \rightarrow u_0$ in $D^{1,2}(\mathbf{R}^N)$. ■

4. Existence of multiple solutions

For sufficiently small values of $\lambda > 0$, the shape of the graph of $h(x)$ may be responsible for multiple solutions.

Suppose that $h(x)$ and $g(x)$ satisfy conditions (H1) and (G1), but further impose restrictions (H2) and (G2). Define

$$M = \{x \in \mathbf{R}^N : h(x) = \|h\|_\infty\}.$$

Since $h(x)$ is continuous, M is closed.

(H2) Suppose $h(x) \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. For some integer k , suppose $M = \bigcup_{j=1}^k \gamma_j$. For each $1 \leq j \leq k$, assume $\gamma_j \subset \Gamma_j$ where $\Gamma_j \subset \mathbf{R}^N$ is an open bounded set satisfying

$$\Gamma_i \cap \Gamma_j = \emptyset \quad \text{for } i \neq j;$$

$$\max\{h(x) : x \in \partial\Gamma_j\} < \|h\|_\infty.$$

In the case that $\text{int}(\gamma_j) = \emptyset$, we must further specify that there exists $x_0 \in \gamma_j$ and constants $\sigma > 1$, $c_1 > 0$ and $\mathcal{R} > 0$ such that for $|x - x_0| < \mathcal{R}$,

$$\begin{aligned} |h(x_0) - h(x)| &\leq c_1 |x - x_0|^2 & \text{if } N = 4 \\ |h(x_0) - h(x)| &\leq c_2 |x - x_0|^{2+\sigma} & \text{if } N \geq 5 \end{aligned}$$

In the case that $\text{int}(\gamma_j) \neq \emptyset$, selecting any $x_0 \in \text{int}(\gamma_j)$ will suffice.

(G2) For the same $x_0 \in \gamma_j$ and $\mathcal{R} > 0$ there exists a constant $g_0 > 0$ such that $g(x_0) \geq g_0 > 0$ for all $|x - x_0| < \mathcal{R}$.

The main result to be derived in this section is the following theorem:

Theorem 4.1. *Suppose that (H2) and (G2) hold. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, problem (1) possesses at least k positive distinct solutions $u_{j,\lambda}$ satisfying*

$$(10) \quad |\nabla u_{j,\lambda}|^2 \rightarrow \|h\|_\infty^{-(N-2)/2} S^{N/2} \delta_{a_j},$$

$$(11) \quad |u_{j,\lambda}|^{2^*} \rightarrow \|h\|_\infty^{-N/2} S^{N/2} \delta_{a_j}$$

weakly in $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$ where $a_j \in \gamma_j$ and δ_{a_j} is a Dirac measure centred at a_j . If $\lambda = 0$, no nontrivial solutions exist with $I_\lambda(u) \leq c^*$.

Define the Nehari manifold

$$(12) \quad A_\lambda = \{u \in D^{1,2} : u \neq 0, \langle I'_\lambda(u), u \rangle = 0\}$$

Analysis in much of this section concentrates on A_0 . Towards the conclusion, properties of A_λ are shown to closely resemble A_0 for λ small. Clearly A_λ is nonempty since for any u such that $\int h|u|^{2^*} > 0$, $\langle I'_\lambda(tu), tu \rangle = 0$ for some $t > 0$. We now estimate $\inf I_\lambda$ on A_λ using the Brézis-Nirenberg estimates [4].

Introduce the truncated Talenti extremal function,

$$(13) \quad u_\varepsilon(x) = \frac{\phi_{\mathcal{R}}(x_0)(x)}{(\varepsilon + |x - x_0|^2)^{(N-2)/2}}$$

where $\phi_{\mathcal{R}}(x_0)$ is defined in (4). Note that $u_\varepsilon(x)$ corresponds to $\phi_{\mathcal{R}}(x_0)(x)T_\varepsilon(x - x_0)$ to within a constant.

Lemma 4.2. *Assume $N \geq 4$ and (G2), (H2) hold. For $\varepsilon > 0$ sufficiently small and $\lambda > 0$, we have*

$$\sup_{t \geq 0} I_\lambda(tv_\varepsilon) < \frac{S^{N/2}}{N h(x_0)^{(N-2)/2}} = \frac{S^{N/2}}{N \|h\|_\infty^{(N-2)/2}} = c^*$$

where $v_\varepsilon(x) = u_\varepsilon(x)/\|u_\varepsilon\|_{2^*}$.

Proof. From Brézis and Nirenberg [4], we have

$$\begin{aligned} \|\nabla u_\varepsilon\|^2 &= \frac{K_1}{\varepsilon^{(N-2)/2}} + O(1) \quad \text{if } N \geq 3 \\ \|u_\varepsilon\|_{2^*}^2 &= \frac{K_2}{\varepsilon^{(N-2)/2}} + O(\varepsilon) \quad \text{if } N \geq 3 \quad \text{and} \\ \|u_\varepsilon\|_2^2 &= \begin{cases} \frac{K_3}{\varepsilon^{(N-4)/2}} + O(1) & \text{if } N \geq 5 \\ K_3 |\log \varepsilon| + O(1) & \text{if } N = 4 \\ K_3 + O(\varepsilon^{1/2}) & \text{if } N = 3 \end{cases} \end{aligned}$$

where K_1, K_2, K_3 are positive constants depending on N and $S = K_1/K_2$. There exists a unique number $t_\varepsilon \in (0, \infty)$ such that

$$I_\lambda(t_\varepsilon v_\varepsilon) = \max_{t \geq 0} I_\lambda(tv_\varepsilon) = \frac{X_\varepsilon^{N/2}}{NV_\varepsilon^{(N-2)/2}}$$

where

$$\begin{aligned} X_\varepsilon &= \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 - \lambda g v_\varepsilon^2) \quad \text{and} \\ V_\varepsilon &= \int h v_\varepsilon^{2^*}. \end{aligned}$$

To estimate X_ε , we write

$$\lambda \int g v_\varepsilon^2 dx \geq \frac{\lambda g_0 \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}$$

to achieve

$$-\lambda \int g v_\varepsilon^2 dx \leq \begin{cases} -\lambda g_0 K_3 \varepsilon / K_2 + O(\varepsilon^{(N-2)/2}) + O(\varepsilon^{3/2}) & \text{if } N \geq 5 \\ -\lambda g_0 K_3 \varepsilon |\log \varepsilon| / K_2 + O(\varepsilon) & \text{if } N = 4 \end{cases}$$

If $N = 4$, then $|h(x) - h(x_0)| \leq c_1 |x - x_0|^2$ when $|x - x_0| < \mathcal{R}$, and this gives the estimate:

$$V_\varepsilon \geq h(x_0) - \frac{C \varepsilon^{1-(N/2)}}{\|u_\varepsilon\|_{2^*}^2} = h(x_0) + O(\varepsilon)$$

for some constant $C > 0$. If $N \geq 5$, we use the inequality $|h(x) - h(x_0)| \leq c_1 |x - x_0|^{2+\sigma}$ to obtain

$$V_\varepsilon \geq h(x_0) + O(\varepsilon^{1+(\sigma/2)}).$$

Combining these estimates, we get

$$(14) \quad I_\lambda(t_\varepsilon v_\varepsilon) \leq \begin{cases} \frac{S^{N/2}[1 - \lambda g(x_0)K_3\varepsilon/(K_2S) + O(\varepsilon^{3/2})]^{N/2}}{Nh(x_0)^{(N-2)/2}[1 + O(\varepsilon^{(1+\sigma)/2})]^{(N-2)/2}} & \text{if } N \geq 5 \\ \frac{S^{N/2}[1 - \lambda g(x_0)K_3\varepsilon|\log \varepsilon|/(K_2S) + O(\varepsilon)]^{N/2}}{Nh(x_0)^{(N-2)/2}[1 + O(\varepsilon)]^{(N-2)/2}} & \text{if } N = 4 \end{cases}$$

Taking $\varepsilon > 0$ small, the assertion follows. $\blacksquare$

In [14] and [16] it is assumed that $|h(x_0) - h(x)| = o(|x - x_0|^k)$ in $B(x_0, 2\mathcal{R})$, where $k = N - 2$. For $N > 5$, our conditions are sharper. No mention is made in [14] or [16] of constraints on N , but we believe that they implicitly assume $N \geq 4$. The cases of $N = 1$ and $N = 2$ provide special cases of the value 2^* . For $N = 3$, in the notation of Lemma 4.2,

$$X_\varepsilon = \int |\nabla v_\varepsilon|^2 - \lambda \int g v_\varepsilon^2 = \frac{K_1}{K_2} + O(\varepsilon^{1/2}) - \lambda g_0 K_3 \varepsilon^{1/2}$$

where $S = K_1/K_2$. Since the terms involving ε are of the same order, we cannot deduce which dominates and we are unable to guarantee the existence of a mountain pass path below the forbidden level. Following the traditions of [4], for λ sufficiently large the conclusions of Lemma 4.2 would follow.

The following lemma is developed analogously to results achieved by Tarantello [22] and proves an ‘openness’ and a ‘smoothness’ property associated with the manifold.

Lemma 4.3. *Let $u \in A_\lambda$. Then there exists $\varepsilon > 0$ and a differentiable function $t : B(0, \varepsilon) \subset D^{1,2}(\mathbf{R}^N) \mapsto \mathbf{R}$ with $t(0) = 1$ such that $t(w)(u - w) \in A_\lambda$ for each $w \in B(0, \varepsilon)$. Furthermore,*

$$(15) \quad \langle t'(0), w \rangle = \frac{\nabla u \nabla w - \lambda \int g u w - \int h |u|^{2^*-2} u w}{\|\nabla u\|^2 - \lambda \int g u^2 - (2^* - 1) \int h |u|^{2^*}}$$

Proof. Define $F : \mathbf{R} \times D^{1,2} \mapsto \mathbf{R}$ by

$$F(t, w) = t \|\nabla(u - w)\|^2 - \lambda t \int g(u - w)^2 - t^{2^*-1} \int h(u - w)^{2^*}$$

From the definition it follows that

$$F(1, 0) = \|\nabla u\|^2 - \lambda \int g u^2 - \int h |u|^{2^*} = 0$$

$$F_t(t, w) = \|\nabla(u - w)\|^2 - \lambda \int g(u - w)^2 - (2^* - 1) t^{2^*-2} \int h(u - w)^{2^*}$$

$$F_t(1, 0) = \|\nabla u\|^2 - \lambda \int g u^2 - (2^* - 1) \int h |u|^{2^*} < 0$$

As a consequence of the implicit function theorem, it now follows that

$$(16) \quad \langle t'(0), w \rangle = \frac{\langle F_w(1, 0), w \rangle}{F_t(1, 0)} = \frac{\int \nabla u \nabla w - \lambda \int guw - \int h|u|^{2^*-2}uw}{\|\nabla u\|^2 - \lambda \int gu^2 - (2^* - 1) \int h|u|^{2^*}}. \quad \blacksquare$$

Since $u \in A_\lambda$, it follows that the denominator in (16) is negative. By contracting the manifold linearly towards the origin, A_λ must be smoothly deformable to segments of $S^1 \subset D^{1,2}(\mathbf{R}^N)$ and hence forms a manifold of codimension 1.

Before proceeding to deeper analysis of the nature of I_λ on A_λ , we introduce a functional to provide information concerning the extent of concentration of a function.

Since we work in $\mathbf{R}^N$ and since $\{\Gamma_j\}_{j=1}^k$ is a finite collection of sets,

$$\inf_{1 \leq j \leq k} \text{dist}(\gamma_j, \partial \Gamma_j) > \delta' > 0.$$

Consequently there exists a function $\psi_j \in C(\mathbf{R}^N, [0, 1])$ such that

$$\psi_j(x) = \begin{cases} 0 & \text{if } x \in \gamma_j \\ 1 & \text{if } x \in \mathbf{R}^N \setminus \Gamma_j \end{cases}$$

For each $u \in D^{1,2}(\mathbf{R}^N)$, define β_j as

$$(17) \quad \beta_j(u) = \frac{\int_{\mathbf{R}^N} \psi_j(x) |\nabla u|^2 dx}{\int_{\mathbf{R}^N} |\nabla u|^2 dx}$$

For small $\delta > 0$, define

$$(18) \quad A_0^j(\delta) = \{u \in A_0 : \beta_j(u) < \delta\}$$

$$(19) \quad m_0^j(\delta) = \inf \{I_0(u) : u \in \overline{A_0^j(\delta)}\}$$

$$(20) \quad \overline{m_0^j}(\delta) = \inf \{I_0(u) : \beta_j(u) = \delta\}$$

Lemma 4.4. *Let $0 \leq \lambda < \lambda_1$, $\{u_m\} \subset A_\lambda$ and suppose that $I_\lambda(u_m) \rightarrow c \neq 0$. Then there exists $r, R > 0$ such that $r \leq \lim_{m \rightarrow \infty} \|\nabla u_m\| \leq R$. No $(PS)_c$ sequences exist for $c < 0$. If $\lambda = 0$ then $r^2 = S^{N/2} / \|h\|_\infty^{(N-2)/2}$.*

Proof. From the conditions on u_m ,

$$(21) \quad \|\nabla u_m\|^2 - \lambda \int gu_m^2 - \int h|u_m|^{2^*} = 0,$$

$$(22) \quad \frac{1}{2} \|\nabla u_m\|^2 - \frac{\lambda}{2} \int gu_m^2 - \frac{1}{2^*} \int h|u_m|^{2^*} = c + o(1)$$

which reveals

$$\frac{1}{N} \int h|u_m|^{2^*} = c + o(1).$$

This will provide an upper bound as

$$N(c + o(1)) = \int |\nabla u_m|^2 - \lambda \int g u_m^2 \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|\nabla u_m\|^2.$$

Clearly $c < 0$ is impossible. For a lower bound,

$$\begin{aligned} \left(1 - \frac{\lambda}{\lambda_1}\right) \|\nabla u_m\|^2 &\leq \int |\nabla u_m|^2 - \lambda \int g u_m^2 \\ &= \int h|u_m|^{2^*} \leq S^{-2^*/2} \|h\|_\infty \|\nabla u_m\|^{2^*} \end{aligned}$$

and consequently $\|\nabla u_m\| \geq r$. If $\lambda = 0$, the final assertion of the lemma follows. $\blacksquare$

Lemma 4.5. *Given sufficiently small $\delta > 0$,*

$$(23) \quad \overline{m_0^j}(\delta) > c^* + \eta$$

for some small $\eta > 0$.

Proof. Suppose to the contrary of expression (23) that there exists $\{u_m\} \subset A_0$ satisfying $\beta_j(u_m) = \delta$ such that $I_0(u_m) \rightarrow c \leq c^*$.

Let

$$(24) \quad s_m = \left(\frac{\int |\nabla u_m|^2}{\int \|h\|_\infty |u_m|^{2^*}} \right)^{1/(2^*-2)}$$

So $v_m = s_m u_m$ satisfies

$$\int |\nabla v_m|^2 = \|h\|_\infty \int |v_m|^{2^*}.$$

Since $\int h|u_m|^{2^*} \leq \|h\|_\infty \int |u_m|^{2^*}$ it follows that $0 < \lim s_m = s_0 \leq 1$. We now show that $s_0 = 1$. From our supposition,

$$c = \lim_{m \rightarrow \infty} I_0(u_m) = \lim_{m \rightarrow \infty} \frac{1}{N} \int |\nabla u_m|^2 \leq c^*$$

Thus

$$\lim_{m \rightarrow \infty} \int |\nabla u_m|^2 = \lim_{m \rightarrow \infty} \int h|u_m|^{2^*} \leq \frac{S^{N/2}}{\|h\|_\infty^{(N-2)/2}}.$$

From Lemma 4.4,

$$\lim_{m \rightarrow \infty} \|\nabla u_m\|^2 \geq \frac{S^{N/2}}{\|h\|_{\infty}^{(N-2)/2}}$$

Hence,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|\nabla u_m\|^2 &= \frac{S^{N/2}}{\|h\|_{\infty}^{(N-2)/2}} = \lim_{m \rightarrow \infty} \int h(x) |u_m|^{2^*} \leq \|h\|_{\infty} \lim_{m \rightarrow \infty} \int |u_m|^{2^*} \\ &\leq \|h\|_{\infty} S^{-N/(N-2)} \lim_{m \rightarrow \infty} \|\nabla u_m\|^{2^*} \\ &= \|h\|_{\infty} S^{-N/(N-2)} \left(\frac{S^{N/2}}{\|h\|_{\infty}^{(N-2)/2}} \right)^{N/(N-2)} = \frac{S^{N/2}}{\|h\|_{\infty}^{(N-2)/2}} \end{aligned}$$

It follows that the chain of inequalities must be equalities, and so

$$(25) \quad \lim_{m \rightarrow \infty} \int h |u_m|^{2^*} = \lim_{m \rightarrow \infty} \|h\|_{\infty} \int |u_m|^{2^*}$$

Let $w_m = v_m / \|v_m\|_{2^*}$, so $\|w_m\|_{2^*} = 1$ and

$$\begin{aligned} \int |\nabla w_m|^2 &= \frac{\int |\nabla v_m|^2}{\|v_m\|_{2^*}^2} = \frac{\int |\nabla u_m|^2}{\left(\int |u_m|^{2^*}\right)^{2/2^*}} = \|h\|_{\infty}^{2/2^*} \frac{\int |\nabla u_m|^2}{\left(\int \|h\|_{\infty} |u_m|^{2^*}\right)^{2/2^*}} \\ &= \|h\|_{\infty}^{2/2^*} \left(\int |\nabla u_m|^2 \right)^{1-(2/2^*)} \frac{\left(\int |\nabla u_m|^2\right)^{2/2^*}}{\left(\|h\|_{\infty} \int |u_m|^{2^*}\right)^{2/2^*}} \\ &= \|h\|_{\infty}^{2/2^*} \left(\int |\nabla u_m|^2 \right)^{1-(2/2^*)} s_m^{2(2^*-2)/2^*} \\ &\leq \|h\|_{\infty}^{2/2^*} \left(\frac{S^{N/2}}{\|h\|_{\infty}^{(N-2)/2}} \right)^{1-(2/2^*)} s_m^{2(2^*-2)/2^*} + o(1) \end{aligned}$$

Consequently

$$\lim_{m \rightarrow \infty} \|\nabla w_m\|^2 \leq S s_m^{2(2^*-2)/2}$$

However, $\|w_m\|_{2^*} = 1$, and as an implication of the Sobolev imbedding theorem, $s_m \rightarrow 1$ and w_m must be a minimising sequence for S . In accordance with Propositions 3.1 and 3.2, $w_m \rightharpoonup w_0$ weakly in $D^{1,2}(\mathbf{R}^N)$ and

$$|\nabla w_m|^2 \rightharpoonup |\nabla w_0|^2 + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_{\infty}; \quad |w_m|^{2^*} \rightharpoonup |w_0|^{2^*} + \sum_{j \in J} v_j \delta_{x_j} + v_{\infty},$$

weakly in the space $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$. Thus,

$$\begin{aligned} S &= \|\nabla w_0\|^2 + \|\mu\| + \mu_\infty \geq S(\|w_0\|_{2^*}^2 + \|v\|^{2/2^*} + v_\infty^{2/2^*}) \\ &\geq S(\|w_0\|_{2^*}^2 + \|v\| + v_\infty)^{2/2^*} = S \end{aligned}$$

Consequently, Proposition 3.3 implies that only one term can be nonzero and only one of (A), (B) or (C) can hold (where $\|v\| = \sum_{j \in J} v_j$):

- (A) $\|w_0\|_{2^*} = 1; \|v\| = 0; v_\infty = 0$ (compactness),
- (B) $w_0 = 0; \|v\| = 1; v_\infty = 0$ (local concentration),
- (C) $w_0 = 0; \|v\| = 0; v_\infty = 1$ (concentration at infinity).

Suppose that (A) holds. Then $w_0 = T_r(x - x_0)$, a Talenti extremal function. However, $h(x) \not\equiv \text{constant}$, and so

$$\int \|h\|_\infty |w_0(x)|^{2^*} > \int |h(x)| |w_0|^{2^*}$$

providing a contradiction to (25).

Suppose that (B) is true. Then by Proposition 3.3, w_m must concentrate at a single point, say $\tilde{x} \in \mathbf{R}^N$, and weakly in $\mathcal{M}(\mathbf{R}^N)$ the following convergence result holds:

$$|\nabla w_m|^2 \rightharpoonup S\delta_{\tilde{x}}; \quad |w_m|^{2^*} \rightharpoonup \delta_{\tilde{x}}.$$

By the definition of weak convergence to measures,

$$\delta = \lim_{m \rightarrow \infty} \beta_j(u_m) = \lim_{m \rightarrow \infty} \frac{\int \psi_j(x) |\nabla u_m|^2}{\|\nabla u_m\|^2} = \lim_{m \rightarrow \infty} \frac{\int \psi_j(x) |\nabla w_m|^2}{\|\nabla w_m\|^2} = \psi_j(\tilde{x})$$

and consequently $\tilde{x} \in \Gamma_j \setminus \gamma_j$. But

$$\lim_{m \rightarrow \infty} \int h(x) |w_m|^{2^*} = h(\tilde{x}) < \|h\|_\infty$$

again contradicting (25).

Suppose that (C) holds. Then $\mu_\infty = S$ and $\|\mu\| = \|v\| = 0$. Define ϕ^R as in (3). Then

$$\delta = \lim_{m \rightarrow \infty} \beta_j(u_m) = \lim_{m \rightarrow \infty} \frac{\int \psi_j |\nabla w_m|^2}{\|\nabla w_m\|^2} = \lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{\int \phi^R |\nabla w_m|^2}{\|\nabla w_m\|^2} = \frac{\mu_\infty}{S} = 1$$

Since δ is small, this provides a contradiction.

In light of these contradictions, it is impossible for $\inf\{I_0(u) : \beta_j(u) = \delta\} \leq c^*$ and so there exists $\eta > 0$ such that $m_0^j > c^* + \eta$. ■

For each $j, 1 \leq j \leq k$, such a δ^j , and η^j exist, satisfying (23). Let $\delta =$

$\inf\{\delta^j, 1 \leq j \leq k\}$, and assign η analogously. Henceforth omit the δ references in definitions (18)–(20).

We now apply the above-derived bounds associated with the Nehari manifold A_0 to A_λ by showing that for sufficiently small λ , and with a restriction to bound $\|\nabla u\|$, the two sets have a small separation distance. The lemma above has shown that $I_0(u) > c^* + \eta$ for u on $\{u \in A_0 : \beta_j(u) = \delta\}$. Retain this notation for η in the sequel.

On the set A_0 , $I_0(u) = N^{-1}\|\nabla u\|^2$. Hence there exists a number $R_{c^*+\eta} = [N(c^* + \eta)]^{1/2}$ such that if $\|\nabla u\| > R_{c^*+\eta}$ then $I_0(u) > c^* + \eta$.

Define a submanifold of A_0^j by:

$$(26) \quad A_0^{j, c^*+\eta} = \{u \in A_0^j : \|\nabla u\| < R_{c^*+\eta}\}.$$

To see $A_0^{j, c^*+\eta}$ is nonempty, we note $\eta > 0$. Let $u_\varepsilon = s_\varepsilon v_\varepsilon \in A_0$ where v_ε is from Lemma 4.2 and

$$s_\varepsilon = \left(\frac{\int |\nabla v_\varepsilon|^2}{\int h(x)|v_\varepsilon|^{2^*}} \right)^{(N-2)/4}.$$

By estimate (14), with $\lambda = 0$, $I_0(u_\varepsilon) \rightarrow c^* < c^* + \eta$ as $\varepsilon \rightarrow 0$.

Lemma 4.6. *For each $u \in A_0^{j, c^*+\eta}$ there exists $s_\lambda(u) > 0$ such that $s_\lambda(u)u \in A_\lambda$ and*

$$\sup_{u \in A_0^{j, c^*+\eta}} |I_\lambda(s_\lambda(u)u) - I_0(u)| < \varepsilon(\lambda)$$

where $\varepsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly for $u \in A_0^{j, c^*+\eta}$.

Proof. For each $u \in A_0$ the existence of $s_\lambda(u) > 0$ such that $s_\lambda(u)u \in A_\lambda$ is guaranteed. Indeed, $s_\lambda(u)$ is expressed in

$$(27) \quad s_\lambda(u)^{2^*-2} = \frac{\|\nabla u\|^2 - \lambda \int gu^2}{\int h|u|^{2^*}}.$$

For $u \in A_0^{j, c^*+\eta}$, it follows that $\|\nabla u\| \leq R_{c^*+\eta}$. An upper bound for $I_0(u)$ in $A_0^{j, c^*+\eta}$ follows from definition (26), that is, $\sup\{I_0(u) : u \in A_0^{j, c^*+\eta}\} < \infty$. As a consequence of Lemma 4.4, $(\int h|u|^{2^*})^{1/2} = \|\nabla u\| > r > 0$.

Now

$$s_\lambda(u)^{2^*-2} = \frac{\|\nabla u\|^2}{\int h|u|^{2^*}} - \lambda \frac{\int gu^2}{\int h|u|^{2^*}} = 1 - \lambda \frac{\int gu^2}{\int h|u|^{2^*}}.$$

By the Sobolev inequality, and Lemma 4.4

$$1 - \frac{\lambda}{r} S^{-1} \|g\|_{N/2} R_{c^*+\eta}^2 \leq s_\lambda(u)^{2^*-2} \leq 1 + \frac{\lambda}{r} S^{-1} \|g\|_{N/2} R_{c^*+\eta}^2.$$

Hence

$$\begin{aligned} I_\lambda(s_\lambda(u)u) &= s_\lambda(u)^2 \left[\frac{1}{N} \|\nabla u\|^2 - \frac{\lambda}{N} \int g u^2 \right] \\ &\geq \left(1 - \frac{\lambda}{S r} \|g\|_{N/2} R_{c^*+\eta}^2 \right)^{2/(2^*-2)} \left[I_0(u) - \frac{\lambda}{N S} \|g\|_{N/2} R_{c^*+\eta}^2 \right]. \end{aligned}$$

Similarly,

$$I_\lambda(s_\lambda(u)u) \leq \left(1 + \frac{\lambda}{r S} \|g\|_{N/2} R_{c^*+\eta}^2 \right)^{2/(2^*-2)} \left[I_0(u) + \frac{\lambda}{N S} \|g\|_{N/2} R_{c^*+\eta}^2 \right].$$

Consequently, $I_\lambda(s_\lambda(u)u)$ can be made arbitrarily close to $I_0(u)$ by selection of small values of λ . The bound $\varepsilon(\lambda)$ holds uniformly over all $u \in A_0^{j, c^*+\eta}$ as a consequence of the uniform bound $\|\nabla u\| \leq R_{c^*+\eta}$. ■

With $s_\lambda(u)$ defined in (27), we now define the set $A_\lambda^{j, c^*+\eta} \subset A_\lambda^j$ as follows:

$$A_\lambda^{j, c^*+\eta} = \{u \in A_\lambda : \exists v \in A_0^{j, c^*+\eta} \text{ with } u = s_\lambda(v)v \in A_\lambda\}.$$

As a corollary to Lemma 4.3, the following will be required:

Lemma 4.7. *Let $t(\cdot)$ be defined as in Lemma 4.3. For each $u \in A_\lambda^{j, c^*+\eta}$ there exists a $\theta^\lambda > 0$ such that for each $w \in D^{1,2}(\mathbf{R}^N)$ with $\|\nabla w\| < \theta^\lambda$, it holds that $t(w)(u - w) \in A_\lambda^{j, c^*+\eta}$.*

Proof. Consider $S_\lambda(u) = s_\lambda(u)u$ as a mapping from A_0 to A_λ . For each $0 \leq \lambda < \lambda_1$, expression (27) shows that S_λ is a one-to-one continuous map between the two manifolds.

Let $u \in A_\lambda^{j, c^*+\eta}$. Then there exists a unique $v \in A_0^{j, c^*+\eta}$ such that $S_\lambda(v) = u$. Let $\mathcal{N}$ be a relatively open neighbourhood of v in $A_0^{j, c^*+\eta}$. Consequently $S_\lambda(\mathcal{N})$ is a relatively open neighbourhood of u in A_λ .

If $B(u, \varepsilon) \cap A_\lambda \subset S_\lambda(\mathcal{N})$, then choosing any $w \in D^{1,2}(\mathbf{R}^N)$ with $\|\nabla w\| < \varepsilon$ means that $t(w)(u - w) \in S_\lambda(\mathcal{N})$. Consequently, $t(w)(u - w) \in A_\lambda^{j, c^*+\eta}$. ■

Define

$$m_\lambda^j = \inf \left\{ I_\lambda(u) : u \in \overline{A_\lambda^{j, c^*+\eta}} \right\}$$

As a consequence of Lemma 4.2, for each $\lambda > 0$ it follows that $m_\lambda^j < c^*$. In accordance with the technique developed in [8], the following proposition produces a locally coercive geometry on the manifold A_λ .

Proposition 4.8. *For each $\lambda \in (0, \lambda_0]$ there exist sequences $\{v_m^j\} \subset A_\lambda^{j, c^*+\eta}$, $j = 1, \dots, k$ such that $v_m^j \geq 0$ and $I_\lambda(v_m^j) \rightarrow m_\lambda^j$ and $I'_\lambda(v_m^j) \rightarrow 0$ in $D^{-1,2}(\mathbf{R}^N)$ as $m \rightarrow \infty$.*

Proof. Following [8], let Υ_j denote the boundary of $A_\lambda^{j, c^* + \eta}$. That is,

$$\begin{aligned}\Upsilon_j &= \{s_\lambda(u)u \in A_\lambda : u \in \overline{A_0^{j, c^* + \eta}}; \\ &\quad \beta_j(u) \leq \delta \text{ and } \|\nabla u\| = R_{c^* + \eta}; \text{ or} \\ &\quad \beta_j(u) = \delta \text{ and } \|\nabla u\| \leq R_{c^* + \eta}\}\end{aligned}$$

Then $\overline{A_\lambda^{j, c^* + \eta}} = A_\lambda^{j, c^* + \eta} \cup \Upsilon_j$.

As a corollary of Lemma 4.5, there exists $\lambda_0 > 0$ such that for each $0 < \lambda < \lambda_0$, $I_\lambda(u) > c^* + (\eta/2)$ for each $u \in \Upsilon_j$.

According to Lemmas 4.2 and 4.5,

$$m_\lambda^j = \inf\{I_\lambda(u) : u \in \overline{A_\lambda^{j, c^* + \eta}}\} < \inf\{I_\lambda(u) : u \in \Upsilon_j\}.$$

Let $\{v_m^j\} \subset \overline{A_\lambda^{j, c^* + \eta}}$ be a minimising sequence for m_λ^j . Replacing v_m^j by $|v_m^j|$ if necessary, we may assume that $v_m^j(x) \geq 0$. By application of Ekeland's variational principle [15], for each $1 \leq j \leq k$, there is a sequence $\{u_m^j\} \subset \overline{A_\lambda^{j, c^* + \eta}}$ such that

$$(28) \quad \begin{cases} I_\lambda(u_m^j) \leq I_\lambda(v_m^j) < m_\lambda^j + (1/m), & \|\nabla(u_m^j - v_m^j)\| \leq 1/m \\ I_\lambda(u_m^j) < I_\lambda(w) + m^{-1}\|\nabla(w - u_m^j)\| \end{cases}$$

for every $w \in \overline{A_\lambda^{j, c^* + \eta}}$ and $w \neq u_m^j$. In the sequel, omit the subscript j for brevity, denoting u_m^j as u_m . From Lemma 4.7, for each u_m there exists θ_m and a function $t_m(w)$ defined on $B(0, \theta_m) \subset D^{1,2}(\mathbf{R}^N)$ such that $t_m(w)(u_m - w) \in A_\lambda^{j, c^* + \eta}$. Let $0 < \theta < \theta_m$, $u \in D^{1,2}$ with $u \neq 0$ and set $w_\theta = \theta u / \|\nabla u\|$. For m fixed, consider $v_\theta = t_m(w_\theta)(u_m - w_\theta)$. Since $v_\theta \in A_\lambda^{j, c^* + \eta}$, by (28) we have

$$I_\lambda(v_\theta) - I_\lambda(u_m) \geq -\frac{1}{m} \|\nabla(v_\theta - u_m)\|.$$

Applying the Taylor expansion reveals

$$\langle I'_\lambda(u_m), v_\theta - u_m \rangle + o(\|\nabla(v_\theta - u_m)\|) \geq -\frac{1}{m} \|\nabla(v_\theta - u_m)\|.$$

Consequently,

$$\begin{aligned}\langle I'_\lambda(u_m), -w_\theta + (t_m(w_\theta) - 1)(u_m - w_\theta) \rangle \\ \geq -\frac{1}{m} \|\nabla(v_\theta - u_m)\| + o(\|\nabla(v_\theta - u_m)\|),\end{aligned}$$

and hence

$$(29) \quad \begin{aligned} & \langle I'_\lambda(u_m), -w_\theta \rangle + (t_m(w_\theta) - 1) \langle I'_\lambda(u_m), u_m - w_\theta \rangle \\ & \geq -\frac{1}{m} \|\nabla(v_\theta - u_m)\| + o(\|\nabla(v_\theta - u_m)\|). \end{aligned}$$

Since $\langle I'_\lambda(v_\theta), t_m(w_\theta)(u_m - w_\theta) \rangle = 0$, equation (29) implies that

$$(30) \quad \begin{aligned} & \langle I'_\lambda(u_m), -w_\theta \rangle + \frac{t_m(w_\theta) - 1}{t_m(w_\theta)} \langle I'_\lambda(u_m) - I'_\lambda(v_\theta), t_m(w_\theta)(u_m - w_\theta) \rangle \\ & \geq -\frac{1}{m} \|\nabla(v_\theta - u_m)\| + o(\|\nabla(v_\theta - u_m)\|). \end{aligned}$$

Rearranging expression (30) gives

$$(31) \quad \begin{aligned} \langle I'_\lambda(u_m), \frac{u}{\|\nabla u\|} \rangle & \leq \frac{1}{m\theta} \|\nabla(v_\theta - u_m)\| + o\left(\frac{\|\nabla(v_\theta - u_m)\|}{\theta}\right) \\ & + \frac{|t_m(w_\theta) - 1|}{\theta} \langle I'_\lambda(u_m) - I'_\lambda(v_\theta), u_m - w_\theta \rangle. \end{aligned}$$

Since $\|\nabla(v_\theta - u_m)\| \leq \theta + C|t_m(w_\theta) - 1|$ and by (15),

$$\lim_{\theta \rightarrow 0} \frac{|t_m(w_\theta) - 1|}{\theta} \leq \|t'_m(0)\| + \tilde{C}$$

for positive constants C and $\tilde{C}$ independent of θ and m . Allowing $\theta \rightarrow 0$ in (31), it follows that $v_\theta \rightarrow u_m$ in $D^{1,2}$ and $I'_\lambda(v_\theta) \rightarrow I'_\lambda(u_m)$ as $\theta \rightarrow 0$ and consequently

$$(32) \quad \left\langle I'_\lambda(u_m), \frac{u_m}{\|\nabla u_m\|} \right\rangle \leq \frac{C}{m}.$$

Inequality (32) yields that $\|I'_\lambda(u_m)\|_{D^{-1,2}} \rightarrow 0$ as $m \rightarrow \infty$. Since $\|\nabla(v_m^j - u_m^j)\| \rightarrow 0$ as $m \rightarrow \infty$, it must be that $\|I'_\lambda(v_m^j)\|_{D^{-1,2}} \rightarrow 0$ as $m \rightarrow \infty$, providing the conclusion. $\blacksquare$

Proof of Theorem 4.1. For each $j, 1 \leq j \leq k$, Proposition 4.8 dictates that there exists $\lambda_0^j > 0$, such that for each $0 < \lambda < \lambda_0^j$ there exists a minimising sequence $\{u_m^j\} \subset A_\lambda^{j, c^* + \eta}$ with $I_\lambda(u_m^j) \rightarrow m_\lambda^j$ and $I'_\lambda(u_m^j) \rightarrow 0$ in $D^{-1,2}(\mathbf{R}^N)$.

Lemmas 4.2 and 4.5 imply that

$$m_\lambda^j < c^* < \inf\{I_\lambda(u) : u \in \Upsilon_j\}, \quad j = 1, \dots, k.$$

By Lemma 3.5, I_λ satisfies the (PS) $_c$ -condition for all $c < c^*$ and consequently a subsequence of u_m^j is strongly convergent in $D^{1,2}$ to, say, u_λ^j , a weak solution to (1). Distinction between the solutions is guaranteed since the sets $A_\lambda^{j, c^* + \eta}$ are disjoint for small δ .

It remains to show that the solutions converge in measure. Fix $\lambda \in (0, \lambda_0)$. Later we shall take the limit $\lambda \rightarrow 0^+$. For each such $\lambda > 0$, there exists a (PS)-sequence, $\{u_k^j\} \subset A_\lambda^{j, c^* + \eta}$. We have derived that the (PS)-sequence is strongly convergent to a solution: $u_k^j \rightarrow u_\lambda^j \in A_\lambda^{j, c^* + \eta}$. Recall firstly that by Lemma 4.4, $\|\nabla u_\lambda^j\|$ is bounded from below over all small λ and that for $\lambda = 0$, we have a specific bound. Let

$$s_\lambda^j = \left(\frac{\int |\nabla u_\lambda^j|^2}{\int \|h\|_\infty |u_\lambda^j|^{2^*}} \right)^{1/(2^*-2)}$$

and set $v_\lambda^j = s_\lambda^j u_\lambda^j$. Then v_λ^j satisfies

$$\int |\nabla v_\lambda^j|^2 = \int \|h\|_\infty |v_\lambda^j|^{2^*}.$$

As previously (see Lemma 4.5) we may derive

$$(33) \quad \lim_{\lambda \rightarrow 0^+} \int |\nabla u_\lambda^j|^2 = \lim_{\lambda \rightarrow 0^+} \int \|h\|_\infty |v_\lambda^j|^{2^*} = \frac{S^{N/2}}{\|h\|_\infty^{(N-2)/2}}$$

and $s_\lambda^j \rightarrow s_0^j \in (0, 1]$. Set $w_\lambda^j = v_\lambda^j / \|v_\lambda^j\|_{2^*}$. Then

$$\int |\nabla w_\lambda^j|^2 = \|h\|_\infty^{2/2^*} \left(\int |\nabla u_\lambda^j|^2 \right)^{1-(2/2^*)} (s_\lambda^j)^{2(2^*-2)/2^*}.$$

Hence $s_0^j = 1$ and $\lim_{\lambda \rightarrow 0^+} \int |\nabla w_\lambda^j|^2 = S$. Following the proof of Lemma 4.5, w_λ^j must satisfy one of (A) compactness, (B) local concentration or (C) concentration at infinity. Similarly, alternatives (A) and (C) may be eliminated immediately. Thus, option (B) remains the only possibility and $|\nabla w_\lambda^j|^2 \rightarrow S\delta_{\tilde{x}}$, and $|w_\lambda^j|^{2^*} \rightarrow \delta_{\tilde{x}}$ weakly in $\mathcal{M}(\mathbf{R}^N)$ as $\lambda \rightarrow 0^+$, for some $\tilde{x} \in \Gamma_j$. From expression (33),

$$\begin{aligned} \frac{S^{N/2}}{\|h\|_\infty^{(N-2)/2}} &= \lim_{\lambda \rightarrow 0^+} \int |\nabla u_\lambda^j|^2 = \lim_{\lambda \rightarrow 0^+} (s_\lambda^j)^{-2^*} \|v_\lambda^j\|_{2^*}^{2^*} \int h(x) |w_\lambda^j|^{2^*} \\ &= \frac{1}{\|h\|_\infty} \frac{S^{N/2}}{\|h\|_\infty^{(N-2)/2}} h(\tilde{x}) \end{aligned}$$

which proves that $\tilde{x} \in \gamma_j$.

Suppose $\lambda = 0$, and $\{u_m\}$ is a $(PS)_c$ -sequence for I_0 with $0 \neq c \leq c^*$. Lemma 4.4 gives $c = c^*$. Following the proof to Lemma 4.5, $\lim_{m \rightarrow \infty} \int h|u_m|^{2^*} = \lim_{m \rightarrow \infty} \|h\|_\infty \int |u_m|^{2^*}$. Again one of (A), (B) or (C) occurs. Since condition (H1) expresses that $h(x) \neq \text{constant}$, option (A) is easily eliminated so $u_m \rightarrow 0$ in $D^{1,2}(\mathbf{R}^N)$. $\blacksquare$

Remark 4.9. We remark that our results do not contradict the achievements in [2]. Referring to their paper, replace the group G with the identity mapping. Then the condition imposed in [2] Theorem 1 becomes

$$\int_{\mathbf{R}^N} h(x) S^{-N/(N-2)} |T_r(x-a)|^{2^*} dx \geq S^{-N/(N-2)} \|h\|_\infty$$

for some real r and a . However, Hölders inequality dictates $\int |T_r(x-a)|^{2^*} \leq \|h\|_\infty S^{-2/2^*}$. Equality can only be achieved in the limit $r \rightarrow 0$. Hence, results in [2] are truly dependent on group symmetries, and do not elicit solutions below the first forbidden level c^* .

Remark 4.10. Suppose that γ_j is a simply connected open set. An open problem concerns where inside of γ_j does the solution concentrate? A delicate play-off occurs between the influence of $h(x)$ outside of γ_j , and the effect of $g(x)$ over $\mathbf{R}^N$. Suppose we further restrict $g(x)$ to be continuous with a unique strict maximum on the interior of γ_j .

We query whether the influence of $g(x)$ dominates, and that as $\lambda \rightarrow 0$, does $|\nabla u_\lambda|^2 \rightarrow S^{N/2} \|h\|_\infty^{(2-N)/2} \delta_{\tilde{x}}$ in $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$ where $\max_{\gamma_j} g(x) = g(\tilde{x})$? Investigations to date have not revealed the truth of this hypothesis.

Indeed, suppose that $g(x) \leq 0$, except a small neighbourhood $\mathcal{N}$ of $\tilde{x} \in \gamma_j$, where $g(x) > 0$. As a preliminary investigation, one wonders whether the solution must concentrate inside $\mathcal{N}$.

5. Category and multiplicity

In this section, we follow [19] and postulate that the Lusternik-Schnirelmann category of

$$M = \{x \in \mathbf{R}^N : h(x) = \|h\|_\infty\}$$

provides an estimate of the number of solutions. Several publications [19], [12] have demonstrated that an investigation of the nonlinearity extends the multiplicity results due to the topology of the domain [1].

Introduce the stronger condition on $g(x)$ and clarify the condition on $h(x)$:

(G3) Suppose that $g \in L^{N/2} \cap L^\infty(\mathbf{R}^N)$ and $g(x) > g_0 > 0$ for all $x \in M$.

(H3) Suppose $h \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, and there exist constants $\sigma > 1$, $c_1 > 0$ and $\mathcal{R} > 0$ such that for $|x - x_0| < \mathcal{R}$, each $x_0 \in M$,

$$|h(x_0) - h(x)| \leq c_1 |x - x_0|^2 \quad \text{if } N = 4$$

$$|h(x_0) - h(x)| \leq c_1 |x - x_0|^{2+\sigma} \quad \text{if } N \geq 5$$

Suppose that for x outside of a sufficiently large ball B_Q , $h(x) < \|h\|_\infty - \varepsilon$ for some $\varepsilon > 0$.

With conditions (G3) and (H3), the set M is compact and for any $\delta > 0$ there exists a set

$$M_\delta = \{x \in \mathbf{R}^N : \text{dist}(x, M) \leq \delta\}$$

We denote by $\text{cat}_{M_\delta}(M)$ the Lusternik-Schnirelmann category of the set M in M_δ .

Suppose Y is a closed subset of a topological space X . Recall that the LS category of Y in X , $\text{cat}_X(Y)$ is the least integer σ such that Y can be covered by σ closed subsets $A_1, \dots, A_\sigma$ of Y such that for all i , A_i is contractible in X . For more details on category, see [25].

The main theorem of this section is the following:

Theorem 5.1. *Suppose $h(x)$ and $g(x)$ satisfy (G3) and (H3). Let $\delta > 0$ be arbitrarily small. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, problem (1) possesses at least $\text{cat}_{M_\delta}(M)$ solutions.*

Firstly we show that as the perturbation coefficient λ diminishes, any function for which the energy lies below c^* must assume a concentrating form.

Define the set

$$(34) \quad A_\lambda^* = \{u \in A_\lambda : I_\lambda(u) < c^*\}.$$

Proposition 5.2. *Let λ_m be a positive sequence with $\lambda_m \rightarrow 0$. Let $u_m \in A_{\lambda_m}^*$. Then for a subsequence, again indexed with m , there exists a sequence of points $\{x_m\}$ in $\mathbf{R}^N$ and a sequence of real numbers $\{r_m\}$ such that $x_m \rightarrow x_0 \in \bar{M}$ and $r_m \rightarrow 0$ with*

$$\begin{aligned} \left\| \nabla \left(u_m - \left(\frac{S}{\|h\|_\infty} \right)^{(N-2)/2} T_{r_m}(x - x_m) \right) \right\| &\rightarrow 0, \\ \left\| u_m - \left(\frac{S}{\|h\|_\infty} \right)^{(N-2)/2} T_{r_m}(x - x_m) \right\|_{2^*} &\rightarrow 0. \end{aligned}$$

Proof. As previously (Lemma 4.5),

$$\lim_{m \rightarrow \infty} \|\nabla u_m\|^2 = \lim_{m \rightarrow \infty} \int h |u_m|^{2^*} = S^{N/2} \|h\|_\infty^{(2-N)/2}.$$

Define s_m as in (24). and set $v_m = s_m u_m$. As previously, we derive that $s_m \rightarrow s_0 \in (0, 1]$. Define $w_m = v_m / \|v_m\|_{2^*}$. It follows that

$$\|\nabla w_m\|^2 = \|h\|_\infty^{2/2^*} \frac{\int |\nabla u_m|^2}{(\int \|h\|_\infty |u_m|^{2^*})^{2/2^*}} \leq S s_m^{2(2^*-2)/2^*}$$

Hence $\lim_{m \rightarrow \infty} \|\nabla w_m\|^2 = s_0^{4/N} S$ and so $s_0 = 1$ and w_m is a sequence providing the best Sobolev constant. We infer from Lions' work, in particular [17] Theorem I.1 and Corollary I.1 that the sequence w_m is compact to within translation and scaling and sign. Without loss of generality, in the sequel we assume $w_m(x) \geq 0$. Thus there exists a sequence of positive real numbers $\{r_m\}$ and points in $\mathbf{R}^N$, $\{x_m\}$ such that

$$\|\nabla(w_m - T_{r_m}(x - x_m))\| \rightarrow 0, \quad \|w_m - T_{r_m}(x - x_m)\|_{2^*} \rightarrow 0.$$

We now show that $r_m \rightarrow 0$. Following the working in Lemma 4.5, we have that

$$(35) \quad \lim_{m \rightarrow \infty} \int h(x) |w_m|^{2^*} = \lim_{m \rightarrow \infty} \|h\|_{\infty} \int |w_m|^{2^*} = \|h\|_{\infty}$$

Further, $w_m \rightharpoonup w_0$ weakly in $D^{1,2}(\mathbf{R}^N)$. Since $\|\nabla w_m\| \rightarrow S$, Proposition 3.3 dictates that, following Lemma 4.5, only one of (A) compactness, (B) local concentration or (C) concentration at infinity may occur.

Suppose that (A) occurs. Then for a subsequence $w_m \rightarrow w_0 = T_r(x - x_0)$ in $D^{1,2}$ for some $x_0 \in \mathbf{R}^N$, $r > 0$. But $h(x)$ is nonconstant, and $\int \|h\|_{\infty} |w_0|^{2^*} > \int |h(x) w_0|^{2^*}$ contradicting (35).

Suppose that (C) occurs. From the definition of the large number $Q \in \mathbf{R}$ in the condition (H3), and from the definition of concentration at infinity,

$$\begin{aligned} \int_{\mathbf{R}^N} h |w_m|^{2^*} dx &= \int_{\mathbf{R}^N \setminus B_Q} h(x) |w_m|^{2^*} dx + o(1) \\ &\leq \sup_{|x| > Q} |h(x)| \int_{\mathbf{R}^N \setminus B_Q} |w_m|^{2^*} dx + o(1) \\ &\leq (\|h\|_{\infty} - \varepsilon) + o(1) \end{aligned}$$

providing a contradiction to (35).

Subsequently, (B) is the only valid option. For local concentration to occur, we must have $r_m \rightarrow 0$ as $|\nabla T_{r_m}(x - \tilde{x})| \rightarrow S \delta_{\tilde{x}}$, $|T_{r_m}(x - \tilde{x})|^{2^*} \rightarrow \delta_{\tilde{x}}$ in $\mathcal{M}(\mathbf{R}^N \cup \{\infty\})$.

It remains to show that $x_m \rightarrow x_0 \in \overline{M}$. Since concentration at infinity has been disallowed, $\{x_m\}$ is a bounded sequence and has a convergent subsequence. Suppose that $x_m \rightarrow x_0 \notin \overline{M}$. Then $h(x_0) < \|h\|_{\infty}$ and $\int h(x) |w_m|^{2^*} \rightarrow h(x_0)$ providing a contradiction.

We now express the derived properties of w_m in terms of u_m . For each $\lambda_m \in (0, \lambda_0]$, $u_m \in A_{\lambda_m}^* = \{u \in A_{\lambda_m} : I_{\lambda_m}(u) < c^*\}$. For each $\lambda \in (0, \lambda_0]$, the set A_{λ}^* is a bounded set, and so $u_m = C_m w_m$, where C_m is bounded from below and above by positive real numbers.

As $\lambda_m \rightarrow 0$, we have $\|\nabla w_m\|^2 \rightarrow S$ and $\|w_m\|_{2^*}^2 = 1$. Further, $\|\nabla u_m\|^2 \rightarrow S^{N/2} \|h\|_\infty^{(2-N)/2}$ and $\|u_m\|_{2^*}^2 \rightarrow (S \|h\|_\infty^{-1})^{N/2}$. Combining these,

$$o(1) = \|\nabla(u_m - (S \|h\|_\infty^{-1})^{(N-2)/2} w_m)\|$$

giving

$$\|\nabla(u_m - (S \|h\|_\infty^{-1})^{(N-2)/2} T_{r_m}(x - x_m))\| \rightarrow 0. \quad \blacksquare$$

From (H3), we know $M \subset B_Q$. Define

$$\zeta_Q(x) = \begin{cases} x & \text{for } |x| \leq Q \\ (x/|x|)Q & \text{for } |x| > Q. \end{cases}$$

Define a ‘barycentre’ α of $u \in D^{1,2}(\mathbf{R}^N)$ by:

$$\alpha(u) = \frac{\int_{\mathbf{R}^N} \zeta_Q |\nabla u|^2 dx}{\|\nabla u\|^2}.$$

Recall from (27) the definition of $s_\lambda(u)$.

In the following, we wish to take advantage of the estimates derived in [4] and expressed in (14). Accordingly, we work with a truncated Talenti function. We introduce the truncator $\phi_{\mathcal{R}}(x_0)$ as defined in (4). Consider s_λ , expressed in (27) as a mapping from $D^{1,2}(\mathbf{R}^N)$ to A_λ .

Define $\Phi_\lambda : \mathbf{R}^N \mapsto D^{1,2}(\mathbf{R}^N)$ as:

$$\Phi_\lambda(x) = s_\lambda(\phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0)) \phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0).$$

As $\lambda \rightarrow 0$, the function $\Phi_\lambda(x)$ concentrates and this allows the working to neglect the $\phi_{\mathcal{R}}(x_0)$ function (formally the estimate is contained in the proof to Lemma 1.1 in [4]).

From the estimate (14), with $x_0 \in M$, we may discern that for fixed $\lambda > 0$ there exists $\varepsilon > 0$ such that $I_\lambda(t_\varepsilon v_\varepsilon) < c^*$. This relationship between λ and ε can be expressed as follows (using the more stringent restrictions imposed in the case $N = 4$). Let $\varepsilon_\lambda = \exp(-b/\lambda)$. There exists suitable $b > 0$ and $\tilde{\lambda} > 0$ such that for $0 < \lambda < \tilde{\lambda}$ and for $x_0 \in M$,

$$\sup_{t \geq 0} I_\lambda(t \phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0)) < c^*.$$

Now,

$$\begin{aligned} & \left(1 - \frac{\lambda}{\lambda_1}\right) \frac{\|\nabla[\phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0)]\|^2}{\int h(x) \phi_{\mathcal{R}}(x_0)^{2^*} T_{\varepsilon_\lambda}(x - x_0)^{2^*}} \\ & \leq s_\lambda(\phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0))^{2^*-2} \\ & \leq \left(1 + \frac{\lambda \|g\|_{N/2}}{S}\right) \frac{\|\nabla[\phi_{\mathcal{R}}(x_0) T_{\varepsilon_\lambda}(x - x_0)]\|^2}{\int h(x) \phi_{\mathcal{R}}(x_0)^{2^*} T_{\varepsilon_\lambda}(x - x_0)^{2^*}} \end{aligned}$$

Accordingly there exists $\tilde{\lambda} > 0$ such that for $0 < \lambda < \tilde{\lambda}$, $s_\lambda(\phi_{\mathcal{R}}(x_0)T_{e_\lambda}(x - x_0))$ is bounded from below and above by positive constants uniformly for all $x_0 \in M$. As $\lambda \rightarrow 0$,

$$\begin{aligned} \Phi_\lambda(x) &\rightarrow 0 \text{ in } L^2_{|g|}(\mathbf{R}^N) \quad \text{and a.e. in } \mathbf{R}^N, \\ (36) \quad |\nabla \Phi_\lambda(x)|^2 &\rightarrow S^{N/2} \|h\|_\infty^{(2-N)/2} \delta_x \quad \text{in } \mathcal{M}(\mathbf{R}^N \cup \{\infty\}), \\ (37) \quad |\Phi_\lambda(x)|^{2^*} &\rightarrow (S \|h\|_\infty^{-1})^{N/2} \delta_x \quad \text{in } \mathcal{M}(\mathbf{R}^N \cup \{\infty\}). \end{aligned}$$

Lemma 5.3. As $\lambda \rightarrow 0$,

- (i) $\alpha(\Phi_\lambda(x)) = x + o(1)$ uniformly for all $x \in B_Q$;
- (ii) $\Phi_\lambda(x) \in A_\lambda$ and $I_\lambda(\Phi_\lambda(x)) = c^* + o(1)$ for all $x \in M$;
- (iii) $\sup\{\text{dist}(\alpha(u), M) : u \in A_\lambda^*\} \rightarrow 0$.

Proof. Part (i) follows simply from (36). To see (ii), use (36) combined with (37). Since $\lambda \int g(x) \Phi_\lambda(x)^2 \rightarrow 0$ as $\lambda \rightarrow 0$ we have, (abbreviating $s_\lambda(\phi_{\mathcal{R}}(x_0)T_{e_\lambda}(x - x_0))$ as s_λ),

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \|\nabla \Phi_\lambda(x)\|^2 &= \lim_{\lambda \rightarrow 0} \int h(x) \Phi_\lambda(x)^{2^*} \\ &\Rightarrow \lim_{\lambda \rightarrow 0} s_\lambda^2 \|\nabla(\phi_{\mathcal{R}}(x_0)T_{e_\lambda}(x - x_0))\|^2 \\ &= \lim_{\lambda \rightarrow 0} s_\lambda^{2^*} \int h(x) \phi_{\mathcal{R}}(x_0)^{2^*} |T_{e_\lambda}(x - x_0)|^{2^*} \\ &\Rightarrow \lim_{\lambda \rightarrow 0} s_\lambda^2 S = \lim_{\lambda \rightarrow 0} s_\lambda^{2^*} \|h\|_\infty \\ &\Rightarrow \lim_{\lambda \rightarrow 0} s_\lambda = \left(\frac{S}{\|h\|_\infty} \right)^{(N-2)/2} \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} I_\lambda(\Phi_\lambda(x)) &= \lim_{\lambda \rightarrow 0} \frac{1}{N} \int |\nabla \Phi_\lambda(x)|^2 \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{N} \left(\frac{S}{\|h\|_\infty} \right)^{(N-2)/2} \int |\nabla \{\phi_{\mathcal{R}}(x_0)T_{e_\lambda}(x - x_0)\}|^2 \\ &= c^* \end{aligned}$$

To prove (iii), take a positive sequence $\lambda_m \rightarrow 0$. Let $u_m \in A_{\lambda_m}^*$. By Proposition 5.2, we have for a subsequence (again indexed with m), $r_m \rightarrow 0$ and $x_m \rightarrow x_0 \in M$ such that $|u_m - T_{r_m}(x - x_m)| \rightarrow 0$ in $D^{1,2}(\mathbf{R}^N)$. By continuity of $\alpha(\cdot)$,

$$\begin{aligned}
\alpha(u_m) &= \frac{\int \zeta_Q(x) |\nabla u_m|^2}{\|\nabla u_m\|^2} \\
&= \frac{\int \zeta_Q(x) |\nabla T_{r_m}(x - x_m)|^2}{\|\nabla T_{r_m}(x - x_m)\|^2} + o(1) = \zeta_Q(x_0) + o(1)
\end{aligned}$$

But since $x_0 \in M \subset B_Q$, $\zeta_Q(x_0) = x_0$. ■

Proof of Theorem 5.1. Set $\delta > 0$. For $\lambda > 0$ sufficiently small, we remark that for all $x \in M$, Lemma 4.2 shows that

$$\Phi_\lambda(x) \in A_\lambda^*$$

From Lemma 5.3, for $\lambda > 0$ sufficiently small,

$$\begin{aligned}
|\alpha(\Phi_\lambda(x)) - x| &< \delta \quad \text{for all } x \in B_Q; \\
\alpha(u) &\in M_\delta \quad \text{for all } u \in A_\lambda^*.
\end{aligned}$$

We claim that

The map $\alpha \circ \Phi_\lambda$ is homotopic to the inclusion $M \mapsto M_\delta$.

To see this, simply consider the homotopy

$$\mathcal{H}(t, x) = x + t(\alpha(\Phi_\lambda(x)) - x).$$

We have

$$\text{dist}(\mathcal{H}(t, x), M) \leq |\alpha(\Phi_\lambda(x)) - x| \leq \delta$$

for every $x \in M$ and $t \in [0, 1]$. That is, $\mathcal{H} : [0, 1] \times M \mapsto M_\delta$.

Recall that I_λ satisfies the $(PS)_c$ -condition for each $c < c^*$. By Lusternik-Schnirelmann theory, to prove Theorem 5.1 it suffices to show that

$$(38) \quad \text{cat}(A_\lambda^*) \geq \text{cat}_{M_\delta}(M).$$

Denote by Φ_λ the restriction of Φ_λ to M , thus

$$\Phi_\lambda : M \mapsto A_\lambda^*.$$

Suppose $A_1, \dots, A_\sigma$ is a closed covering of A_λ^* such that for every i , there exists a homotopy $H_i : [0, 1] \times A_i \mapsto A_\lambda^*$ with

$$\begin{aligned}
H_i(0, u) &= u \quad \text{for } u \in A_i \quad \text{and} \\
H_i(1, \cdot) &= \text{constant for } i = 1, \dots, \sigma.
\end{aligned}$$

Set $C_i = \Phi_\lambda^{-1}(A_i)$. Then C_i is closed in M for all i and $M \subset \bigcup_i C_i$. To show (38), we need only show that C_i are contractible in M_δ . To achieve this, use

the homotopy $h_i : [0, 1] \times C_i \mapsto M_\delta$ defined by

$$h_i(t, x) = \alpha H_i(t, \Phi_\lambda(x)). \quad \blacksquare$$

Remark 5.4. We anticipate that the condition (G3) is excessive. Given $\delta > 0$, let $\text{cat}_{M_\delta}(M) = \sigma$. We postulate that a sufficient requirement on $g(x)$ be that for some $\mathcal{R} > 0$, $g(x) > g_0 > 0$ for all $|x - x_i| < \mathcal{R}$ where $x_i \in \mathcal{C}_i$, $i = 1, \dots, \sigma$ are contractible sets and $\bigcup_{i=1}^\sigma \mathcal{C}_i$ is a closed covering of M .

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References

- [1] Bahri, A. and Coron, J. M., On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.*, **41** (1988), 253–294.
- [2] Bianchi, G., Chabrowski, J., Szulkin, A., On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent, *Nonlinear Anal.*, **25** (1995), 41–59.
- [3] Brézis, H. and Lieb, E., A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, **88** (1983), 486–490.
- [4] Brézis, H. and Nirenberg, L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437–477.
- [5] Brown, K. J., Cosner, C. and Fleckinger, J., Principal eigenvalues for problems with indefinite weight function on $\mathbf{R}^N$, *Proc. Amer. Math. Soc.*, **109** (1990), 147–155.
- [6] Brown, K. J. and Stavrakakis, N. M., Global bifurcation results for a semilinear elliptic equation on all of $\mathbf{R}^N$, *Duke Math. J.*, **85** (1996), 77–94.
- [7] Brown, K. J. and Stavrakakis, N. M., On the construction of super and subsolutions for elliptic equations on all of $\mathbf{R}^N$, *Nonlinear Anal.*, **32** (1998), 87–95.
- [8] Cao, D. and Chabrowski, J., Multiple solutions of nonhomogeneous elliptic equation with critical nonlinearity, *Differential Integral Equations*, **10** (1997), 797–814.
- [9] Cao, D. and Noussair, E. S., Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbf{R}^N$, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **13** (1995), 567–588.
- [10] Cao, D. and Zhong, X., Multiplicity of positive solutions for semilinear elliptic equations involving the critical Sobolev exponents, *Nonlinear Anal.*, **29(4)** (1997), 461–483.
- [11] Chabrowski, J., Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations*, **3** (1995), 493–512.
- [12] Congolani, S. and Lazzo, M., Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations, *Top. Methods Nonlinear Anal.*, **10** (1997), 1–13.
- [13] Ding, W. Y. and Ni, W. M., On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics, *Duke Math. J.*, **52** (1985), 485–506.
- [14] Drábek, P. and Huang, Y. X., Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbf{R}^N$ with critical Sobolev exponent, *J. Differential Equations*, **140** (1997), 106–132.

- [15] Ekeland, I., On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353.
- [16] Huang, Y., Multiple positive solutions of nonhomogeneous equations involving the p -Laplacian, preprint.
- [17] Lions, P. L., The concentration-compactness principle in the Calculus of Variations. The limit case, I, II, *Rev. Mat. Iberoamericana*, **1** (1985), 145–201 and **2** (1985), 45–121.
- [18] Lions, P. L., The concentration-compactness principle in the Calculus of Variations. The locally compact case, part I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.
- [19] Musina, R., Multiple positive solutions of a scalar field equation in $\mathbf{R}^n$, *Top. Methods Nonlinear Anal.*, **7** (1996), 171–185.
- [20] Rey, O., Concentration of solutions to elliptic equations with critical nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 201–218.
- [21] Serrin, J., Local behaviour of solutions of quasilinear equations, *Acta. Math.*, **111** (1964), 247–302.
- [22] Tarantello, G., On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. Henri Poincaré*, **9** (1992), 281–304.
- [23] Talenti, G., Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **101** (1976), 353–372.
- [24] Tertikas, A., Critical phenomena in linear elliptic problems, *J. Func. Anal.*, **154** (1998), 42–66.
- [25] Willem, M., *Minimax Theorems*, Progress in nonlinear differential equations and their applications **24**, Birkhäuser, Boston, 1996.
- [26] Xie, Z., Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbf{R}^N$, *Proc. Royal Soc. Edinburgh*, **128A** (1998), 1069–1097.

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