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Comparison Principle and Lipschitz Regularity for Viscosity Solutions of Some Classes of Nonlinear Partial Differential Equations

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1. Introduction

In [KK1] we investigated the class of fully nonlinear elliptic equations

$$(1) \quad F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \subset \mathbf{R}^n$$

which are proper and strictly elliptic but not uniformly elliptic and not uniformly proper. Under this main assumption we proved a strong maximum principle for semicontinuous viscosity sub- and supersolutions of (1). However, the comparison principle given in [KK1] is valid only when one of the semicontinuous viscosity sub- or supersolutions is a piecewise C^2 smooth function. In the present paper we investigate only the class of autonomous elliptic equations

$$(2) \quad F(u, Du, D^2u) = 0 \quad \text{in } \Omega$$

or degenerate parabolic equations

$$(3) \quad G(Du)u_t + F(u, Du, D^2u) = 0 \quad \text{in } Q = \Omega \times (0, T), \quad \text{with } G(p) \geq 0,$$

and show that under the same assumptions as in [KK1] an unrestricted comparison principle for semicontinuous viscosity sub- and supersolutions of (2) and (3) holds. In other words, the C^2 assumption can be dropped. By means of the Perron procedure this comparison principle guarantees existence of a unique continuous viscosity solution to the Dirichlet problems

$$(4) \quad F(u, Du, D^2u) = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega,$$

or

$$(5) \quad \begin{aligned} G(Du)u_t + F(u, Du, D^2u) &= 0 \quad \text{in } Q, \\ u &= g \quad \text{on } \Gamma := (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T]) \end{aligned}$$

provided (2) and (3) have a sub- and supersolution satisfying the boundary data. If additionally a continuous viscosity solution is Lipschitz continuous on the

boundary $\partial\Omega$ or on Γ , then the solution is Lipschitz continuous in the whole domain, i.e. regularity is inherited from the boundary to the interior of the domain.

One motivation to consider this problem is to better understand so-called interior blow-up phenomena as in [AF] and [G1]. In those papers the spatial gradient of classical solutions of (3) blows up for a finite time at some interior point and not on the boundary, while our results imply that blow up, once it occurs, must happen on the boundary. Since all conditions of Theorem 6 below are fulfilled in [AF] and [G1], except the properness (7) of operator (3), it is clear that the wrong monotonicity of (3) with respect to u is responsible for the interior gradient blow up in the counterexamples of Angenent, Fila and Giga. Of course, nonproperness of (3) is by itself not sufficient for interior gradient blow-up. It must be combined with some structural assumptions such as a supergrowth of the lower order terms, as in [AG], [G1], in order to cause this blow-up phenomenon.

Another reason to investigate the above problems is to relax the uniform monotonicity condition (6)

$$(6) \quad \gamma(r - s) \leq F(r, p, X) - F(s, p, X)$$

for every $r \geq s$ and for some positive constant $\gamma > 0$ to the nonuniform case, i.e., when $\gamma = 0$. Monotonicity condition (6) is a basic assumption in the general theory of viscosity solutions, for the current state of art see [CIL], Sections [3, 9, 12]. Only in [J1, Thm. 3.1], [BM, Thm. 2.6] and [Tr3, Thm. 1.1] is γ allowed to be zero, but at the expense of replacing the degenerate ellipticity condition (9) by a global uniform ellipticity condition and by an additional $C(\bar{\Omega}) \cap C^{0,1}(\Omega)$ regularity assumption of the sub- and supersolutions. As we show in Theorems 1 and 2 below, the comparison principle holds for semicontinuous sub- and supersolutions of nonuniformly elliptic and parabolic equations (2), (3) without the extra conditions in [J1], [Tr3]. In fact, in Theorems 5 and 6 we show that continuous viscosity solutions of (2), (3) are Lipschitz continuous if they have the same regularity on the boundary. This is a consequence of the comparison results in Theorems 1 and 2.

Let us mention a few words about the method of viscosity solutions of M. Crandall and P.-L. Lions, which is crucial in the present paper. We work in the class of continuous viscosity solutions. This allows us to focus our attention only on their Lipschitz regularity, disregarding any further (C^1 or C^2) smoothness. In this way we can separately investigate the Lipschitz and classical regularity of viscosity solutions. This is important in the fully nonlinear situation, because there are different reasons for the lack of Lipschitz resp. classical regularity.

Finally let us recall that interior and global Hölder estimates of viscosity

solutions of (1) as well as of their gradient was proved in [Tr1], [Tr2] under stronger assumptions than in the present paper, namely, global uniform ellipticity of (1) and global sublinear growth conditions for $F(x, r, p, X)$ with respect to the p -variables, see [Tr2, Thms. 2.1 and 2.3]. The intermediate case of Lipschitz a priori estimates for viscosity solutions of (1) was done in [B1], where the author extended the classical Bernstein method for C^2 -smooth solutions to continuous viscosity solutions. However, the assumptions in [B1, Thm. 1] hold for (2) under some stronger structure conditions than in our Theorem 5. They are for example satisfied when the operator F is uniformly proper, i.e. when (6) holds.

As for degenerate parabolic equations, the standard reference on viscosity solutions is [CIL, Ch. 8], in which $G \equiv 1$ and thus the important condition (6) holds without loss of generality. There are also treatments of special degenerate equations [CGG, GGIS, ES, IS, OS], and we shall explain them in more detail after Theorem 2. There is little overlap between their results and ours. Other investigations on degenerate parabolic equations (in which G depends on u and not on Du) were done in classes of weak solutions. They were motivated by nonlinear filtration problems for instance in [AL, K, S] or from medical applications [M], and they lead to elliptic-parabolic equations as well.

2. Main results and definitions

Let us give precise assumptions and recall some definitions. We call the operator $F(r, p, X)$ in (2) and (3) *proper* if

$$(7) \quad F(s, p, X) \leq F(r, p, X) \quad \text{for every } s \leq r, \\ s, r \in \mathbf{R} \quad \text{and} \quad (p, X) \in \mathbf{R}^n \times S^n.$$

Here S^n denotes the space of all symmetric $n \times n$ matrices.

Further on we suppose that (2) is *locally strictly elliptic*, i.e.

$$(8) \quad \lambda_K \operatorname{trace}(X - Y) \leq F(r, p, Y) - F(r, p, X)$$

for every $X \geq Y$ and every $|r|, |p|, \|X\|, \|Y\| \leq K$ and for some positive constant λ_K depending on K . Note that (8) defines a class of nondegenerate elliptic and parabolic equations, which includes the mean curvature operator. However it does not include the explicitly degenerate ones, such as the p -Laplace operator, for which the more general degenerate ellipticity assumption

$$(9) \quad F(r, p, X) \leq F(r, p, Y)$$

for every $X \geq Y$ with $X, Y \in S^n$ and every $(r, p) \in \mathbf{R} \times \mathbf{R}^n$, holds.

As for the regularity of F and G in (2) and (3), we assume throughout that $F(r, p, X)$ and $G(p)$ are continuous functions of all their variables, and locally Lipschitz continuous in the p variable, i.e.

$$(10) \quad \begin{aligned} |F(r, p, X) - F(r, q, X)| &\leq B_K |p - q| \\ |G(p) - G(q)| &\leq C_K |p - q| \end{aligned}$$

for every $|r|, |p|, |q|, \|X\| \leq K$ and for some positive constants B_K and C_K depending on K .

Definition 1. Suppose $F \in C(\mathbf{R} \times \mathbf{R}^n \times S^n)$ satisfies (7) and (9). An upper semicontinuous function $u \in USC(\Omega)$ is a viscosity subsolution of (2) if $F(u(x_0), p, X) \leq 0$ for every $x_0 \in \Omega$ and for every $(p, X) \in J_{\Omega}^{2,+}u(x_0)$. Here the second order superjet $J_{\Omega}^{2,+}u(x_0)$ is defined as the set of those $(p, X) \in \mathbf{R}^n \times S^n$ for which $u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \langle X(x - x_0), x - x_0 \rangle / 2 + o(|x - x_0|^2)$ as $x \rightarrow x_0$.

Analogously the definition of a lower semicontinuous viscosity supersolution $v \in LSC(\Omega)$ is given by means of the opposite inequality $F(v(y_0), q, Y) \geq 0$ for every $y_0 \in \Omega$ and for every $(q, Y) \in J_{\Omega}^{2,-}v(y_0)$, where $J_{\Omega}^{2,-}v(y_0) = J_{\Omega}^{2,+}(-v(y_0))$.

Finally, a continuous function in Ω is a viscosity solution of (2) if it is both a viscosity sub- and supersolution of (2).

The definition of viscosity sub- and supersolutions of parabolic equations (3) is the same as for elliptic equations (see for example section 8 in [CIL]) and we omit it.

Theorem 1 (Elliptic comparison principle). Suppose $\Omega \subset \mathbf{R}^n$ is a bounded domain and $F \in C(\mathbf{R} \times \mathbf{R}^n \times S^n)$ satisfies (7), (8) and (10). If $u \in USC(\bar{\Omega})$, $v \in LSC(\bar{\Omega})$ are bounded viscosity sub- and supersolutions of (2) then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in $\bar{\Omega}$.

Theorem 2 (Parabolic comparison principle). Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $F \in C(\mathbf{R} \times \mathbf{R}^n \times S^n)$, $G \in C(\mathbf{R}^n)$, $G(p) \geq 0$ satisfy (7), (8) and (10). If $u \in USC(\bar{Q})$, $v \in LSC(\bar{Q})$ are bounded viscosity sub- and supersolutions of (3) then $u(x, t) \leq v(x, t)$ on Γ implies $u \leq v$ in $\bar{Q}$.

All assumptions in Theorem 1 are sharp as the following counterexamples show. If the strict ellipticity condition (8) fails then it was proved in [IR] that the equation

$$|Du(x)| = |x|$$

in the unit ball with zero boundary data has infinitely many Lipschitz viscosity solutions which violate the comparison principle. [CH] gave a different counterexample of an equation which degenerates in an isolated point and has more

than one viscosity solution with the same boundary data, violating again the comparison principle.

If condition (7) does not hold, i.e. the operator is not proper, then simple counterexamples are provided by the eigenfunctions of every linear elliptic equation.

Finally, an example illustrating the necessity of the Lipschitz continuity in (10) was given in [BDD] by the equation

$$-\Delta u - |Du|^m = 0 \quad \text{in } B = \{|x| < R\} \subset \mathbf{R}^n, \quad u = 0 \quad \text{on } \partial B$$

for $m \in (0, 1)$. The above problem has one trivial and one nontrivial classical solution violating again the comparison principle.

Remark 1. As in [KK1] we can relax the Lipschitz regularity of F and G in the p variable at those points $P := (r, p, Y)$ where the modulus of ellipticity is better than the Lipschitz one in (8). More precisely, suppose that

$$(8') \quad \omega_{1,P}(\lambda_K \operatorname{trace}(X - Y)) \leq F(r, p, Y) - F(r, p, X)$$

for every fixed $P = (r, p, Y)$ and every $X \geq Y$ and $|r|, |p|, \|X\|, \|Y\| \leq K$ and for some positive constant λ_K depending on K . As a modulus of continuity the real function $\omega_{1,P}(s)$ satisfies the usual conditions $\omega_{1,P}(s) \in C[0, \infty)$, $\omega_{1,P}(s) > \omega_{1,P}(t)$ for $s > t > 0$ and $\omega_{1,P}(0) = 0$. Note that the special choice $\omega_{1,P}(s) = s$ turns (8') into the usual local strict ellipticity condition (8).

In this case we can relax (10) to the weaker condition (10')

$$(10') \quad \begin{aligned} |F(r, p, Y) - F(r, q, Y)| &\leq \omega_{2,P}(B_K |p - q|) \\ |G(p) - G(q)| &\leq \omega_{3,P}(C_K |p - q|), \end{aligned}$$

i.e. for every fixed $P = (r, p, Y)$ and every $|r|, |p|, |q|, \|Y\| \leq K$ there are some positive constants B_K and C_K depending on K , such that (10') holds. Here $\omega_{2,P}$ and $\omega_{3,P}$ are some moduli of continuity with $\omega_{1,P}$ dominating $\omega_{2,P}$ and $\omega_{3,P}$ in the sense that $\omega_{1,P}(s) \geq \max\{\omega_{2,P}(s), \omega_{3,P}(s)\}$ in $[0, s_0)$ for some positive s_0 .

If (8') and (10') hold for (r, p, Y) in a neighborhood of the nullset of F , then the results in Theorems 1 and 2 remain true. The proofs are virtually identical, except that now the previous Lipschitz moduli $\omega_i(s) = s$, $i = 1, 2, 3$, need to be adjusted in a straightforward way.

Typical examples illustrating the above case are $-(\Delta u)^{1/3} + f(u, Du) = 0$ or $-(\Delta u)^{1/3} - (\Delta u - 1)^{1/3} + f(u, Du) = 0$ with appropriate f .

Let us note that Theorem 2 does not cover the level set formulation of mean curvature flow

$$u_t - |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0,$$

see [GGIS], [ES] and the references therein, nor the (more general) singular parabolic equation

$$u_t - \operatorname{div}(|Du|^{q-2} Du) = 0, \quad \text{for } q \in (1, 2),$$

see [IS], [OS]. For these equations we have

$$G(p) = |p|^{2-q}, \quad F(p, X) = \operatorname{trace} \left\{ \left(I + (q-2) \frac{p \otimes p}{|p|^2} \right) X \right\},$$

and $F(p, X)$ is discontinuous at $p = 0$. In [GGIS] this difficulty was overcome using a continuous extension of $F(p, X)$ at $(0, 0)$, while in [IS], [OS] a new definition of viscosity solutions was suggested. As in [KK1] we could cover even the case when $F(r, \cdot, X)$ has discontinuities on some set N_0 with n -dim. Hausdorff measure zero, if we are willing to redefine F on N_0 in a suitable way. Since this would only complicate the exposition, we will not pursue this issue any further in the present paper.

However, for the following version of the mean curvature flow equation

$$(11) \quad u_t - \sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = H(u, Du),$$

see [H], [OU] and their references, Theorem 2 gives immediately a comparison principle for viscosity sub- and supersolutions.

Corollary 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, and let $H(r, p) \in C(\mathbf{R} \times \mathbf{R}^n)$ be locally Lipschitz in the p variables and nonincreasing with respect to r . If $u \in USC(\bar{Q})$, $v \in LSC(\bar{Q})$ are bounded viscosity sub- and supersolutions of (11) then $u(x, t) \leq v(x, t)$ on Γ implies $u \leq v$ in $\bar{Q}$.*

This result might already be known, but we could not find it in the literature. There are comparison principles for weak Lipschitz solutions in [G] or for weak solutions from the class $L^\infty((0, T); W^{1,1}(\Omega))$ in [H], [OU], but none in the class of semicontinuous viscosity solutions. The advantage of our comparison principle lies in its applicability to the Perron method, which leads to existence results for equations without divergence structure.

As a consequence of the general comparison principle formulated in Theorems 1 and 2, we get immediately by means of Perron's method the existence of a continuous viscosity solution to the Dirichlet problem (4) resp. (5), (see [12]). To this end we recall the definition of the upper and lower semicontinuous envelopes u^* and u_* of u .

$$u^*(x) := \lim_{r \rightarrow 0} \sup \{u(y) | y \in \Omega, |y - x| \leq r\}$$

$$u_*(x) := \lim_{r \rightarrow 0} \inf \{u(y) | y \in \Omega, |y - x| \leq r\}$$

Theorem 3 (Existence). *Suppose $\Omega \subset \mathbf{R}^n$ is a bounded domain, $g \in C(\partial\Omega)$ and $F \in C(\mathbf{R} \times \mathbf{R}^n \times S^n)$ satisfies (7), (8) and (10). If $v \in USC(\bar{\Omega})$, $w \in LSC(\bar{\Omega})$ are viscosity sub- and supersolutions of (2) such that $v_* = w^* = g$ on $\partial\Omega$, then there exists a continuous viscosity solution $u \in C(\bar{\Omega})$ of (2). Moreover $v \leq u \leq w$ in $\bar{\Omega}$ and in particular $u = g$ on $\partial\Omega$.*

Theorem 4. *Suppose that all assumptions of Theorem 2 hold and that $g \in C(Q \cup \Gamma)$. If $v \in USC(\bar{Q})$, $w \in LSC(\bar{Q})$ are viscosity sub- and supersolutions of (3) such that $v_*(x, t) = w^*(x, t)$ on Γ , then there exists a continuous viscosity solution $u \in C(\bar{Q})$ of (3). Moreover $v \leq u \leq w$ in $\bar{Q}$ and in particular $u = g$ on Γ .*

Finally, another consequence of the comparison results in Theorems 1 and 2 is the Lipschitz regularity of the continuous viscosity solution of (2) or (3), provided they are Lipschitz continuous on the boundary $\partial\Omega$ or Γ .

Theorem 5 (Lipschitz regularity). *Suppose $\Omega \subset \mathbf{R}^n$ is a bounded domain and $F \in C(\mathbf{R} \times \mathbf{R}^n \times S^n)$ satisfies (7), (8) and (10). If $u \in C(\bar{\Omega})$ is a viscosity solution of (2) which is Lipschitz continuous on $\partial\Omega$ with Lipschitz constant L , i.e.*

$$(12) \quad |u(x) - u(y)| \leq L|x - y|$$

for every $x \in \partial\Omega$, $y \in \bar{\Omega}$ (or equivalently for every $(x, y) \in \partial(\Omega \times \Omega)$), then u is Lipschitz continuous in $\bar{\Omega}$ with Lipschitz constant L .

Theorem 6. *Suppose that all assumptions of Theorem 2 hold and that $u(x, t) \in C(\bar{Q})$ is a viscosity solution of (3) such that $u(x, t)$ is Lipschitz continuous at Γ with Lipschitz constant L , i.e. $|u(x, t) - u(y, s)| \leq L\{|x - y|^2 + |t - s|^2\}^{1/2}$ for every $(x, t) \in \Gamma$ and $(y, s) \in \bar{Q}$.*

Then $u(x, t)$ is Lipschitz continuous in $\bar{Q} \times [0, T)$ with Lipschitz constant L .

Remark 2. An easily checkable sufficient condition for the Lipschitz continuity of u on the boundary $\partial\Omega$ is the existence of a Lipschitz sub- and supersolution $v, w \in C^{0,1}(\bar{\Omega})$ of (2) such that $v = w = g$ on $\partial\Omega$. Analogously, the same remark applies to the parabolic equation (3). However, in the present paper we do not discuss when the Lipschitz assumptions of Theorems 5 and 6 are satisfied. In fact, sometimes they are violated and there is a boundary gradient blow up, see [FL] or [KK0]. The reasons for such a phenomenon are quite different from those for interior gradient blow up. In general they originate from the geometry of the domain, from boundary data and the nonlinearity of the equation.

3. Proofs of the main results

The proof of Theorems 1 and 2 uses the same idea of doubling the number of independent variables as in the proof of the comparison principle for uniformly proper operators, i.e. satisfying (6) (see for example [CIL-Thm. 3.3]). However, to overcome the difficulties that (2), (3) do not satisfy the important condition (6) but the weaker one (7) we need some different barrier technique inspired by [KK1].

Proof of Theorem 1. Suppose that the claim in Theorem 1 is not true and $\sup\{[u(x) - v(x)] \mid x \in \bar{\Omega}\} = M > 0$. Without loss of generality we assume that for some positive r, R we have $\bar{\Omega} \subset \{r < |x| < R\}$. For every $\alpha > 1$ we consider the function

$$\phi(x, y) = u(x) - v(y) - \alpha|x - y|^2/2 + e^{-\beta|x|^2/2}$$

on $\bar{\Omega} \times \bar{\Omega}$. Here the positive constant $\beta = \beta(\alpha) > 0$ is chosen sufficiently large, so that the following inequalities hold

$$(13) \quad \begin{aligned} \sup_{\bar{\Omega}} [e^{-\beta|x|^2/2} + |De^{-\beta|x|^2/2}| + \|D^2e^{-\beta|x|^2/2}\|] &\leq 1, \\ \max\{\alpha r^2 \lambda_K, 2B_K R\} &< r^2 \lambda_K \beta, \\ n + 4n^4 \beta^3 e^{-\beta r^2/2} &\leq \beta r^2/2, \end{aligned}$$

and the constants λ_K, B_K are defined in (8), (10) for $K = (81n^4 + 2R)\alpha + \sup|u| + \sup|v|$. It is clear from the upper semicontinuity of $\phi(x, y)$ that

$$\sup_{\bar{\Omega} \times \bar{\Omega}} \phi(x, y) = \phi(x_\alpha, y_\alpha) = M_\alpha \geq M > 0 \quad \text{for some } x_\alpha, y_\alpha \in \bar{\Omega}.$$

Moreover, as in [CIL, Lemma 3.1], it is easy to check that

$$(14) \quad \begin{aligned} \text{i)} \quad &\alpha|x_\alpha - y_\alpha|^2 \rightarrow 0 \\ \text{ii)} \quad &x_\alpha, y_\alpha \rightarrow z \in \bar{\Omega}, \quad M_\alpha \rightarrow u(z) - v(z) = M \end{aligned} \quad \text{as } \alpha \rightarrow \infty.$$

Since $u(x) - v(x) \leq 0$ for $x \in \partial\Omega$ it follows that z is an interior point of Ω , $z \in \Omega$, so that x_α, y_α are also interior points of Ω , $x_\alpha, y_\alpha \in \Omega$, for all sufficiently large values of $\alpha \geq \alpha_0$. Moreover, (14) ii) guarantees that

$$u(x_\alpha) - v(y_\alpha) \geq 0 \quad \text{for } \alpha \geq \alpha_0.$$

We fix now $\alpha \geq \alpha_0$. From Theorem 3.2 in [CIL] for the function $\phi(x, y)$ and $\varepsilon = 1/\alpha$ there exist matrices X, Y such that

$$(\alpha(x_\alpha - y_\alpha) + \beta x_\alpha e^{-\beta|x_\alpha|^2/2}, X) \in \bar{J}_\Omega^{2,+} u(x_\alpha), \quad (\alpha(x_\alpha - y_\alpha), Y) \in \bar{J}_\Omega^{2,-} v(y_\alpha),$$

where

$$(15) \quad -(\alpha + \|A\|)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + A^2/\alpha$$

and

$$A = \begin{pmatrix} \alpha I + B & -\alpha I \\ -\alpha I & \alpha I \end{pmatrix}, \quad B = \beta e^{-\beta|x_\alpha|^2/2} \{I - \beta x_\alpha \otimes x_\alpha\}.$$

Since

$$A^2/\alpha = 2 \begin{pmatrix} \alpha I + B + B^2/2\alpha & -\alpha I - B/2 \\ -\alpha I - B/2 & \alpha I \end{pmatrix}$$

from (13), (15), multiplying the left side of (15) from the right and from the left with the vectors $(e_i, e_i), (e_i, e_i)^T$, $e_i = (0, \dots, 1, \dots, 0)$ with 1 at the i th place and summing i from 1 to n , we get the two estimates

$$(16) \quad |u(x_\alpha)|, |v(y_\alpha)|, |\alpha(x_\alpha - y_\alpha) + \beta x_\alpha e^{-\beta|x_\alpha|^2/2}|, |\alpha(x_\alpha - y_\alpha)|, \|X\|, \|Y\| \\ \leq (81n^4 + 2R)\alpha + \sup|u| + \sup|v|$$

and

$$(17) \quad \text{trace}(X - Y) \leq \text{trace}(B + B^2/\alpha) \\ \leq \beta e^{-\beta|x_\alpha|^2/2} (n - \beta|x_\alpha|^2 + 4n^4\beta^3 e^{-\beta r^2/2}) \\ \leq -\beta^2(r^2/2) e^{-\beta|x_\alpha|^2/2}.$$

Now, Definition 1 and (7), (8), (10), (16), (17) give us the following impossible chain of inequalities for $K = (81n^4 + 2R)\alpha + \sup|u| + \sup|v|$

$$0 \geq F(u(x_\alpha), \alpha(x_\alpha - y_\alpha) + \beta x_\alpha e^{-\beta|x_\alpha|^2/2}, X) \\ \geq F(v(y_\alpha), \alpha(x_\alpha - y_\alpha), X) - B_K \beta |x_\alpha| e^{-\beta|x_\alpha|^2/2} \\ \geq F(v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) + \lambda_K \beta^2 r^2 e^{-\beta|x_\alpha|^2/2} / 2 \\ - B_K \beta R e^{-\beta|x_\alpha|^2/2} \\ > 0$$

which proves Theorem 1.

Proof of Theorem 2. Since (5) is degenerate and G can vanish, it is not possible to apply [CIL, Thm. 8.2]. In fact u_t can not be recovered from (5).

Therefore we follow the idea in [OS, Thm. 3.1] for an analogue of [CIL, Thm. 8.2] in the degenerate setting. Suppose that $\sup_{\bar{Q}}[u(x, t) - v(x, t)] = M > 0$.

Without loss of generality we can assume that $\bar{Q} \subset \{x \in \mathbf{R}^n \mid 0 < r < |x| < R\}$. For every fixed $\alpha > 1$ and sufficiently small $\gamma > 0$ we consider in $\bar{Q}$ the function

$$\varphi(x, y, t, s) = u(x, t) - v(y, s) - \frac{\alpha}{2}(|x - y|^2 + (t - s)^2) + e^{-\beta|x|^2/2} - \frac{\gamma}{T-t} - \frac{\gamma}{T-s},$$

where the choice of β is almost the same as in (13). The additional terms $\gamma/(T-t)$ and $\gamma/(T-s)$ eliminate a possible maximum of $\varphi(x, y, t, s)$ at the top of $\bar{Q}$. The rest of the proof is almost the same as the proof of Theorem 1 above. It uses some auxiliary statements from the proof of Theorem 3.1 in [OS] instead of [CIL, Thm. 8.2], and we will only sketch it. If ϕ attains its positive maximum M_α at $(x_\alpha, y_\alpha, t_\alpha, s_\alpha)$, we can show as in Propositions 3.2, 3.3 and 3.4 of [OS] that $|x_\alpha - y_\alpha| \rightarrow 0$, $|t_\alpha - s_\alpha| \rightarrow 0$ as $\alpha \rightarrow \infty$ uniformly in $\gamma \in (0, \gamma_0)$ and $\beta \geq \beta_0$, where $\gamma_0 > 0$ and $\beta_0 > 0$. Moreover, (x_α, t_α) and (y_α, s_α) are interior points of $\bar{Q}$ for every $\alpha \geq \alpha_0$ uniformly in $\gamma \in (0, \gamma_0)$. Applying Theorem 3.2 in [CIL] and Corollary 3.6 in [OS] to the function $\phi(x, y, s, t)$ and setting $\varepsilon = 1/\alpha$ we conclude that there exist matrices $X, Y \in S^n$ such that $(\Phi_t(\xi_\alpha, \eta_\alpha), \Phi_x(\xi_\alpha, \eta_\alpha), X) \in \mathcal{P}^{2,+}u(\xi_\alpha)$, $(\Phi_s(\xi_\alpha, \eta_\alpha), \Phi_y(\xi_\alpha, \eta_\alpha), Y) \in \mathcal{P}^{2,-}v(\eta_\alpha)$ where $\Phi(x, y, t, s) = \frac{\alpha}{2}(|x - y|^2 + (t - s)^2) - e^{-\beta|x|^2/2} + \frac{\gamma}{T-t} + \frac{\gamma}{T-s}$ and $\xi_\alpha = (x_\alpha, t_\alpha)$, $\eta_\alpha = (y_\alpha, s_\alpha)$. Moreover we have

$$\begin{aligned} & |u(\xi_\alpha)| + |v(\eta_\alpha)| + \alpha|t_\alpha - s_\alpha| + \frac{\gamma}{(T-t_\alpha)^2} + \frac{\gamma}{(T-s_\alpha)^2} \\ & + \beta|x_\alpha|e^{-\beta|x_\alpha|^2/2} + \|X\| + \|Y\| \leq d(\alpha, \gamma, r, R, T) \end{aligned}$$

with a positive constant d independent of $\beta \geq \beta_0$, and the following inequality holds.

$$\begin{pmatrix} X_1 & 0 \\ 0 & -Y_1 \end{pmatrix} \leq A + A^2/\alpha.$$

Here $X_1 = \begin{pmatrix} a & \ell^T \\ \ell & X \end{pmatrix}$, $Y_1 = \begin{pmatrix} b & k^T \\ k & Y \end{pmatrix}$ for some $a, b \in \mathbf{R}$, $\ell, k \in \mathbf{R}^n$ and $A = \begin{pmatrix} \Phi_{\xi\xi}(\xi_\alpha, \eta_\alpha) & \Phi_{\xi\eta}(\xi_\alpha, \eta_\alpha) \\ \Phi_{\eta\xi}(\xi_\alpha, \eta_\alpha) & \Phi_{\eta\eta}(\xi_\alpha, \eta_\alpha) \end{pmatrix}$ where $\xi = (x, t)$, $\eta = (y, s)$. From the above inequality, as in Theorem 1, we get the estimate

$$\text{trace}(X - Y) \leq -\beta^2(r^2/2)e^{-\beta|x_\alpha|^2/2}.$$

Now from the definition of viscosity sub- and supersolutions and the assumptions of Theorem 2 we have for fixed α, γ and large β and $K = d(\alpha, \gamma, r, R, T)$ the

following impossible chain of inequalities

$$\begin{aligned}
0 &\geq G(\alpha(x_\alpha - y_\alpha) + \beta x_\alpha e^{-\beta|x_\alpha|^2/2})[\alpha(t_\alpha - s_\alpha) + \gamma/(T - t_\alpha)^2] \\
&\quad - G(\alpha(x_\alpha - y_\alpha))[\alpha(t_\alpha - s_\alpha) - \gamma/(T - s_\alpha)^2] \\
&\quad + F(u(x_\alpha, t_\alpha), \alpha(x_\alpha - y_\alpha) + \beta x_\alpha e^{-\beta|x_\alpha|^2/2}, X) \\
&\quad - F(v(y_\alpha, s_\alpha), \alpha(x_\alpha - y_\alpha), Y) \\
&\geq \lambda_K \beta^2 r^2 e^{-\beta|x_\alpha|^2/2} - B_K \beta R e^{-\beta|x_\alpha|^2/2} - d C_K \beta R e^{-\beta|x_\alpha|^2/2} \\
&\geq \lambda_K \beta^2 r^2 e^{-\beta|x_\alpha|^2/2} - (d+1) \beta R (B_K + C_K) e^{-\beta|x_\alpha|^2/2} \\
&> 0
\end{aligned}$$

which proves Theorem 2.

As for the *proof of Theorems 3 and 4*, it follows immediately from the Perron's procedure because of the unrestricted comparison principle, i.e. Theorems 1 and 2, for semicontinuous viscosity sub- and supersolutions of (2) and (3) (see for example [I2] or [CIL, Sec. 4]).

Proof of Theorem 5. Since (2) is translation invariant, if $u(x)$ is a viscosity solution of (2) in Ω , then for $h \in \mathbf{R}^n$ the function $v(x) = u(x - h)$ is a viscosity solution of (2) in $\Omega_h := \{x \in \mathbf{R}^n \mid x - h \in \Omega\}$. Now from (12) it follows that $|u(x) - v(x)| \leq L|h|$ on $\partial(\Omega \cap \Omega_h)$. In particular $u(x) \leq v(x) + L|h|$ on $\partial(\Omega \cap \Omega_h)$. Thus from the comparison principle, Theorem 1, $u(x) \leq v(x) + L|h|$ in $\Omega \cap \Omega_h$. By the same argument $v(x) - L|h| \leq u(x)$ in $\Omega \cap \Omega_h$, so that $|u(x) - v(x)| \leq L|h|$ in $\Omega \cap \Omega_h$. Note that we have used the fact that $v \pm L|h|$ is a super- resp. sub-solution of (2). Since h was arbitrary, Theorem 5 follows.

The *proof of Theorem 6* is the same as the proof of Theorem 5, and we omit it.

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