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Positive Decreasing Solutions of Systems of Second Order Quasilinear Differential Equations

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0. Introduction

This paper is concerned with systems of second order quasilinear differential equations of the type

$$(A) \quad \begin{cases} (p(t)|y'|^{\alpha-1}y')' = \varphi(t)|z|^{\lambda-1}z \\ (q(t)|z'|^{\beta-1}z')' = \psi(t)|y|^{\mu-1}y, \end{cases}$$

where α, β, λ and μ are positive constants, and $p(t), q(t), \varphi(t)$ and $\psi(t)$ are positive continuous functions defined on $[a, \infty)$, $a \geq 0$.

By a solution of (A) on $[a, \infty)$ we mean a positive vector functions (y, z) which have the property that $y, z, p|y'|^{\alpha-1}y'$ and $q|z'|^{\beta-1}z'$ are continuously differentiable on $[a, \infty)$ and satisfy the system (A) at every point of $[a, \infty)$.

Our objective is to study the existence of positive decreasing solution of (A), that is, those solutions (y, z) of (A) whose components y and z are both positive and decreasing on $[a, \infty)$. There are two possible cases: either both components y, z are positive on $[a, \infty)$ or there exists a finite $T > a$ such that

$$y(t) > 0 \quad \text{and} \quad z(t) > 0 \quad \text{on} \quad [a, T) \quad \text{and} \quad y(t) = z(t) = 0 \quad \text{on} \quad [T, \infty).$$

In the former case (y, z) is called a *decreasing proper solution* of (A), and in the latter case (y, z) is called an *extinct singular solution* of (A).

The question of existence of extinct singular solution for (A) is discussed in Section 1. The main result therein is Theorem 1.1 which asserts that the differential system (A) possesses an extinct singular solution if $\alpha\beta > \lambda\mu$. Sections 2 and 3 concern the existence of positive decreasing proper solutions for (A). Motivated by the previous papers [2, 4] on single differential equations of the form

$$(B) \quad (p(t)|y'|^{\alpha-1}y')' = \varphi(t)|y|^{\lambda-1}y,$$

we take up the two cases:

$$(0.1) \quad \int_a^\infty (p(t))^{-1/\alpha} dt = \infty \quad \text{and} \quad \int_a^\infty (q(t))^{-1/\beta} dt = \infty$$

and

$$(0.2) \quad \int_a^\infty (p(t))^{-1/\alpha} dt < \infty \quad \text{and} \quad \int_a^\infty (q(t))^{-1/\beta} dt < \infty,$$

and examine if the system (A) admits the following two types of decreasing proper solutions (y, z) of (A):

$$(I-1) \quad \lim_{t \rightarrow \infty} y(t) = \text{const} > 0, \quad \lim_{t \rightarrow \infty} z(t) = \text{const} > 0,$$

and

$$(II-1) \quad \lim_{t \rightarrow \infty} y(t) = 0, \quad \lim_{t \rightarrow \infty} z(t) = 0$$

in case the condition (0.1) holds, and

$$(I-2) \quad \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = \text{const} > 0, \quad \lim_{t \rightarrow \infty} \frac{z(t)}{\rho(t)} = \text{const} > 0,$$

and

$$(II-2) \quad \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = 0, \quad \lim_{t \rightarrow \infty} \frac{z(t)}{\rho(t)} = 0$$

in case the condition (0.2) holds, where $\pi(t)$ and $\rho(t)$ are defined by

$$(0.3) \quad \pi(t) = \int_t^\infty (p(s))^{-1/\alpha} ds, \quad \rho(t) = \int_t^\infty (q(s))^{-1/\beta} ds, \quad t \geq a.$$

A solution of the type (I-1) or (I-2) is called a *moderately decreasing solution* or a *moderately decaying solution*, while that of the type (II-1) or (II-2) is called a *strongly decreasing solution* or a *strongly decaying solution*. The cases (0.1) and (0.2) are studied separately in Sections 2 and 3, respectively. In each of them we first establish necessary and sufficient conditions for the existence of moderately decreasing solutions or moderately decaying solutions of (A), and then present sufficient conditions for (A) to possess strongly decreasing solutions or strongly decaying solutions.

The motivation of the present work is to extend some of the results [2, 4] obtained for the single quasilinear differential equation (B).

Because of the generality of the functions $p(t)$ and $q(t)$ in (A) our results for (A) can be applied to derive nontrivial information about the existence of positive radial solutions of systems of quasilinear partial differential equations of the type

$$(C) \quad \begin{cases} \operatorname{div}(|Du|^{m-2}Du) = f(|x|)|v|^{\lambda-1}v \\ \operatorname{div}(|Dv|^{n-2}Dv) = g(|x|)|u|^{\mu-1}u, \end{cases}$$

in exterior domains E_a in $\mathbf{R}^N$, $N \geq 2$, where $m > 1$, $n > 1$, $\lambda > 0$ and $\mu > 0$ are constants, D denotes the gradient operator $(\partial/\partial x_1, \dots, \partial/\partial x_N)$, $E_a = \{x \in \mathbf{R}^N : |x| \geq a\}$, and $f(t)$ and $g(t)$ are positive continuous functions on $[a, \infty)$, $a > 0$. The results thus obtained for (C) are presented in Section 4.

1. Existence and nonexistence of extinct singular solutions

In our recent paper [4] we have proved that in case $\alpha > \lambda$ the single differential equation (B) possesses an extinct singular solution, while in case $\alpha < \lambda$ the equation (B) admits no extinct singular solution. An extension of this result to differential systems of the form (A) is given below.

Theorem 1.1. *Suppose that*

$$(1.1) \quad \alpha\beta > \lambda\mu$$

Then, for any $T > a$, the system (A) possesses a singular solution (y, z) such that

$$y(t) > 0 \quad \text{and} \quad z(t) > 0 \quad \text{on } [a, T) \quad \text{and} \quad y(t) = z(t) = 0 \quad \text{on } [T, \infty).$$

Proof. Let k and l denote the positive constants

$$(1.2) \quad k = \frac{\beta(\alpha+1) + \lambda(\beta+1)}{\alpha\beta - \lambda\mu}, \quad l = \frac{\alpha(\beta+1) + \mu(\alpha+1)}{\alpha\beta - \lambda\mu}.$$

Define the positive constants K_1 , K_2 , L_1 and L_2 by

$$(1.3) \quad \begin{aligned} K_1 &= \frac{1}{k} \left(\frac{\varphi^*/p^*}{1 + \lambda l} \right)^{1/\alpha}, & K_2 &= \frac{1}{k} \left(\frac{\varphi^*/p^*}{1 + \lambda l} \right)^{1/\alpha}, \\ L_1 &= \frac{1}{l} \left(\frac{\psi^*/q^*}{1 + k\mu} \right)^{1/\beta}, & L_2 &= \frac{1}{l} \left(\frac{\psi^*/q^*}{1 + k\mu} \right)^{1/\beta}, \end{aligned}$$

where we have used the notation

$$f^* = \max_{t \in [a, T]} f(t), \quad f_* = \min_{t \in [a, T]} f(t).$$

Letting

$$(1.4) \quad \begin{aligned} c_1 &= (K_1 L_1^{\lambda/\alpha})^{\alpha\beta/(\alpha\beta - \lambda\mu)}, & c_2 &= (K_2 L_2^{\lambda/\alpha})^{\alpha\beta/(\alpha\beta - \lambda\mu)}, \\ d_1 &= (L_1 K_1^{\mu/\beta})^{\alpha\beta/(\alpha\beta - \lambda\mu)}, & d_2 &= (L_2 K_2^{\mu/\beta})^{\alpha\beta/(\alpha\beta - \lambda\mu)} \end{aligned}$$

and noting that $K_1 \leq K_2$ and $L_1 \leq L_2$ implies $c_1 \leq c_2$ and $d_1 \leq d_2$, we consider the set $\mathcal{Y} \subset C[a, T] \times C[a, T]$ consisting of all vector functions $(y(t), z(t))$ satisfying

$$(1.5) \quad c_1(T-t)^k \leq y(t) \leq c_2(T-t)^k, \quad d_1(T-t)^l \leq z(t) \leq d_2(T-t)^l, \quad t \in [a, T],$$

and the mapping $\mathcal{F} : \mathcal{Y} \rightarrow C[a, T] \times C[a, T]$ defined by

$$(1.6) \quad \mathcal{F}(y, z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad t \in [a, T],$$

where

$$(1.7) \quad \mathcal{G}z(t) = \int_t^T \left[(p(s))^{-1} \int_s^T \varphi(r)(z(r))^\lambda dr \right]^{1/\alpha} ds,$$

and

$$(1.8) \quad \mathcal{H}y(t) = \int_t^T \left[(q(s))^{-1} \int_s^T \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds.$$

As is easily verified, $(y, z) \in \mathcal{Y}$ implies that

$$K_1 d_1^{\lambda/\alpha} (T-t)^k \leq \mathcal{G}z(t) \leq K_2 d_2^{\lambda/\alpha} (T-t)^k,$$

$$L_1 c_1^{\mu/\beta} (T-t)^l \leq \mathcal{H}y(t) \leq L_2 c_2^{\mu/\beta} (T-t)^l, \quad t \in [a, T].$$

Since from (1.3) and (1.4) we have $K_1 d_1^{\lambda/\alpha} = c_1$, $K_2 d_2^{\lambda/\alpha} = c_2$, $L_1 c_1^{\mu/\beta} = d_1$ and $L_2 c_2^{\mu/\beta} = d_2$, we see that $\mathcal{F}(y, z) \in \mathcal{Y}$. Thus $\mathcal{F}$ maps $\mathcal{Y}$ into itself. Let $\{(y_n(t), z_n(t))\}_{n=1}^\infty$ be a sequence of vector functions of $\mathcal{Y}$ converging to $(y(t), z(t))$ uniformly on $[a, T]$ as $n \rightarrow \infty$. Then $(y, z) \in \mathcal{Y}$ and it can be shown with the help of the Lebesgue convergence theorem that $\mathcal{F}(y_n, z_n)(t)$ converges to $\mathcal{F}(y, z)(t)$ uniformly on $[a, T]$ as $n \rightarrow \infty$. This shows that $\mathcal{F}$ is a continuous mapping. Finally, using (1.5), (1.7) and (1.8), we find that

$$|(\mathcal{G}z)'(t)| \leq \left(\frac{\varphi^*/p_*}{1 + \lambda l} \right)^{1/\alpha} d_2^{\lambda/\alpha} (T-t)^{k-1}$$

$$|(\mathcal{H}y)'(t)| \leq \left(\frac{\psi^*/q_*}{1 + k\mu} \right)^{1/\beta} c_2^{\mu/\beta} (T-t)^{l-1}, \quad t \in [a, T],$$

from which it follows that the set $\mathcal{F}(\mathcal{Y})$ is compact in $C[a, T] \times C[a, T]$.

Thus, Schauder's fixed point theorem applies and there exists a vector function $(y, z) \in \mathcal{Y}$ such that $(y, z) = \mathcal{F}(y, z)$, that is,

$$\begin{aligned}
 (1.9) \quad y(t) &= \int_t^T \left[(p(s))^{-1} \int_s^T \varphi(r)(z(r))^\lambda dr \right]^{1/\alpha} ds, \\
 z(t) &= \int_t^T \left[(q(s))^{-1} \int_s^T \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds, \quad t \in [a, T].
 \end{aligned}$$

Differentiating the system of integral equations (1.9), we see that $(y(t), z(t))$ is a solution of (A) on $[a, T]$ which is positive on $[a, T)$ and decreases to zero as $t \rightarrow T - 0$. Continuing $y(t)$ and $z(t)$ on $[T, \infty)$ by letting $y(t) = z(t) = 0$ there provides a desired extinct singular solution of the system (A). This completes the proof.

Theorem 1.2. *If $\alpha\beta < \lambda\mu$, then the system (A) has no extinct singular solution.*

Proof. Assume to the contrary that the system (A) has an extinct singular solution (y, z) such that

$$(1.10) \quad y(t) > 0, \quad z(t) > 0 \quad \text{on } [a, T) \quad \text{and} \quad y(t) = z(t) = 0 \quad \text{on } [T, \infty)$$

Integrating (A) from t to T gives

$$p(t)(-y'(t))^\alpha = \int_t^T \varphi(s)(z(s))^\lambda ds, \quad q(t)(-z'(t))^\beta = \int_t^T \psi(s)(y(s))^\mu ds,$$

which implies that

$$\begin{aligned}
 0 \leq -y'(t) &\leq \left((p(t))^{-1} \int_t^T \varphi(s)(z(s))^\lambda ds \right)^{1/\alpha}, \\
 0 \leq -z'(t) &\leq \left((q(t))^{-1} \int_t^T \psi(s)(y(s))^\mu ds \right)^{1/\beta}, \quad t \in [a, T).
 \end{aligned}$$

Integrating the above inequality from t to T , we have

$$\begin{aligned}
 (1.11) \quad 0 \leq y(t) &\leq \int_t^T \left((p(s))^{-1} \int_s^T \varphi(r)(z(r))^\lambda dr \right)^{1/\alpha} ds \\
 0 \leq z(t) &\leq \int_t^T \left((q(s))^{-1} \int_s^T \psi(r)(y(r))^\mu dr \right)^{1/\beta} ds, \quad t \in [a, T).
 \end{aligned}$$

Since $y(t)$ and $z(t)$ are decreasing on $[a, T)$, we see from (1.11) that

$$(1.12) \quad y(t) \leq (z(t))^{\lambda/\alpha} \Phi(t), \quad z(t) \leq (y(t))^{\mu/\beta} \Psi(t), \quad t \in [a, T),$$

where $\Phi(t)$ and $\Psi(t)$ are defined by

$$\Phi(t) = \int_t^T \left((p(s))^{-1} \int_s^T \varphi(r) dr \right)^{1/\alpha} ds,$$

$$\Psi(t) = \int_t^T \left((q(s))^{-1} \int_s^T \psi(r) dr \right)^{1/\beta} ds.$$

From (1.11) we easily obtain

$$(1.13) \quad \begin{aligned} (y(t))^{(\alpha\beta-\lambda\mu)/\alpha\beta} &\leq \Phi(t)(\Psi(t))^{\lambda/\alpha}, \\ (z(t))^{(\alpha\beta-\lambda\mu)/\alpha\beta} &\leq (\Phi(t))^{\mu/\beta}\Psi(t), \quad t \in [a, T]. \end{aligned}$$

We let $t \rightarrow T - 0$. The left hand sides of (1.13) tend to ∞ because of the assumption $\alpha\beta < \lambda\mu$, while the right hand sides of (1.13) approach zero. This contradiction proves that (A) cannot have an extinct singular solution.

Example 1.2. Consider the differential system

$$(1.14) \quad \begin{cases} (|y'|^{\alpha-1}y')' = \alpha(k-1)k^\alpha|z|^{\lambda-1}z \\ (|z'|^{\beta-1}z')' = \beta(l-1)l^\beta|y|^{\mu-1}y, \end{cases}$$

on $[0, \infty)$, where λ and μ satisfy (1.1) and k and l are given by (1.2). It is clear that $k > 1$ and $l > 1$. According to Theorem 1.1 the system (1.14) has an extinct singular solution. One such solution is given by $(y(t), z(t)) = ((T-t)^k, (T-t)^l)$.

2. Existence of positive decreasing proper solutions

This section is devoted to the study of positive decreasing solutions of (A) existing on the entire interval $[a, \infty)$. We begin by assuming that the functions $p(t)$ and $q(t)$ in (A) satisfy the condition (0.1), and direct our attention to the two types of positive decreasing solutions (y, z) of (A) on $[a, \infty)$ such that

$$(2.1) \quad \lim_{t \rightarrow \infty} y(t) = \text{const} > 0, \quad \lim_{t \rightarrow \infty} z(t) = \text{const} > 0$$

and

$$(2.2) \quad \lim_{t \rightarrow \infty} y(t) = 0, \quad \lim_{t \rightarrow \infty} z(t) = 0.$$

A solution (y, z) satisfying (2.1) or (2.2) is called a moderately decreasing solution or a strongly decreasing solution of (A).

Let (y, z) be a positive decreasing solution of (A) on $[a, \infty)$. We notice that the system (A) for this (y, z) is written as

$$(2.3) \quad \begin{cases} (-p(t)(-y'(t))^\alpha)' = \varphi(t)(z(t))^\lambda \\ (-q(t)(-z'(t))^\beta)' = \psi(t)(y(t))^\mu, \end{cases} \quad t \geq a.$$

We assert that

$$(2.4) \quad \lim_{t \rightarrow \infty} p(t)(-y'(t))^\alpha = \lim_{t \rightarrow \infty} q(t)(-z'(t))^\beta = 0.$$

In fact, if $\lim_{t \rightarrow \infty} p(t)(-y'(t))^\alpha = k > 0$, then

$$p(t)(-y'(t))^\alpha \geq k \quad \text{or} \quad -y'(t) \geq k^{1/\alpha}(p(t))^{-1/\alpha}, \quad t \geq a,$$

and integrating the last inequality gives

$$y(t) - y(a) \leq -k^{1/\alpha} \int_a^t (p(s))^{-1/\alpha} ds, \quad t \geq a.$$

Letting $t \rightarrow \infty$ in the above and using (0.1), we have $\lim_{t \rightarrow \infty} y(t) = -\infty$, which contradicts the assumed positivity of $y(t)$. Consequently, we must have $\lim_{t \rightarrow \infty} p(t)(-y'(t))^\alpha = 0$. Similarly, it is shown that $\lim_{t \rightarrow \infty} q(t)(-z'(t))^\beta = 0$.

Integrating (2.3) from t to ∞ , we obtain in view of (2.4)

$$(2.5) \quad \begin{cases} -y'(t) = \left[(p(t))^{-1} \int_t^\infty \varphi(s)(z(s))^\lambda ds \right]^{1/\alpha} \\ -z'(t) = \left[(q(t))^{-1} \int_t^\infty \psi(s)(y(s))^\mu ds \right]^{1/\beta}, \end{cases} \quad t \geq a.$$

One more integration of (2.5) yields the following system of integral equations for a moderately decreasing solution (y, z) of (A):

$$(2.6) \quad \begin{cases} y(t) = y(\infty) + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^\lambda dr \right]^{1/\alpha} ds \\ z(t) = z(\infty) + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds, \end{cases} \quad t \geq a,$$

where $y(\infty) = \lim_{t \rightarrow \infty} y(t)$ and $z(\infty) = \lim_{t \rightarrow \infty} z(t)$. On the basis of (2.6) one can characterize the existence of moderately decreasing solutions of (A).

Theorem 2.1. Suppose that (0.1) holds and that $\alpha\beta \neq \lambda\mu$. There exists a positive decreasing proper solution (y, z) of (A) satisfying (2.1) if and only if

$$(2.7) \quad \int_a^\infty \left[(p(t))^{-1} \int_t^\infty \varphi(s) ds \right]^{1/\alpha} dt < \infty$$

and

$$(2.8) \quad \int_a^\infty \left[(q(t))^{-1} \int_t^\infty \psi(s) ds \right]^{1/\beta} dt < \infty.$$

Proof. (The “only if” part) Suppose that (A) has a moderately decreasing solution (y, z) . Integrating (2.3) twice from t to ∞ and using (2.4), we obtain (2.6). Naturally, two repeated integrals in (2.6) must converge for all $t \geq a$. This fact combined with (2.1) implies that both (2.7) and (2.8) are satisfied.

(The “if” part) Suppose that (2.7) and (2.8) are satisfied. Choose $c > 0$ and $d > 0$ so that

$$(2.9) \quad \int_a^\infty \left[(p(t))^{-1} \int_t^\infty \varphi(s) ds \right]^{1/\alpha} dt \leq (2d)^{-\lambda/\alpha} c$$

and

$$(2.10) \quad \int_a^\infty \left[(q(t))^{-1} \int_t^\infty \psi(s) ds \right]^{1/\beta} dt \leq (2c)^{-\mu/\beta} d.$$

Such a choice of c, d is possible because of the assumption $\alpha\beta \neq \lambda\mu$. Let $\mathcal{Y}$ denote the set of all vector functions $(y, z) \in C[a, \infty) \times C[a, \infty)$ such that

$$(2.11) \quad c \leq y(t) \leq 2c \quad \text{and} \quad d \leq z(t) \leq 2d \quad \text{for } t \geq a.$$

Clearly, $\mathcal{Y}$ is a closed convex subset of the Fréchet space $C[a, \infty) \times C[a, \infty)$ with the usual product metric topology. Define the mapping $\mathcal{F} : \mathcal{Y} \rightarrow C[a, \infty) \times C[a, \infty)$ by

$$\mathcal{F}(y, z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad (y, z) \in \mathcal{Y},$$

where $\mathcal{G}$ and $\mathcal{H}$ are the integral operators given by

$$(2.12) \quad \mathcal{G}z(t) = c + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(2.13) \quad \mathcal{H}y(t) = d + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq a.$$

Using the conditions (2.9) and (2.10), we see that $(y, z) \in \mathcal{Y}$ implies

$$c \leq \mathcal{G}z(t) \leq 2c, \quad d \leq \mathcal{H}y(t) \leq 2d, \quad t \geq a,$$

which shows that $\mathcal{F}$ maps $\mathcal{Y}$ into itself. Let $(y_n, z_n) \in \mathcal{Y}$, $n = 1, 2, \dots$, be a sequence converging to $(y, z) \in \mathcal{Y}$ in the topology of $C[a, \infty) \times C[a, \infty)$. Then, with the help of the Lebesgue convergence theorem it can be shown that $\mathcal{F}(y_n, z_n)$ converges to $\mathcal{F}(y, z)$ in the same topology, which ensures that the

mapping $\mathcal{F}$ is continuous. Next, for any $(y, z) \in \mathcal{Y}$ we have from (2.11), (2.12) and (2.13)

$$\begin{aligned} 0 \leq -(\mathcal{G}z)'(t) &= \left[(p(t))^{-1} \int_t^\infty \varphi(s)(z(s))^\lambda ds \right]^{1/\alpha} \\ &\leq (2d)^{\lambda/\alpha} \left[(p(t))^{-1} \int_t^\infty \varphi(s) ds \right]^{1/\alpha}, \\ 0 \leq -(\mathcal{H}y)'(t) &= \left[(q(t))^{-1} \int_t^\infty \psi(s)(y(s))^\mu ds \right]^{1/\beta} \\ &\leq (2c)^{\mu/\beta} \left[(q(t))^{-1} \int_t^\infty \psi(s) ds \right]^{1/\beta}, \quad t \geq a, \end{aligned}$$

from which it readily follows that the set $\mathcal{F}(\mathcal{Y})$ is a relatively compact subset of $C[a, \infty) \times C[a, \infty)$.

Thus all the hypotheses of the Schauder–Tychonoff fixed point theorem are satisfied, and so there exists an element $(y, z) \in \mathcal{Y}$ such that $(y, z) = \mathcal{F}(y, z)$, which satisfies the system of integral equations

$$(2.14) \quad \begin{cases} y(t) = c + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^\lambda ds \right]^{1/\alpha} ds \\ z(t) = d + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq a. \end{cases}$$

Differentiating (2.14) twice, we conclude that the vector function (y, z) is a moderately decreasing solution of (A) on the entire interval $[a, \infty)$ such that $y(\infty) = c$ and $z(\infty) = d$. This completes the proof.

Theorem 2.2. Suppose that (0.1) and (1.1) hold. If (2.7) and (2.8) hold, then there exists a positive decreasing proper solution (y, z) of (A) satisfying (2.2)

Proof. Choose $t_0 > a$ so that

$$(2.17) \quad \int_{t_0}^\infty \left[(p(t))^{-1} \int_t^\infty \varphi(s) ds \right]^{1/\alpha} dt \leq 2^{-\lambda/\alpha}$$

and

$$(2.18) \quad \int_{t_0}^\infty \left[(q(t))^{-1} \int_t^\infty \psi(s) ds \right]^{1/\beta} dt \leq 2^{-\mu/\beta}$$

For every $n \in \mathbb{N}$ let $\mathcal{Y}_n$ denote the set of functions $(y, z) \in C[t_0, \infty) \times C[t_0, \infty)$ satisfying

$$(2.19) \quad \frac{1}{n} \leq y(t) \leq 2, \quad \frac{1}{n} \leq z(t) \leq 2, \quad t \geq t_0,$$

and consider the mapping $\mathcal{F}_n : \mathcal{Y}_n \rightarrow C[t_0, \infty) \times C[t_0, \infty)$ defined by

$$(2.20) \quad \mathcal{F}_n(y, z) = (\mathcal{G}_n z, \mathcal{H}_n y), \quad (y, z) \in \mathcal{Y}_n,$$

where

$$(2.21) \quad \mathcal{G}_n z(t) = \frac{1}{n} + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(2.22) \quad \mathcal{H}_n y(t) = \frac{1}{n} + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq t_0.$$

Applying the Schauder–Tychonoff fixed point theorem, we can show that there exists a fixed point (y_n, z_n) of $\mathcal{F}_n$ in $\mathcal{Y}_n$, which gives rise to a sequence of moderately decreasing solutions of (A), $\{(y_n, z_n)\}_{n=1}^\infty$, defined on the interval $[t_0, \infty)$. Since (y_n, z_n) satisfies

$$(2.23) \quad y_n(t) = \frac{1}{n} + \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z_n(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(2.24) \quad z_n(t) = \frac{1}{n} + \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y_n(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq t_0$$

and

$$(2.25) \quad -y'_n(t) = \left[(p(t))^{-1} \int_t^\infty \varphi(s)(z_n(s))^\lambda ds \right]^{1/\alpha},$$

$$(2.26) \quad -z'_n(t) = \left[(q(t))^{-1} \int_t^\infty \psi(s)(y_n(s))^\mu ds \right]^{1/\beta}, \quad t \geq t_0$$

we see that the sequence $\{(y_n(t), z_n(t))\}_{n=1}^\infty$ is uniformly bounded and locally equicontinuous on $[t_0, \infty)$, so that there exists a subsequence of $\{(y_n(t), z_n(t))\}_{n=1}^\infty$ which converges to a limit function $(y_*(t), z_*(t))$ uniformly on compact subintervals of $[t_0, \infty)$. It is easy to verify that $(y_*(t), z_*(t))$ satisfies

$$(2.27) \quad y_*(t) = \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z_*(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(2.28) \quad z_*(t) = \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y_*(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq t_0,$$

which shows that (y_*, z_*) is a solution of the system (A) on $[t_0, \infty)$.

It remains to prove that $y_*(t) > 0$ and $z_*(t) > 0$ on $[t_0, \infty)$. Because of (1.1), Theorem 1.1 guarantees the existence of an extinct singular solution $(y_T(t), z_T(t))$ such that

$$(2.29) \quad y_T(t) > 0 \quad \text{and} \quad z_T(t) > 0 \quad \text{on} \quad [t_0, T)$$

$$\text{and} \quad y_T(t) = z_T(t) = 0 \quad \text{on} \quad [T, \infty)$$

for any $T > t_0$. Clearly, $(y_T(t), z_T(t))$ satisfies

$$(2.30) \quad y_T(t) = \int_t^T \left[(p(s))^{-1} \int_s^T \varphi(r) (z_T(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(2.31) \quad z_T(t) = \int_t^T \left[(q(s))^{-1} \int_s^T \psi(r) (y_T(r))^\mu dr \right]^{1/\beta} ds, \quad t \in [t_0, T].$$

We claim that

$$(2.32) \quad y_n(t) > y_T(t) \quad \text{and} \quad z_n(t) > z_T(t) \quad \text{on} \quad [t_0, T]$$

for every $n \in N$. To verify this by contradiction assume that (2.32) fails to hold for some $n \in N$. Then, neither of the two inequalities in (2.32) can be true, because is one of them, say $y_n(t) > y_T(t)$ would hold on $[t_0, T]$, then, comparison of (2.24) and (2.31) would imply that $z_n(t) > z_T(t)$ on $[t_0, T]$. So, there exist t_1 and t_2 in $[t_0, T)$ such that

$$y_n(t_1) = y_T(t_1) \quad \text{and} \quad y_n(t) > y_T(t) \quad \text{on} \quad (t_1, T],$$

$$z_n(t_2) = z_T(t_2) \quad \text{and} \quad z_n(t) > z_T(t) \quad \text{on} \quad (t_2, T].$$

If $t_1 \geq t_2$, then $y_n(t) > y_T(t)$ on $(t_2, T]$, and this implies, in view of (2.24) and (2.31), that

$$0 = z_n(t_2) - z_T(t_2)$$

$$> \int_{t_2}^T \left[(q(s))^{-1} \int_s^T \psi(r) (y_n(r))^\mu dr \right]^{1/\beta} ds$$

$$- \int_{t_2}^T \left[(q(s))^{-1} \int_s^T \psi(r) (y_T(r))^\mu dr \right]^{1/\beta} ds > 0,$$

which is a contradiction. Likewise, if $t_1 \leq t_2$, then (2.23) and (2.30) generate the contradiction: $0 = y_n(t_1) - y_T(t_1) > 0$. Therefore (2.32) must hold for all $n \in N$ and we see that $y_*(t) > 0$ and $z_*(t) > 0$ on $[t_0, T)$. Since $T > t_0$ is arbitrary, we conclude that (y_*, z_*) is a positive proper solution of (A) existing on $[t_0, \infty)$ and satisfying (2.2).

In order to complete the proof we have to show that the solution (y_*, z_*) constructed above can be extended to the entire interval $[a, \infty)$. To this end, we continue (y_*, z_*) to the left of t_0 as a solution of (A), and let J be the maximal interval of existence of this continuation, again denoted by (y_*, z_*) . Assume that J does not coincide with $[a, \infty)$. We then have $J = (\tau_0, \infty)$ with $a \leq \tau_0 < t_0$, and

$$(2.34) \quad \begin{aligned} \lim_{t \rightarrow \tau_0+0} y_*(t) &= \lim_{t \rightarrow \tau_0+0} p(t)(-y'_*(t))^\alpha = \infty, \\ \lim_{t \rightarrow \tau_0+0} z_*(t) &= \lim_{t \rightarrow \tau_0+0} q(t)(-z'_*(t))^\beta = \infty. \end{aligned}$$

Define the functions $u(t)$ and $v(t)$ by

$$(2.35) \quad u(t) = p(t)(-y'_*(t))^\alpha \quad \text{and} \quad v(t) = q(t)(-z'_*(t))^\beta, \quad t \in J$$

which, in view of (A), satisfy

$$(2.36) \quad -u'(t) = \varphi(t)(z_*(t))^\lambda \quad \text{and} \quad -v'(t) = \psi(t)(y_*(t))^\mu, \quad t \in J.$$

Integrating (2.36) from t to t_0 and using the equations

$$\begin{aligned} y_*(t) &= y_*(t_0) + \int_t^{t_0} (p(s))^{-1/\alpha} (u(s))^{1/\alpha} ds, \\ z_*(t) &= z_*(t_0) + \int_t^{t_0} (q(s))^{-1/\beta} (v(s))^{1/\beta} ds, \end{aligned}$$

which readily follow from (2.35), we obtain

$$(2.37) \quad \begin{aligned} u(t) &= u(t_0) + \int_t^{t_0} \varphi(s) \left[z_*(t_0) + \int_s^{t_0} (q(r))^{-1/\beta} (v(r))^{1/\beta} dr \right]^\lambda ds, \\ v(t) &= v(t_0) + \int_t^{t_0} \psi(s) \left[y_*(t_0) + \int_s^{t_0} (p(r))^{-1/\alpha} (u(r))^{1/\alpha} dr \right]^\mu ds, \quad t \in (\tau_0, t_0]. \end{aligned}$$

Noting that $u(t)$ and $v(t)$ are decreasing on $(\tau_0, t_0]$ we see from (2.37) that

$$(2.38) \quad \begin{aligned} u(t) &\leq u(t_0) + \int_t^{t_0} \varphi(s) \left[z_*(t_0) + (v(s))^{1/\beta} \int_a^{t_0} (q(r))^{-1/\beta} dr \right]^\lambda ds, \\ v(t) &\leq v(t_0) + \int_t^{t_0} \psi(s) \left[y_*(t_0) + (u(s))^{1/\alpha} \int_a^{t_0} (p(r))^{-1/\alpha} dr \right]^\mu ds, \quad t \in (\tau_0, t_0], \end{aligned}$$

which clearly implies that

$$\begin{aligned}
 (2.39) \quad & u(t) \leq w_0 + \int_t^{t_0} \chi(s) [\zeta_0 + \rho(v(s))^{1/\beta}]^\lambda ds \leq w_0 + c [\zeta_0 + \rho(v(t))^{1/\beta}]^\lambda \\
 & v(t) \leq w_0 + \int_t^{t_0} \chi(s) [\zeta_0 + \rho(u(s))^{1/\alpha}]^\mu ds \leq w_0 + c [\zeta_0 + \rho(u(t))^{1/\alpha}]^\mu, \\
 & t \in (\tau_0, t_0],
 \end{aligned}$$

where we have put

$$\begin{aligned}
 w_0 &= \max\{u(t_0), v(t_0)\}, \quad \zeta_0 = \max\{y_*(t_0), z_*(t_0)\}, \\
 \rho &= \max\left\{\int_a^{t_0} (p(s))^{-1/\alpha} ds, \int_a^{t_0} (q(s))^{-1/\beta} ds\right\}, \\
 \chi(t) &= \max\{\varphi(t), \psi(t)\}, \quad c = \int_a^{t_0} \chi(s) ds.
 \end{aligned}$$

From (2.39) it follows that

$$\begin{aligned}
 (2.40) \quad & u(t) \leq w_0 + c [\zeta_0 + \rho(w_0 + c(\zeta_0 + \rho(u(t))^{1/\alpha})^\mu)^{1/\beta}]^\lambda \\
 & v(t) \leq w_0 + c [\zeta_0 + \rho(w_0 + c(\zeta_0 + \rho(v(t))^{1/\beta})^\lambda)^{1/\alpha}]^\mu
 \end{aligned}$$

for $t \in (\tau_0, t_0]$. Since $\frac{1}{\alpha}\mu\frac{1}{\beta}\lambda = \frac{1}{\beta}\lambda\frac{1}{\alpha}\mu = \frac{\lambda\mu}{\alpha\beta} < 1$ (2.40) shows that both $u(t)$ and $v(t)$ must remain bounded as $t \rightarrow \tau_0 + 0$. This, however, is a contradiction to (2.34) which implies $\lim_{t \rightarrow \tau_0 + 0} u(t) = \lim_{t \rightarrow \tau_0 + 0} v(t) = \infty$. Therefore, J must coincide with $[a, \infty)$, and we conclude that (A) has a strongly decreasing solution existing on the entire interval $[a, \infty)$.

Example 2.1. Consider the differential system

$$(2.41) \quad \begin{cases} (e^{-\alpha t} |y'|^{\alpha-1} y')' = 2\alpha e^{-(2\alpha-\lambda)t} z^\lambda, \\ (e^{-\beta t} |z'|^{\beta-1} z')' = 2\beta e^{-(2\beta-\mu)t} y^\mu, \end{cases} \quad t \geq 0,$$

where α, β, λ and μ are positive constants. The condition (0.1) holds for this system. Simple computation shows that the conditions (2.7) and (2.8) are satisfied if $\alpha > \lambda$ and $\beta > \mu$. Note that the last inequalities imply (1.1). From Theorems 2.1 and 2.2 it follows that if $\alpha > \lambda$ and $\beta > \mu$, the system (2.41) possesses a moderately decreasing positive solution (y_m, z_m) such that

$$\lim_{t \rightarrow \infty} y_m(t) = c \quad \text{and} \quad \lim_{t \rightarrow \infty} z_m(t) = d$$

for any given positive constants c and d , as well as a strongly decreasing positive solution (y_s, z_s) such that

$$\lim_{t \rightarrow \infty} y_s(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} z_s(t) = 0.$$

We notice that (2.41) has a solution (e^{-t}, e^{-t}) , which is a concrete example of a strongly decreasing solution. No requirement on α , β , λ and μ such as (1.1) is necessary here.

3. Existence of decaying positive proper solutions

Let us turn our attention to positive decreasing proper solutions of the system (A) in which $p(t)$ and $q(t)$ satisfy the condition (0.2). Extensive use is made of the decaying functions $\pi(t)$ and $\rho(t)$ defined by

$$(3.1) \quad \pi(t) = \int_t^\infty (p(s))^{-1/\alpha} ds, \quad \rho(t) = \int_t^\infty (q(s))^{-1/\beta} ds, \quad t \geq a.$$

Our purpose here is to examine the existence of a positive decreasing proper solution (y, z) of (A) such that either

$$(3.2) \quad \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = \text{const} > 0, \quad \lim_{t \rightarrow \infty} \frac{z(t)}{\rho(t)} = \text{const} > 0$$

or

$$(3.3) \quad \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = 0, \quad \lim_{t \rightarrow \infty} \frac{z(t)}{\rho(t)} = 0.$$

A solution of (A) satisfying (3.2) or (3.3) is called a moderately decaying solution or strongly decaying solution, respectively.

Let (y, z) be a moderately decaying solution of (A) on $[a, \infty)$. Integrating (2.3) from t to ∞ , we have

$$(3.4) \quad \begin{aligned} -y'(t) &= \left[(p(t))^{-1} \left(\eta^\alpha + \int_t^\infty \varphi(s)(z(s))^\lambda ds \right) \right]^{1/\alpha}, \\ -z'(t) &= \left[(q(t))^{-1} \left(\zeta^\beta + \int_t^\infty \psi(s)(y(s))^\mu ds \right) \right]^{1/\beta}, \quad t \geq a, \end{aligned}$$

where η and ζ are positive constants given by

$$\eta = - \lim_{t \rightarrow \infty} (p(t))^{1/\alpha} y'(t) = \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)}, \quad \zeta = - \lim_{t \rightarrow \infty} (q(t))^{1/\beta} z'(t) = \lim_{t \rightarrow \infty} \frac{z(t)}{\rho(t)}.$$

Integration of (3.4) from t to ∞ then yields

$$(3.5) \quad \begin{aligned} y(t) &= \int_t^\infty \left[(p(s))^{-1} \left(\eta^\alpha + \int_s^\infty \varphi(r)(z(r))^\lambda dr \right) \right]^{1/\alpha} ds, \\ z(t) &= \int_t^\infty \left[(q(s))^{-1} \left(\zeta^\beta + \int_s^\infty \psi(r)(y(r))^\mu dr \right) \right]^{1/\beta} ds, \quad t \geq a. \end{aligned}$$

Naturally, letting $\eta = \zeta = 0$ in (3.5), we obtain the integral equations for a strongly decaying solution of (A).

The existence of a moderately decreasing solution for (A) can be characterized as the following theorem indicates.

Theorem 3.1. *Suppose that (0.2) holds and that $\alpha\beta \neq \lambda\mu$. Then system (A) has a positive decreasing proper solution (y, z) satisfying (3.2) if and only if*

$$(3.6) \quad \int_a^\infty \varphi(t)(\rho(t))^\lambda dt < \infty \quad \text{and} \quad \int_a^\infty \psi(t)(\pi(t))^\mu dt < \infty.$$

Proof. (The “only if” part) In deriving (3.4) we found the convergence of the integrals

$$\int_t^\infty \varphi(s)(z(s))^\lambda ds \quad \text{and} \quad \int_t^\infty \psi(s)(y(s))^\mu ds$$

for all $t \geq a$. This fact combined with (3.2) implies the truth of (3.6).

(The “if” part) Suppose that (3.6) holds. Choose η and ζ so that

$$(3.7) \quad \begin{aligned} \int_a^\infty \varphi(t)(\rho(t))^\lambda dt &\leq (2^\alpha - 1)\eta^\alpha(2\zeta)^{-\lambda} \quad \text{and} \\ \int_a^\infty \psi(t)(\pi(t))^\mu dt &\leq (2^\beta - 1)\zeta^\beta(2\eta)^{-\mu}. \end{aligned}$$

Denote by $\mathcal{Y}$ the set of all vector functions $(y, z) \in C[a, \infty) \times C[a, \infty)$ such that

$$(3.8) \quad \eta\rho(t) \leq y(t) \leq 2\eta\rho(t), \quad \zeta\rho(t) \leq z(t) \leq 2\zeta\rho(t), \quad t \geq a,$$

and let $\mathcal{F}$ denote the mapping

$$(3.9) \quad \mathcal{F}(y, z)(t) = (\mathcal{G}z(t), \mathcal{H}y(t)), \quad (y, z) \in \mathcal{Y},$$

where $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$(3.10) \quad \mathcal{G}z(t) = \int_t^\infty \left[(p(s))^{-1} \left(\eta^\alpha + \int_s^\infty \varphi(r)(z(r))^\lambda dr \right) \right]^{1/\alpha} ds,$$

$$(3.11) \quad \mathcal{H}y(t) = \int_t^\infty \left[(q(s))^{-1} \left(\zeta^\beta + \int_s^\infty \psi(r)(y(r))^\mu dr \right) \right]^{1/\beta} ds, \quad t \geq a.$$

It can be shown that (i) $\mathcal{F}$ map $\mathcal{Y}$ into itself, (ii) $\mathcal{F}$ is a continuous mapping, and (iii) $\mathcal{F}(\mathcal{Y})$ is a relatively compact subset of $C[a, \infty) \times C[a, \infty)$. Therefore, by the Schauder–Tychonoff theorem, there exists a fixed element $(y, z) \in \mathcal{Y}$ of $\mathcal{F}$, which satisfies the system (3.5) of integral equations for $t \geq a$. Hence (y, z) is a moderately decaying solution of (A) defined on $[a, \infty)$ and satisfying $\lim_{t \rightarrow \infty} y(t)/\pi(t) = \eta$ and $\lim_{t \rightarrow \infty} z(t)/\rho(t) = \zeta$. This completes the proof.

Theorem 3.2. *Suppose that (0.2) holds. If (1.1) and (3.6) hold, then the system (A) possesses a positive decreasing proper solution (y, z) satisfying (3.3).*

Proof. Choose $t_0 \geq a$ so that

$$(3.12) \quad \int_{t_0}^{\infty} \varphi(t)(\rho(t))^{\lambda} dt \leq 2^{-\lambda/\beta}$$

and

$$(3.13) \quad \int_{t_0}^{\infty} \psi(t)(\pi(t))^{\mu} dt \leq 2^{-\mu/\alpha}.$$

For every $n \in \mathbb{N}$ let $\mathcal{Y}_n$ denote the set of functions $(y, z) \in C[t_0, \infty) \times C[t_0, \infty)$ satisfying

$$(3.14) \quad \frac{1}{n}\pi(t) \leq y(t) \leq 2\pi(t), \quad \frac{1}{n}\rho(t) \leq z(t) \leq 2\rho(t), \quad t \geq t_0,$$

and consider the mapping $\mathcal{F}_n : \mathcal{Y}_n \rightarrow C[t_0, \infty) \times C[t_0, \infty)$ defined by

$$(3.15) \quad \mathcal{F}_n(y, z) = (\mathcal{G}_n z, \mathcal{H}_n y), \quad (y, z) \in \mathcal{Y}_n,$$

where

$$(3.16) \quad \mathcal{G}_n z(t) = \int_t^{\infty} \left[(p(s))^{-1} \left(\frac{1}{n^{\alpha}} + \int_s^{\infty} \varphi(r)(z(r))^{\lambda} dr \right) \right]^{1/\alpha} ds,$$

$$(3.17) \quad \mathcal{H}_n y(t) = \int_t^{\infty} \left[(q(s))^{-1} \left(\frac{1}{n^{\beta}} + \int_s^{\infty} \psi(r)(y(r))^{\mu} dr \right) \right]^{1/\beta} ds, \quad t \geq t_0.$$

It can be verified without difficulty that the Schauder–Tychonoff fixed point theorem applies to $\mathcal{F}_n$, giving rise to a sequence of moderately decaying solutions of (A), $\{(y_n(t), z_n(t))\}_{n=1}^{\infty}$, defined on the interval $[t_0, \infty)$. The sequence is clearly relatively compact in $C[t_0, \infty) \times C[t_0, \infty)$, and hence it contains a subsequence converging to a limit function $(y_*(t), z_*(t))$ in $C[t_0, \infty) \times C[t_0, \infty)$. Since $(y_n(t), z_n(t))$ satisfies

$$(3.18) \quad y_n(t) = \int_t^\infty \left[(p(s))^{-1} \left(\frac{1}{n^\alpha} + \int_s^\infty \varphi(r)(z_n(r))^\lambda dr \right) \right]^{1/\alpha} ds,$$

$$(3.19) \quad z_n(t) = \int_t^\infty \left[(q(s))^{-1} \left(\frac{1}{n^\beta} + \int_s^\infty \psi(r)(y_n(r))^\mu dr \right) \right]^{1/\beta} ds, \quad t \geq t_0,$$

it follows that

$$(3.20) \quad y_*(t) = \int_t^\infty \left[(p(s))^{-1} \int_s^\infty \varphi(r)(z_*(r))^\lambda dr \right]^{1/\alpha} ds,$$

$$(3.21) \quad z_*(t) = \int_t^\infty \left[(q(s))^{-1} \int_s^\infty \psi(r)(y_*(r))^\mu dr \right]^{1/\beta} ds, \quad t \geq t_0,$$

which shows that (y_*, z_*) is a solution of the system (A) on $[t_0, \infty)$.

The positivity of (y_*, z_*) on $[t_0, \infty)$ is verified in the same manner as in the proof of Theorem 2.2. The condition (1.1) is required to prove this.

In order to complete the proof it suffices to continue (y_*, z_*) as a solution of (A) to the entire interval $[a, \infty)$. The procedure is exactly the same as that mentioned in the proof of Theorem 2.2.

Example 3.3. The above results will be illustrated by the example

$$(3.22) \quad \begin{cases} ((\cosh t)^\alpha |y'|^{\alpha-1} y')' = \alpha 2^{1-\alpha} e^{(\lambda-2)t} (1 + e^{-2t})^{\alpha-1} |z|^{\lambda-1} z \\ ((\cosh t)^\beta |z'|^{\beta-1} z')' = \beta 2^{1-\beta} e^{(\mu-2)t} (1 + e^{-2t})^{\beta-1} |y|^{\mu-1} y, \end{cases} \quad t \geq 0,$$

where α, β, λ and μ are positive constants. Clearly, $p(t) = (\cosh t)^\alpha$ and $q(t) = (\cosh t)^\beta$ satisfy (0.2) and the corresponding functions $\pi(t)$ and $\rho(t)$ (cf. (3.1)) are given by $\pi(t) = \rho(t) = \pi - 2 \tan^{-1} e^t$, $t \geq 0$. In calculating the integrals in (3.6), we can use $2e^{-t}$ in place of $\pi(t)$ and $\rho(t)$, since $\lim_{t \rightarrow \infty} e^t \pi(t) = \lim_{t \rightarrow \infty} e^t \rho(t) = 2$, and regard the functions

$$\varphi(t) = \alpha 2^{1-\alpha} e^{(\lambda-2)t} (1 + e^{-2t})^{\alpha-1} \quad \text{and} \quad \psi(t) = \beta 2^{1-\beta} e^{(\mu-2)t} (1 + e^{-2t})^{\beta-1}$$

to be positive constant multiples of $e^{(\lambda-2)t}$ and $e^{(\mu-2)t}$, respectively. Taking these facts into account, we find that (3.6) certainly holds.

Theorem 3.1 then implies that if $\alpha\beta \neq \lambda\mu$, then for any given constants $\eta > 0$ and $\zeta > 0$, the system (3.22) has a moderately decaying solution (y_m, z_m) such that

$$\lim_{t \rightarrow \infty} e^t y_m(t) = \eta \quad \text{and} \quad \lim_{t \rightarrow \infty} e^t z_m(t) = \zeta > 0.$$

A concrete example of such solutions is (e^{-t}, e^{-t}) . Applying Theorem 3.2, we see that, if (1.1) holds, then there exists a strongly decaying solution (y_s, z_s) of

(3.22) such that

$$\lim_{t \rightarrow \infty} e^t y_s(t) = \lim_{t \rightarrow \infty} e^t z_s(t) = 0.$$

4. Application

We will show that the above results for (A) can be applied to the study of positive radial solutions to systems of partial differential equations of the form (C) in an exterior domain E_a , $a > 0$.

A radial function $(u, v) = (y(|x|), z(|x|))$ is a solution of (C) in E_a if and only if $(y(t), z(t))$ satisfies the ordinary differential system

$$(4.1) \quad \begin{cases} (t^{N-1}|y'|^{m-2}y')' = t^{N-1}f(t)|z|^{\lambda-1}z \\ (t^{N-1}|z'|^{n-2}z')' = t^{N-1}g(t)|y|^{\mu-1}y, \quad t \geq a, \end{cases}$$

which is a special case of (A) with $\alpha = m - 1$, $\beta = n - 1$, $p(t) = q(t) = t^{N-1}$, $\varphi(t) = t^{N-1}f(t)$ and $\psi(t) = t^{N-1}g(t)$. It is assumed that $m > 1$, $n > 1$, $\lambda > 0$, $\mu > 0$ and $f(t)$ and $g(t)$ are positive and continuous on $[a, \infty)$.

First, we have the following statement from Theorem 1.1 applied to (4.1).

I. Suppose that

$$(4.2) \quad (m-1)(n-1) > \lambda\mu.$$

Then, for any fixed b , $b > a$, there exists a positive radial solution (u, v) of (C) which is positive on $E(a, b) = \{x \in \mathbf{R}^N : a \leq |x| < b\}$ and satisfies

$$(4.3) \quad u(x) = v(x) = 0 \quad \text{on } S_b = \{x \in \mathbf{R}^N : |x| = b\}.$$

We now restrict our attention to the following special case of (C)

$$(D) \quad \begin{cases} \operatorname{div}(|Du|^{m-2}Du) = |x|^{-k}|v|^{\lambda-1}v \\ \operatorname{div}(|Dv|^{n-2}Dv) = |x|^{-l}|u|^{\mu-1}u, \end{cases}$$

k and l being constants, and examine the existence of positive radial solutions of (D) defined on E_a in the case where

$$(4.4) \quad N > m \quad \text{and} \quad N > n.$$

In this case $p(t) = q(t) = t^{N-1}$ satisfy the condition (0.2), and the functions $\pi(t)$ and $\rho(t)$ in (3.1) reduce, respectively, to

$$(4.5) \quad \pi(t) = \frac{m-1}{N-m} t^{-(N-m)/(m-1)}, \quad \rho(t) = \frac{n-1}{N-n} t^{-(N-n)/(n-1)}, \quad t \geq a.$$

The condition (3.6) for $\varphi(t) = t^{N-1-k}$ and $\psi(t) = t^{N-1-l}$ reads as follows:

$$(4.6) \quad \int_a^\infty \varphi(t)(\rho(t))^\lambda dt < \infty \quad \Leftrightarrow \quad k > N - \frac{\lambda(N-n)}{n-1},$$

$$(4.7) \quad \int_a^\infty \psi(t)(\pi(t))^\mu dt < \infty \quad \Leftrightarrow \quad l > N - \frac{\mu(N-m)}{m-1}.$$

Theorem 3.1 regarding the moderately decaying solutions of (A) yields the following result for (D).

II. Suppose that $(m-1)(n-1) \neq \lambda\mu$. If (4.4), (4.6) and (4.7) hold, then there exists a positive radial solution (u, v) of (D) on E_a satisfying

$$(4.8) \quad \lim_{|x| \rightarrow \infty} |x|^{(N-m)/(m-1)} u(x) = \text{const} > 0, \quad \lim_{|x| \rightarrow \infty} |x|^{(N-n)/(n-1)} v(x) = \text{const} > 0.$$

We present via Theorem 3.2 sufficient conditions for (D) to have a positive radial solution (u, v) such that

$$(4.9) \quad \lim_{|x| \rightarrow \infty} |x|^{(N-m)/(m-1)} u(x) = 0, \quad \lim_{|x| \rightarrow \infty} |x|^{(N-n)/(n-1)} v(x) = 0.$$

III. Suppose that (4.2), (4.4), (4.6) and (4.7) are satisfied. Then, (D) has a positive radial solution (u, v) on E_a satisfying (4.9).

The results of Section 2 could also be applied to formulate two propositions, analogues of the above II and III, for the elliptic system (D) in the case where

$$N \leq m \quad \text{and} \quad N \leq n.$$

We notice here that the qualitative study of elliptic systems of the type (C) slightly different angles has been made in the papers [1, 3, 5].

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