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Shin, Jong Son

Naito, Toshiki

Van Minh, Nguyen

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On Stability of Solutions in Linear Autonomous Functional Differential Equations

By

Jong Son SHIN¹, Toshiki NAITO² and Nguyen VAN MINH³

(¹Korea University and ^{2,3}The University of Electro-Communications, Japan)

1. Introduction

Let $\mathbf{R}$ denote the real line and E an infinite dimensional Banach space with norm $|\cdot|$. If $x : (-\infty, b) \rightarrow E$, then the function $x_t : (-\infty, 0] \rightarrow E, t \in (-\infty, b)$, is defined by $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$. We deal with the linear autonomous functional differential equation with infinite delay in the Banach space E :

$$(AL) \quad \frac{dx(t)}{dt} = Ax(t) + L(x_t).$$

The phase space $\mathcal{B}$ and some hypotheses in Eq. (AL) are demonstrated in Section 3.

The aim of this paper is to investigate stability properties of the zero solution for Eq. (AL). Recently, spectral properties of the generator of the solution semigroup to Eq. (AL) were studied in [5] (see [6] for the complete proof of these). Moreover, using these properties, authors analyzed the stability of the zero solution of some integro-differential equation. Note that the manner employed in [5] is based upon a concrete equation. Also, the stability properties of the zero solution for a special case of Eq. (AL) with finite delay was discussed in [10]. Therefore, in this paper we present criteria on stability properties of the zero solution to Eq. (AL) which is given in the general form.

The present paper is organized with the following three parts. First of all, we shall investigate stability properties in the general setting of C_0 -semigroups in Section 2, because the solution operators of Eq. (AL) form a C_0 -semigroup on $\mathcal{B}$. In general, there are two concepts for a general C_0 -semigroup $T(t)$ on E :

- (1) $T(t)$ is asymptotically stable if $|T(t)x| \rightarrow 0$ as $t \rightarrow \infty$ for every $x \in E$;
- (2) $T(t)$ is uniformly asymptotically stable if $\|T(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

If E is of finite dimension, these two concepts are equivalent. In general, if E

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is of infinite dimension, these two concepts are different from each other (see [3, Examples 1.2, pp. 100–101] or [7, p. 5]). Hence, when E is of infinite dimension, we shall give a sufficient condition to ensure the equivalence of these two concepts. It is related to the condition that the growth bound is really greater than the essential growth bound of the C_0 -semigroup. Moreover, under the above same assumption, we give a necessary and sufficient condition for the stability of a C_0 -semigroup. Next, we will apply the results obtained in Section 2 to find stability criteria of the zero solution for Eq. (AL) with uniform fading memory space in Section 3. In particular, using the representation of the resolvent of the generator of the solution semigroup to Eq. (AL) and properties of the characteristic operator we find a criterion on the stability of the zero solution for Eq. (AL). Our approach follows closely that in [10]. Moreover, we give a sufficient condition for the asynchronous exponential growth of the solution semigroup to Eq. (AL) (see [7, 12]). Finally, the results obtained in Section 3 are illustrated in an integro-differential equation in Section 4.

2. Stability properties of a C_0 -semigroup

Let $T(t)$ be a C_0 -semigroup on E and A its infinitesimal generator throughout this section. We consider an asymptotic behavior of $T(t)$, which is concerned with informations about its essential spectrum. To do so, we will prepare the following lemma, which is easily shown by induction.

Lemma 2.1. *If x belongs to the null space $N((A - \lambda I)^m)$, I being the identity, then*

$$T(t)x = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - \lambda I)^k x.$$

The growth bound $\omega_s(T)$, and the essential growth bound $\omega_e(T)$ of $T(t)$ are defined by

$$\begin{aligned} \omega_s(T) &:= \lim_{t \rightarrow \infty} \frac{\log \|T(t)\|}{t} = \inf_{t > 0} \frac{\log \|T(t)\|}{t}, \\ \omega_e(T) &:= \lim_{t \rightarrow \infty} \frac{\log \alpha(T(t))}{t} = \inf_{t > 0} \frac{\log \alpha(T(t))}{t}, \end{aligned}$$

where $\|T(t)\|$ stands for the operator norm of $T(t)$ and $\alpha(T(t))$ is the measure of non-compactness of $T(t)$ which is described by the Kuratowskii measure of noncompactness of bounded sets in E (cf. [11]). Moreover, we define a quantity $\hat{\omega}_e(T)$ as

$$\hat{\omega}_e(T) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \overline{\lim}_{s \rightarrow t-0} \alpha(T(s)).$$

Then we have that $\omega_e(T) \leq \hat{\omega}_e(T)$. In fact, if $0 < \varepsilon < t$, then

$$\alpha(T(t)) \leq \alpha(T(t-\varepsilon))\alpha(T(\varepsilon)) \leq \sup_{t-\varepsilon \leq s < t} \alpha(T(s))\|T(\varepsilon)\|.$$

Since $\|T(t)\|$ is locally bounded for $t \geq 0$, we have the inequality $\omega_e(T) \leq \hat{\omega}_e(T)$.

Furthermore, the spectral radius $r_s(T(t))$ and the essential spectral radius $r_e(T(t))$ are given as $r_s(T(t)) = \exp(t\omega_s(T))$ and $r_e(T(t)) = \exp(t\omega_e(T))$. Let $\rho(A)$ be the resolvent set of A , $\sigma(A)$ the spectrum of A , $P_\sigma(A)$ the point spectrum of A , $E_\sigma(A)$ the essential spectrum of A , and set $N_\sigma(A) := \sigma(A) \setminus E_\sigma(A)$. The points in $N_\sigma(A)$ are called normal eigenvalues of A . Recall that, by definition, λ is a normal eigenvalue of A if it is an isolated point of $\sigma(A)$, the range $R(A - \lambda I)$ is closed and the generalized eigenspace $M_\lambda(A)$ is of finite dimension.

Then the following relations hold (cf. [11]):

$$(1) \quad \sup\{\Re\lambda : \lambda \in \sigma(A)\} \leq \omega_s(T), \quad \sup\{\Re\lambda : \lambda \in E_\sigma(A)\} \leq \omega_e(T)$$

and

$$\omega_s(T) = \max\{\omega_e(T), s(A)\},$$

where

$$s(A) := \sup\{\Re\lambda : \lambda \in N_\sigma(A)\}.$$

Put $N_0(A) = \{\lambda \in N_\sigma(A) : \Re\lambda = 0\}$ and $R(\mu, A) = (\mu I - A)^{-1}$.

Proposition 2.2. *The following results hold true.*

1) Assume that $\omega_e(T) < \omega_s(T)$. Then there exist a $\lambda \in C$ and an $x_0 \in N(A - \lambda I)$, $x_0 \neq 0$ such that $\Re\lambda = \omega_s(T)$ and

$$|T(t)x_0| = e^{\omega_s(T)t}|x_0|, \quad t \geq 0.$$

2) Assume that $\omega_e(T) < \omega_s(T) = 0$. Then the following statements hold.

(i) If $R(\mu, A)$ has a pole of order 1 at $\mu = \lambda$ for all $\lambda \in N_0(A)$, then there is a constant $H > 0$ such that

$$|T(t)x| \leq H|x|, \quad t \geq 0, \quad x \in E.$$

(ii) If there is a $\lambda_0 \in N_0(A)$ such that $R(\mu, A)$ has a pole of order $m \geq 2$ at $\mu = \lambda_0$, then there exists an $x_0 \in E$ such that

$$|T(t)x_0| \rightarrow \infty \quad \text{as } t \rightarrow +\infty.$$

Proof. 1) Since $\omega_e(T) < \omega_s(T)$, it follows from Theorem 2 in [5] that there is a $\lambda_0 \in N_\sigma(A)$ such that $\Re\lambda_0 = \omega_s(T)$. Take an $x_0 \in N(A - \lambda_0 I)$, $x_0 \neq 0$. By Lemma 2.1 we have $T(t)x_0 = e^{\lambda_0 t}x_0$, and hence $|T(t)x_0| = e^{\omega_s(T)t}|x_0|$.

2) From Theorem 2 in [5] we see that $N_0(A)$ is finite; denote it by $\{\lambda_1, \lambda_2, \dots, \lambda_q\}$. Let m_j be the index of λ_j , and set $M_j = N((A - \lambda_j I)^{m_j})$, $j = 1, 2, \dots, q$. Then $E = M \oplus M_0$, where $M = M_1 \oplus M_2 \oplus \dots \oplus M_q$, and $M_0 = \bigcap_{j=1}^q R((A - \lambda_j I)^{m_j})$. Let P_j be the projection in E such that $P_j E = M_j$, $(I - P_j)E = R((A - \lambda_j I)^{m_j})$; and set $P = P_1 + P_2 + \dots + P_q$, $P_0 = I - P$.

(i) By Theorem 3 in [5] we have

$$(2) \quad |T(t)P_0 x| \leq K e^{-\varepsilon t} \|P_0\| |x|$$

for some $K \geq 1$ and $\varepsilon > 0$. Since $P_j x \in M_j$ and $m_j = 1, 1 \leq j \leq q$, it follows from Lemma 2.1 that $T(t)P_j x = e^{\lambda_j t} P_j x$. Noting that $\Re \lambda_j = 0$, we see that

$$(3) \quad |T(t)P_j x| \leq |P_j x| \leq \|P_j\| |x|, \quad t \geq 0, \quad x \in E.$$

Combining (2) and (3) we obtain the following inequality

$$\begin{aligned} |T(t)x| &\leq |T(t)Px| + |T(t)P_0 x| \\ &\leq \sum_{j=1}^q |T(t)P_j x| + |T(t)P_0 x| \\ &\leq H|x|, \end{aligned}$$

where $H = \sum_{j=1}^q \|P_j\| + K\|P_0\|$.

(ii) Let j be an index such that $R(\mu, A)$ has a pole of order $m \geq 2$ at $\mu = \lambda_j$. From the definition of M_j we see that there is a $x_0 \in M_j$ such that $(A - \lambda_j I)^{m-1} x_0 \neq 0$. Since $m \geq 2$, it follows from Lemma 2.1 that

$$\begin{aligned} |T(t)x_0| &= \left| \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - \lambda_j I)^k x_0 \right| \\ &= t^{m-1} \left| \sum_{k=0}^{m-1} \frac{1}{k! t^{m-1-k}} (A - \lambda_j I)^k x_0 \right| \\ &\rightarrow \infty (t \rightarrow \infty). \end{aligned}$$

Therefore the proof is complete.

We note that, for $x_0 \in D(A)$, $T(t)x_0$ is a solution of Eq. (AL) with $L = 0$; that is,

$$(A) \quad \frac{dx(t)}{dt} = Ax(t).$$

Definition.

1) $T(t)$ (or the zero solution of Eq. (A)) is said to be stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x| < \delta, x \in E$, then $|T(t)x| < \varepsilon$ for all $t \in [0, \infty)$.

2) $T(t)$ (or the zero solution of Eq. (A)) is said to be asymptotically stable if for any $x \in E$, $|T(t)x| \rightarrow 0$ as $t \rightarrow +\infty$.

3) $T(t)$ (or the zero solution of Eq. (A)) is said to be uniformly asymptotically stable if $\|T(t)\| \rightarrow 0$ as $t \rightarrow +\infty$.

4) $T(t)$ (or the zero solution of Eq. (A)) is said to be exponentially asymptotically stable if there are an $\omega_0 > 0$ and a $K \geq 1$ such that

$$|T(t)x| \leq Ke^{-\omega_0 t}|x|, \quad t \geq 0, \quad x \in E.$$

Remark. The following statements are equivalent.

(i) $\omega_s(T) < 0$.

(ii) $T(t)$ is uniformly asymptotically stable.

(iii) $T(t)$ is exponentially asymptotically stable.

The inequality $r_s(T(t)) = e^{\omega_s(T)t} \leq \|T(t)\|$ holds for $t \geq 0$ from the definition of the growth bound $\omega_s(T)$: (ii) implies (i). The other implications are trivial.

Theorem 2.3. Assume that $\omega_e(T) < \omega_s(T)$. Then the following statements are equivalent.

1) $T(t)$ is asymptotically stable.

2) $T(t)$ is exponentially asymptotically stable.

3) For any $\lambda \in P_\sigma(A)$, $\Re \lambda$ is negative.

Proof. We will derive the assertion 3) from the assertion 1), because the other parts are obvious. To do so, it suffices to show that the inequality $\omega_s(T) < 0$ holds. Assume, for a contradiction, that $\omega_s(T) \geq 0$. Since $\omega_e(T) < \omega_s(T)$, from the assertion 1) in Proposition 2.2 it follows that there exist a $\lambda \in C$ and an $x_0 \in N(A - \lambda I)$, $x_0 \neq 0$, such that $\Re \lambda = \omega_s(T)$ and

$$|T(t)x_0| = \begin{cases} e^{\omega_s(T)t}|x_0|, & \text{if } \omega_s(T) > 0 \\ |x_0|, & \text{if } \omega_s(T) = 0. \end{cases}$$

This contradicts the assertion 1). Hence the proof is complete.

Remark. If $\omega_e(T) = \omega_s(T)$, then, in general, the two concepts of asymptotic stability and exponential asymptotic stability for C_0 -semigroup $T(t)$ are different. Of course, if $T(t)$ is a C_0 -compact semigroup, these two concepts are equivalent, because $\omega_e(T) = -\infty$.

Theorem 2.4. Assume that $\omega_e(T) < \omega_s(T) = 0$. Then the following statements are equivalent.

1) $T(t)$ is stable.

2) Each $\lambda \in N_0(A)$ is a simple pole of $R(\mu, A)$.

Proof. From the assumption we see that $N_0(A)$ consists of finite normal eigenvalues of A . Hence if $\lambda_0 \in N_0(A)$, then λ_0 is a pole of $R(\mu, A)$ ([11,

Proposition 4.11]). We will prove the assertion 2) from the assertion 1). Now, assume that the assertion 2) does not hold. Then there is $\lambda_0 \in N_0(A)$ such that $R(\mu, A)$ has a pole of order $m \geq 2$ at $\mu = \lambda_0$. From the assertion 2) (ii) in Proposition 2.2 there is an $x_0 \in E$ such that $|T(t)x_0| \rightarrow \infty$ as $t \rightarrow \infty$, which contradicts the assumption. The remainder are easily shown by Proposition 2.2.

We note that $T(t)$ is stable if and only if there is an $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \in [0, \infty)$.

3. Stability of the zero solution in Eq. (AL)

In this section we will apply the results obtained in the previous section to Eq. (AL). Let $\mathcal{B}$ be a Banach space consisting of some functions mapping $(-\infty, 0]$ into E ; the norm in $\mathcal{B}$ is denoted by $|\cdot|$. For a complex number λ and for an $x \in E$ we define a function $\varepsilon_\lambda \otimes x : (-\infty, 0] \rightarrow E$ by $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x$, $\theta \in (-\infty, 0]$. We assume that the phase space $\mathcal{B}$ satisfies the following axioms (B-1,2,3).

(B-1) If a function $x : (-\infty, \sigma + a) \rightarrow E$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then

(i) $x_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + a)$ and x_t is continuous in $t \in [\sigma, \sigma + a)$;

(ii) $H^{-1}|x(t)| \leq |x_t| \leq K(t - \sigma) \sup\{|x(s)| : \sigma \leq s \leq t\} + M(t - \sigma)|x_\sigma|$ for all $t \in [\sigma, \sigma + a)$, where $H > 0$ is constant, $K : [0, \infty) \rightarrow [0, \infty)$ is continuous, $M : [0, \infty) \rightarrow [0, \infty)$ is locally bounded and they are independent of x .

(B-2) If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$, and if the sequence $\{\phi^n(\theta)\}$ converges to a function $\phi(\theta)$ uniformly on every compact interval of $(-\infty, 0]$, then ϕ lies in $\mathcal{B}$ and $\lim_{n \rightarrow \infty} |\phi^n - \phi| = 0$.

(B-3) There exists a constant γ_0 such that $\varepsilon_\lambda \otimes x \in \mathcal{B}$ for $\Re \lambda > \gamma_0$ and $x \in E$, and that

$$\|\varepsilon_\lambda\| := \sup\{|\varepsilon_\lambda \otimes x| : x \in E, |x| \leq 1\}$$

is finite for each λ with $\Re \lambda > \gamma_0$, and bounded for $\Re \lambda > \gamma_1$ for some $\gamma_1 \geq \gamma_0$.

We introduce the trivial C_0 -semigroup $S(t)$ on $\mathcal{B}$ defined as

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & t + \theta \geq 0 \\ \phi(t + \theta) & t + \theta < 0, \end{cases}$$

and denote by B its infinitesimal generator. Let $S_0(t)$ be the restriction of $S(t)$ to the subspace $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$. We call the constant γ_0 in (B-3) the abscissa of the exponent of the space $\mathcal{B}$. Then $\gamma_0 \leq \omega_s(S_0) = \omega_e(S_0)$ (cf. [5], [6]). Let C_{00} be the set of continuous functions mapping $(-\infty, 0]$ into E , with compact support. The following axiom for $\mathcal{B}$ is used later on.

(C) If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ uniformly on every compact set of $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $\lim_{n \rightarrow \infty} |\phi^n - \phi| = 0$.

If the phase space $\mathcal{B}$ satisfies the above axiom (C) and E is an infinite dimensional space, then $\gamma_0 \leq \omega_e(S_0) = \omega_s(S_0) \leq \omega_e(S) = \omega_s(S)$ and $0 \leq \omega_s(S)$ (cf. [9]).

According to [2], we have the following facts for the phase space $\mathcal{B}$ with an additional axiom (C):

(1) $\mathcal{B}$ is a fading memory space if and only if $S_0(t)$ is asymptotically stable.

(2) $\mathcal{B}$ is a uniform fading memory space if and only if $S_0(t)$ is exponentially asymptotically stable.

Let $\mathcal{B} = E \times \mathcal{L}_\gamma$ be the quotient space with respect to the seminorm defined by

$$\|\phi\| = |\phi(0)| + \int_{-\infty}^0 e^{-\gamma\theta} |\phi(\theta)| d\theta$$

for a measurable function $\phi : (-\infty, 0] \rightarrow E$ such that $e^{-\gamma\theta} |\phi(\theta)|$ is integrable on $(-\infty, 0]$. Then this space is a Banach space satisfying (B-1,2,3), γ is the abscissa of the exponent, and $\omega_e(S_0) = \omega_s(S_0) = \gamma$. If $\gamma < 0$, then $\mathcal{B}$ is a uniform fading memory space (cf. [2]).

We assume that Eq. (AL) always satisfies the following hypotheses throughout this section:

(H-1) $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a C_0 -compact semigroup $T(t), t \geq 0$, on E ;

(H-2) $L : \mathcal{B} \rightarrow E$ is a bounded linear operator.

From the hypotheses (H-1) and (H-2) we see that a mild solution of Eq. (AL) through $(0, \phi), \phi \in \mathcal{B}$, exists uniquely on $[0, \infty)$ (cf. [6], [8]). Denote by $u(t)$ or $u(t, \phi)$ this mild solution. It has the following properties:

(i) $u_0 = \phi \in \mathcal{B}$ and u is continuous on $[0, \infty)$.

$$(ii) \quad u(t) = T(t)\phi(0) + \int_0^t T(t-s)L(u_s)ds, \quad t \geq 0.$$

Define the solution operator $U_L(t)$ on $\mathcal{B}$ by

$$(U_L(t)\phi)(\theta) = u(t+\theta, \phi), \quad \theta \in (-\infty, 0].$$

By using the axioms of the phase space, we see that $U_L(t)$ is a C_0 -semigroup on $\mathcal{B}$. In the particular case that $L \equiv 0, U_0(t)$ is given by

$$(U_0(t)\phi)(\theta) = \begin{cases} T(t+\theta)\phi(0) & -t < \theta \leq 0 \\ \phi(t+\theta) & \theta \leq -t. \end{cases}$$

Denote by A_L and A_0 the infinitesimal generators of $U_L(t)$ and $U_0(t)$, respectively, and set $K_L(t) = U_L(t) - U_0(t)$.

To characterize the point spectrum $P_\sigma(A_L)$ of A_L , we will use the characteristic operator $\Delta(\lambda)$ given as

$$(4) \quad \Delta(\lambda)x = (\lambda I - A - L_\lambda)x, \quad x \in D(A),$$

where $L_\lambda x = L(\varepsilon_\lambda \otimes x)$. It is a closed linear operator which is well defined for $\Re \lambda > \gamma_0$.

The following two lemmas can be found in [5], [6], [9].

Lemma 3.1. *If $\Re \lambda > \gamma_0$, then $\lambda \in P_\sigma(A_L)$ if and only if the null space $N(\Delta(\lambda))$ is nontrivial.*

Since the original semigroup $T(t)$ is a C_0 -compact semigroup on E or $L : \mathcal{B} \rightarrow E$ is a compact operator, $K_L(t)$ is a compact operator for $t > 0$. This implies that $\alpha(U_L(t)) = \alpha(U_0(t))$ for $t > 0$. From this we can obtain the following estimate of the essential growth bound of $U_L(t)$.

Lemma 3.2. *If the phase space $\mathcal{B}$ satisfies the axiom (C), then*

$$\omega_e(U_0) = \omega_e(U_L) \leq \hat{\omega}_e(S_0).$$

Put

$$P_\sigma(\Delta) = \{\lambda \in C : \Re \lambda > \gamma_0 \text{ and } N(\Delta(\lambda)) \text{ is nontrivial}\},$$

$$P_0(\Delta) = \{\lambda \in P_\sigma(\Delta) : \Re \lambda = 0\},$$

$$D^\circ = \{\mu \in C : \Re \mu > \hat{\omega}_e(S_0)\}, \quad D_-^\circ = D^\circ \setminus (P_\sigma(\Delta) \cup P_\sigma(A)),$$

$$D = \{\mu \in C : \Re \mu > \hat{\omega}_e(S)\}, \quad D_- = D \setminus (P_\sigma(\Delta) \cup P_\sigma(A)),$$

$$C^\circ = \{\mu \in C : \Re \mu > \gamma_0\}, \quad \text{and} \quad C_-^\circ = C^\circ \setminus (P_\sigma(\Delta) \cup P_\sigma(A)).$$

Using Lemma 3.2, we have the following result.

Theorem 3.3. *Let the phase space $\mathcal{B}$ satisfy the axiom (C). Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$. Then the following statements are equivalent.*

- 1) *The zero solution of Eq. (AL) is asymptotically stable.*
- 2) *The zero solution of Eq. (AL) is exponentially asymptotically stable.*
- 3) *For any $\lambda \in P_\sigma(\Delta)$, $\Re \lambda$ is negative.*

Proof. From the assumptions in the theorem and Lemma 3.2 it follows that $\omega_e(U_0) = \omega_e(U_L) \leq \hat{\omega}_e(S_0) < \omega_s(U_L)$. Therefore, Theorem 3.3 follows from Theorem 2.3 and Lemma 3.1.

Remark. Let the assumptions of Theorem 3.3 be satisfied. Then the phase space $\mathcal{B}$ is a uniform fading memory space, if the zero solution of Eq. (AL) is asymptotically stable.

Lemma 3.4. For $\lambda \in C_-^\circ$, the inverse $\Delta(\lambda)^{-1}$ exists as a bounded linear operator, and it is analytic for λ .

Proof. Let $\lambda \in C_-^\circ$. Since $R(\lambda, A)$ exists, we can write as $\Delta(\lambda) = (I - L_\lambda R(\lambda, A))(\lambda I - A)$. Suppose that $(I - L_\lambda R(\lambda, A))y = 0$. Since $\lambda \in \rho(A)$, $y = (\lambda I - A)x$ for some $x \in D(A)$; hence $\Delta(\lambda)x = 0$. Since $\lambda \notin P_\sigma(A_L)$, it follows from Lemma 3.1 that $N(\Delta(\lambda)) = \{0\}$; hence $x = 0$. This implies $y = 0$, which shows that one does not lie in $P_\sigma(L_\lambda R(\lambda, A))$. Since $T(t)$ is a C_0 -compact semigroup, $R(\lambda, A)$ is compact. This implies that $L_\lambda R(\lambda, A)$ is compact, and that $\sigma(L_\lambda R(\lambda, A)) \setminus \{0\}$ consists of the point spectrum. Hence one lie in $\rho(L_\lambda R(\lambda, A))$. Therefore, $I - L_\lambda R(\lambda, A)$ is a bijection on E , and $\Delta(\lambda)$ maps $D(A)$ onto E . Since it is one to one for $\lambda \notin P_\sigma(A)$, the inverse $\Delta(\lambda)^{-1}$ exists as a bounded linear operator on E onto $D(A)$ from the closed graph theorem. It is indeed given by

$$\Delta(\lambda)^{-1} = R(\lambda, A)(I - L_\lambda R(\lambda, A))^{-1}.$$

Set $F(\lambda) = I - L_\lambda R(\lambda, A)$, and let $\mu \in C_-^\circ$. Then there exists an $r > 0$ such that if $|\lambda - \mu| < r$, then $\lambda \in C_-^\circ$. Define $G(\lambda) = (F(\lambda) - F(\mu))/(\lambda - \mu)$ for $|\lambda - \mu| < r, \lambda \neq \mu$, and $G(\mu) = F'(\mu)$. Then $G(\lambda)$ is analytic, and

$$F(\lambda) = F(\mu) + (\lambda - \mu)G(\lambda) = F(\mu)(I + (\lambda - \mu)F(\mu)^{-1}G(\lambda)).$$

Thus, it follows that

$$F(\lambda)^{-1} = \sum_{n=0}^{\infty} (-(\lambda - \mu))^n (F(\mu)^{-1}G(\lambda))^n F(\mu)^{-1}$$

if $|\lambda - \mu|$ is sufficiently small. Hence $F(\lambda)^{-1}$ is analytic. As a result, $\Delta(\lambda)^{-1}$ is analytic for $\lambda \in C_-^\circ$.

The formula (5) in the following lemma was already given on a narrow domain in [5]. To discuss the stability for Eq. (AL), we need to extend the domain where it holds.

Lemma 3.5. Let the phase space $\mathcal{B}$ satisfy the axiom (C).

1) Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$. For each $\lambda \in D_-^\circ$,

$$(5) \quad R(\lambda, A_L)\psi = R(\lambda, A_0)\psi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}L(R(\lambda, A_0)\psi), \quad \psi \in \mathcal{B}.$$

2) Assume that $\hat{\omega}_e(S) < \omega_s(U_0)$. For each $\lambda \in D \setminus P_\sigma(A)$,

$$(6) \quad R(\lambda, A_0)\psi = \varepsilon_\lambda \otimes R(\lambda, A)\psi(0) + M_\lambda\psi, \quad \psi \in \mathcal{B},$$

where

$$M_\lambda \psi = R(\lambda, B)\psi - \lambda^{-1} \varepsilon_\lambda \otimes \psi(0).$$

3) Assume that $\hat{\omega}_e(S) < \min\{\omega_s(U_0), \omega_s(U_L)\}$. For each $\lambda \in D_-$,

$$(7) \quad R(\lambda, A_L)\psi = \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}(\psi(0) + L(M_\lambda \psi)) + M_\lambda \psi, \quad \psi \in \mathcal{B}.$$

Proof. 1) From the proof of Theorem 3.3 we have $\omega_e(U_L) \leq \hat{\omega}_e(S_0) < \omega_s(U_L)$. Moreover, the abscissa γ_0 of the exponent of the phase space $\mathcal{B}$, $\omega_e(S_0)$ and $\omega_s(S_0)$ are related as $\gamma_0 \leq \omega_e(S_0) = \omega_s(S_0)$. Let ε be any number such that $0 < \varepsilon < \omega_s(U_L) - \hat{\omega}_e(S_0)$. Put $D^\circ(\varepsilon) = \{\mu \in C : \Re \mu > \hat{\omega}_e(S_0) + \varepsilon\}$ and $D_-^\circ(\varepsilon) = D^\circ(\varepsilon) \setminus (P_\sigma(A) \cup P_\sigma(A_L))$. In view of the relation (1), we see that $P_\sigma(A_0)$ and $P_\sigma(A_L)$ consist of at most finite normal eigenvalues in $D^\circ(\varepsilon)$. Of course, $P_\sigma(A) = P_\sigma(A_0)$ and $P_\sigma(A_L) = P_\sigma(A)$, because of Lemma 3.1. Hence, $R(\lambda, A)$, $R(\lambda, A_0)$ and $R(\lambda, A_L)$ are holomorphic in $D_-^\circ(\varepsilon)$, and so $LR(\lambda, A_0)$ and $LR(\lambda, A_L)$ are also holomorphic in $D_-^\circ(\varepsilon)$. On the other hand, $\Delta(\lambda)$ is well defined on $D_-^\circ(\varepsilon)$. Let $\lambda \in D_-^\circ(\varepsilon)$. Then the inverse $\Delta(\lambda)^{-1}$ is a bounded linear operator, because of Lemma 3.4. Using the above facts and the proof in [6, Theorem 4.2], we have

$$(8) \quad R(\lambda, A_L)\psi = R(\lambda, A_0)\psi + \varepsilon_\lambda \otimes R(\lambda, A)L(R(\lambda, A_L)\psi), \quad \psi \in \mathcal{B},$$

and

$$(9) \quad \Delta(\lambda)R(\lambda, A)L(R(\lambda, A_L)\psi) = L(R(\lambda, A_0)\psi), \quad \psi \in \mathcal{B}.$$

Hence the relation (9) becomes

$$(10) \quad R(\lambda, A)L(R(\lambda, A_L)\psi) = \Delta(\lambda)^{-1}L(R(\lambda, A_0)\psi), \quad \psi \in \mathcal{B}.$$

By substituting (10) into (8), we can see that the relation (5) holds for all $\lambda \in D_-^\circ(\varepsilon)$. Since ε is arbitrary, the proof of the assertion 1) is complete.

2) Using Lemma 4.3 in [6] and the same argument as above, we can easily prove the assertion 2).

3) Combining the assertion 1) and 2), we have that for $\lambda \in D_-$.

$$\begin{aligned} & R(\lambda, A_0)\psi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}L(R(\lambda, A_0)\psi) \\ &= \varepsilon_\lambda \otimes R(\lambda, A)\psi(0) + M_\lambda \psi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}L(\varepsilon_\lambda \otimes R(\lambda, A)\psi(0) + M_\lambda \psi) \\ &= M_\lambda \psi + \varepsilon_\lambda \otimes (R(\lambda, A)\psi(0) + \Delta(\lambda)^{-1}(L_\lambda(R(\lambda, A)\psi(0)) + L(M_\lambda \psi))) \\ &= M_\lambda \psi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}((\lambda I - A - L_\lambda)R(\lambda, A)\psi(0) \\ &\quad + L_\lambda(R(\lambda, A)\psi(0)) + L(M_\lambda \psi)) \\ &= M_\lambda \psi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}(\psi(0) + L(M_\lambda \psi)). \end{aligned}$$

This is the desired result.

Remark. The formula (6) is already found in [1], which is the finite delay case with phase space of continuous functions. Our result is an extension to the infinite delay case with the abstract phase space.

Proposition 3.6. *Let the phase space $\mathcal{B}$ satisfy the axiom (C).*

1) *Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$. Let $\lambda_0 \in (P_\sigma(\mathcal{A}) \setminus P_\sigma(A)) \cap D^\circ$. If $\Delta(\lambda)^{-1}$ has a pole of order n at λ_0 , then $R(\lambda, A_L)$ has a pole of at most order n at λ_0 .*

2) *Assume that $\hat{\omega}_e(S_0) < \min\{\omega_s(U_0), \omega_s(U_L)\}$. Let $\lambda_0 \in (P_\sigma(\mathcal{A}) \setminus P_\sigma(A)) \cap D$. Then $\Delta(\lambda)^{-1}$ has a pole of order n at λ_0 if and only if $R(\lambda, A_L)$ has a pole of order n at λ_0 .*

Proof. The assertion 1) is easily shown by the relation (5). The assertion 2) is shown by combining the relation (7) and the fact that the map $\psi \mapsto \psi(0) + L(M_\lambda \psi)$ is surjective, cf. the proof of [1, Proposition 3.4].

The following is an immediate result of Proposition 3.6

Corollary 3.7. *Let the phase space $\mathcal{B}$ satisfy the axiom (C). Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$. Let $\lambda_0 \in (P_\sigma(\mathcal{A}) \setminus P_\sigma(A)) \cap D^\circ$. Then λ_0 is a simple pole of $\Delta(\lambda)^{-1}$ if and only if it is a simple pole of $R(\lambda, A_L)$.*

Using Corollary 3.7 and Theorem 2.4, we have the following result.

Theorem 3.8. *Let the phase space $\mathcal{B}$ be a uniform fading memory space. Assume that $P_\sigma(A) \cap P_0(\mathcal{A}) = \emptyset$, and $\omega_s(U_L) = 0$. If each $\lambda \in P_0(\mathcal{A})$ is a simple pole of $\Delta(\mu)^{-1}$, then the zero solution of Eq. (AL) is stable.*

Recall that a C_0 -semigroup $H(t)$ on E is said to have *asynchronous exponential growth with intrinsic growth constant* $\lambda_0 \in \mathbb{R}$ provided there exists a rank one projection P_0 in E such that $\lim_{t \rightarrow \infty} e^{-\lambda_0 t} H(t) = P_0$ (see [7, 12]). Put $P\sigma_0(\mathcal{A}) = \{\mu \in P_\sigma(\mathcal{A}) : \Re \mu = \sup\{\Re \lambda : \lambda \in P_\sigma(\mathcal{A})\}\}$. Then the following result holds.

Theorem 3.9. *Let the phase space $\mathcal{B}$ satisfy the axiom (C). Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$ and that $\lambda_0 \in \mathbb{R}$ satisfies the following conditions:*

- 1) $P\sigma_0(\mathcal{A}) = \{\lambda_0\}$ and λ_0 belongs to the resolvent set of A .
- 2) λ_0 is a simple pole of $\Delta(\lambda)^{-1}$.

Then the solution semigroup $U_L(t)$ of Eq. (AL) has asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in \mathbb{R}$.

The proof follows immediately from Proposition 2.3 in [12] (or Theorem 3.6.2 in [7]), Lemma 3.1, Lemma 3.2 and Corollary 3.7.

4. An example

Let $E = L^2([0, \pi], C)$, the set of square integrable functions on $[0, \pi]$. Consider the equation

$$(11) \quad u'(t) = Au(t) + b \int_{-\infty}^t e^{-c(t-s)} u(s) ds,$$

where A is defined as $Af = f''$ for $f \in E$ such that f is continuously differentiable, the derivative f' is absolutely continuous, $f'' \in E$, and that $f(0) = f(\pi) = 0$. It is well known that A is a closed linear operator with dense domain. It is self adjoint, the spectrum of A consists of only point spectrum $\lambda = -n^2$, $n = 1, 2, \dots$, $R(\lambda, A)$ has a pole of order 1 at these points, and $|R(\lambda, A)| \leq 1/|\lambda + 1|$ for $\Re \lambda > -1$. Hence A is the infinitesimal generator of a C_0 -semigroup $T(t)$ such that $|T(t)| \leq e^{-t}$ for $t \geq 0$. Furthermore, $T(t)$ is a C_0 -compact semigroup.

Set

$$L(\phi) = b \int_{-\infty}^0 e^{c\theta} \phi(\theta) d\theta.$$

Then we have that $|L(\phi)| \leq |b| \|\phi\|$ for $\phi \in E \times \mathcal{L}_{-c}$. Hence, we will take $\mathcal{B} = E \times \mathcal{L}_{-c}$ as the phase space of Eq. (11). We note that if $c > 0$, then $\mathcal{B}$ is a uniform fading memory space. The solution semigroup of Eq. (11) and its infinitesimal generator are denoted by $U_L(t)$ and A_L , respectively.

The characteristic operator $\Delta(\lambda)$, given by (4), now becomes

$$\Delta(\lambda)f = \lambda f - Af - b \int_{-\infty}^0 e^{(c+\lambda)\theta} d\theta f \quad f \in D(A), \quad \Re \lambda > -c.$$

If we set $h(\lambda) = \lambda - b/(c + \lambda)$ for $\Re \lambda > -c$, then we can write $\Delta(\lambda) = h(\lambda)I - A$. Moreover, the equation $h(\lambda) = -n^2$ has the roots $\kappa_n = [-(c + n^2) - \sqrt{D}]/2$, $\lambda_n = [-(c + n^2) + \sqrt{D}]/2$, where $D = (c + n^2)^2 - 4(cn^2 - b)$.

Define a curve $b = \chi(c)$ in the c-b plane by

$$\chi(c) = \begin{cases} 0 & \text{for } c \leq 1 \\ -(c-1)^2/4 & \text{for } c > 1, \end{cases}$$

and divide the c-b plane into the subregions as

$$\begin{aligned} \Pi_1 : b > \chi(c), \quad -\infty < c < \infty, \quad \Pi_2 : b \leq \chi(c), \quad c > 1, \\ \Pi_3 : b \leq \chi(c), \quad c \leq 1. \end{aligned}$$

The following result can be found in [5].

Lemma 4.1. *The abscissa of the exponent of the phase space $\mathcal{B}$, and the growth bound and the essential growth bound of U_L become as follows:*

$$\hat{\omega}_e(S_0) = \omega_e(S_0) = \omega_s(S_0) = \omega_e(U_L) = -c = \gamma_0,$$

$$\omega_s(U_L) = \begin{cases} \lambda_1 & \text{if } (c, b) \in \Pi_1 \\ -(c+1)/2 & \text{if } (c, b) \in \Pi_2 \\ -c & \text{if } (c, b) \in \Pi_3. \end{cases}$$

Proposition 4.2. 1) *The zero solution of Eq. (11) is exponentially asymptotically stable if and only if $c > 0$ and $b < c$.*

2) *Assume that $(c, b) \in \Pi_1$ and $c > 0$. The following statements are equivalent.*

- (i) *The zero solution of Eq. (11) is asymptotically stable.*
- (ii) *The zero solution of Eq. (11) is exponentially asymptotically stable.*
- (iii) *$b < c$.*

3) *Assume that $(c, b) \in \Pi_2$. Then, the zero solution of Eq. (11) is asymptotically stable if and only if it is exponentially asymptotically stable.*

Proof. The assertion 1) was obtained in [5]. If $(c, b) \in \Pi_1$, then $-c < \lambda_1$. This implies the inequality $\omega_e(U_L) < \omega_s(U_L)$, because of Lemma 4.1. If $b < c$, then λ_1 is negative. Hence the assertion 2) follows from Lemma 4.1 and Theorem 3.3. Since $(c, b) \in \Pi_2$, we have $c > 1$, from which it follows that the inequality $\omega_e(U_L) < \omega_s(U_L)$ holds. Thus Theorem 3.3 implies the assertion 3).

The following result was already shown in [5] and its proof is strongly dependent on the concrete form of Eq. (11). In contrast with it, our manner is based on Theorem 3.8.

Proposition 4.3. *Assume that $c > 0$. Then the zero solution of Eq. (11) is stable but not asymptotically stable if and only if $b = c$.*

Proof. Assume that the zero solution of Eq. (11) is stable but not asymptotically stable. Since $c > 0$, it follows from 1) in Proposition 4.2 that $b \geq c$. If $b > c$, then $\lambda_1 > 0$. Thus $e^{\lambda_1 t} \leq \|U_L(t)\| \rightarrow \infty$ as $t \rightarrow \infty$. Hence $b = c$.

Next, we assume that $b = c$. Then $\lambda_1 = 0$. Thus we have $-c = \omega_e(U_L) < \omega_s(U_L) = 0$ from Lemma 4.1. Of course, $P_\sigma(A) \cap P_0(A_L) = \emptyset$. For the characteristic operator $A(\lambda) = h(\lambda)I - A$, we see that $N(A(\lambda_1)) \neq \{0\}$, and $P_0(A_L) = \{\lambda_1\}$. Moreover, since $T(t)$ is a C_0 -compact semigroup, there exists an $\varepsilon_0, 0 < \varepsilon_0 < \omega_s(U_L) - \hat{\omega}_e(S_0)$ such that λ_1 is a unique pole of $A(\mu)^{-1}$ within $H(\varepsilon_0) := \{\mu \in C : \Re \mu \geq -\varepsilon_0\}$.

Finally, we shall show that λ_1 is a simple pole of $\Delta(\mu)^{-1}$. Set $g(w) = R(w, A)$. Then $g(h(z)) = \Delta(z)^{-1}$ and $h(0) = -1$. Since $g(w)$ has a pole of order 1 at $w = -1$, we have $g(w) = \hat{g}(w)/(w+1)$, where $\hat{g}(w)$ is holomorphic at $w = -1$ and $\hat{g}(-1) \neq 0$. Hence we have

$$g(h(z)) = \frac{\hat{g}(h(z))}{h(z) + 1} = \frac{\hat{g}(h(z))(z+c)}{z(z+c+1)},$$

from which $g(h(z))$ has a pole of order 1 at $z = 0$. Thus $\lambda_1 (= z = 0)$ is a simple pole of $\Delta(z)^{-1}$. Therefore the proof of the theorem follows from Theorem 3.8.

Proposition 4.4. *Assume that $b = c > 0$. Then the solution semigroup $U_L(t)$ of Eq. (11) has asynchronous exponential growth with intrinsic growth constant $0 \in R$.*

The proof follows immediately from Theorem 3.9 and the proof of Proposition 4.3.

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nuna adreso:

Jong Son Shin
Department of Mathematics
Korea University
Ogawa, Kodaira, Tokyo 187-8560
Japan
E-mail: shinjs@tech.korea-u.ac.jp

Toshiki Naito
The University of Electro-Communications
Chofu, Tokyo 182-8585,
Japan
E-mail: naito@matha.e-one.uec.ac.jp

Nguyen Van Minh
The University of Electro-Communications
Chofu, Tokyo 182-8585,
Japan
E-mail: minh@matha.e-one.uec.ac.jp

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