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On the d'Alembert Type Functional Equation in Hilbert Algebras

By

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1. Introduction

The classical d'Alembert's functional equation has the form:

$$(1) \quad f(x+y) + f(x-y) = 2f(x)f(y),$$

where x and y run over an Abelian group G and the unknown function f is defined on G and assumes values in the complex field $\mathbf{C}$. At present the theory of d'Alembert's equation is extensively developed (see, i.e. the references in [2]). The basic result for the study of Eq. (1) in the scalar case is a result obtained by Kannappan [8] (see also the monograph [1] by Aczél and Dhombres for results and references). It says that every solution $f \neq 0$ of d'Alembert's functional equation (1) has the form:

$$f(x) = \frac{1}{2}(m(x) + m(-x)), \quad x \in G,$$

where $m : G \rightarrow \mathbf{C}$ is an exponential function, i.e. satisfies the following equation

$$m(x+y) = m(x)m(y), \quad x, y \in G.$$

In particular, all non-zero bounded continuous solution of Eq. (1) on a locally compact Abelian group G are represented by the formula:

$$f(x) = \operatorname{Re} \gamma(x), \quad x \in G$$

with some continuous character γ of the group G (see Ger [6]). So, on the real line, i.e. when $G = \mathbf{R}$, all non-zero bounded continuous solution of Eq. (1) can be expressed in the form: $f(x) = \cos \alpha x$, $x \in \mathbf{R}$, where α is a real constant.

In the operator valued version of Eq. (1) let H denotes a Hilbert space and let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H . A cosine function on an Abelian group G with values in $\mathcal{L}(H)$ is a map $f : G \rightarrow \mathcal{L}(H)$ such that f satisfies d'Alembert's equation (1) and the following condition:

$$(2) \quad f(0) = I$$

where 0 stands for the neutral element of G and I denotes the identity operator on H .

The cosine function on the real line are related to the Cauchy problem for 2nd order differential equation (see Fattorini [5]).

For matrices this problem was solved by Székelyhidi [15] and Rukhin [13] for Abelian groups which are divisible by 2.

Chojnacki [2] and Stetkaer [14] (for a generalize functional equation) proved that any uniformly bounded, normal, weakly continuous cosine function f on a locally compact Abelian group G has the form:

$$(3) \quad f(x) = \frac{1}{2}(U(x) + U(-x)), \quad x \in G,$$

where U is a strongly continuous unitary representation of G on H . Dispensing with the normality condition Fattorini [4], Kurepa [9], and Stetkaer [14] showed that there is an inner product on H , equivalent with the original one, with respect to which the value $f(x)$ is a hermitan operator, for all $x \in G$. As a consequence of the Stetkaer result [14], Theorem VII.3, (containing also the Dixmier theorem on a uniformly bounded representation of an Abelian group on a Hilbert space [3]) we have that for every uniformly bounded cosine function $f : G \rightarrow \mathcal{L}(H)$ there exists an inner product on H , equivalent with the original one, with respect to which $f(x)^* = f(-x)$, for all $x \in G$.

As a continuation of these investigations in the present paper we examine the following functional equation:

$$(4) \quad f(x+y) + f(x-y) = 2f(x)f(y)^*$$

for the function f fulfilling the following condition (compare with (2)):

$$(5) \quad \|f(0)\| = 1,$$

where f is defined on an Abelian group G with values in a $*$ -algebra $\mathcal{A}$.

Let us recall (see Rickard [12]) that a mapping $u \rightarrow u^*$ of a complex algebra $\mathcal{A}$ onto itself is called an *involution* if

$$(6) \quad (u^*)^* = u;$$

$$(7) \quad (u+v)^* = u^* + v^*;$$

$$(8) \quad (u \cdot v)^* = v^* \cdot u^*;$$

$$(9) \quad (\alpha u)^* = \bar{\alpha} u^*,$$

for all $u, v \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

An algebra with an involution is called a $*$ -algebra.

A $\mathcal{U}$ - $*$ -algebra [$\mathcal{H}$ - $*$ -algebra] is a complex inner product [Hilbert] space which is at same time a normed [Banach] $*$ -algebra and

$$(10) \quad (uv|w) = (v|u^*w), \quad (uv|w) = (v|wu^*),$$

for all $u, v, w \in \mathcal{A}$.

The purpose of this paper is determine the general (Theorem 1) and the continuous (Theorem 2) solution of Eq. (4) under assumption (5). Moreover, we describe the form of a continuous exponential function defined on a topological group G with values in a $\mathcal{H}^*$ -algebra in terms of unitary representations of G (see Lemma 3).

The main results, Theorems 1 and 2, are formulated for an Abelian group. A form of a solution of Eq. (4) in the non-commutative case is not known.

2. A general solution

We start with the following

Lemma 1. *Let G be a group, let $\mathcal{A}$ be a $*$ -algebra and let $f : G \rightarrow \mathcal{A}$ satisfies Eq. (4). Then*

- (i) $f(-x) = f(x)$, $x \in G$;
- (ii) $f(x)^* = f(x)$, $x \in G$;
- (iii) $f(x+y) = f(y+x)$, $x, y \in G$;
- (iv) $f(x)f(y) = f(y)f(x)$, $x, y \in G$;
- (v) $f(x) = f(x)f(0) = f(0)f(x)$, $x \in G$.

Proof. Inserting $x = 0$ and $y = x$ to (4), we get

$$\begin{aligned} f(x) + f(-x) &= 2f(0)f(x)^* = (2f(x)f(0)^*)^* = (f(x+0) + f(x-0))^* \\ &= 2f(x)^*, \quad x \in G, \end{aligned}$$

whence

$$(11) \quad 2f(x)^* = f(x) + f(-x), \quad x \in G$$

and

$$2f(-x)^* = f(-x) + f(x), \quad x \in G.$$

Therefore,

$$f(x)^* = f(-x)^*, \quad x \in G$$

and

$$f(x) = f(-x), \quad x \in G.$$

Applying (i) to (11), we obtain

$$2f(x)^* = f(x) + f(x) = 2f(x), \quad x \in G,$$

which means that (ii) holds.

Now, using (ii) and (i), from Eq. (4) we have

$$\begin{aligned}
 f(y+x) + f(y-x) &= 2f(y)f(x)^* = (2f(x)f(y)^*)^* \\
 &= (f(x+y) + f(x-y))^* = f(x+y)^* + f(x-y)^* \\
 &= f(x+y) + f(x-y) = f(x+y) + f(-(y-x)) \\
 &= f(x+y) + f(y-x), \quad x, y \in G
 \end{aligned}$$

and condition (iii) holds.

Assertion (iv) results from (ii), (4), (iii) and (i). Indeed,

$$\begin{aligned}
 2f(x)f(y) &= 2f(x)f(y)^* = f(x+y) + f(x-y) = f(y+x) + f(-(y-x)) \\
 &= f(y+x) + f(y-x) = 2f(y)f(x)^* \\
 &= 2f(y)f(x), \quad x, y \in G.
 \end{aligned}$$

Finally, substituting $y = 0$ to (4) and using (i) and (iv) we infer that

$$f(x) + f(x) = 2f(x)f(0)^* = 2f(x)f(0) = 2f(0)f(x), \quad x \in G,$$

which completes the proof.

Lemma 2. *Let G be an Abelian group, let $\mathcal{A}$ be a $*$ -algebra. Assume that $f : G \rightarrow \mathcal{A}$ satisfies Eq. (4) and the following condition*

$$(12) \quad f(x)f(x) = f(0), \quad x \in G.$$

Then

$$f(x+y) = f(x)f(y)^* \quad \text{and} \quad f(x-y) = f(x)f(y)^*, \quad x, y \in G.$$

Proof. Inserting $y = x$ to (4) our assumption (12) implies that

$$f(2x) + f(0) = 2f(x)f(x)^* = 2f(x)f(x) = 2f(0), \quad x \in G,$$

whence

$$(13) \quad f(2x) = f(0), \quad x \in G.$$

Moreover, taking $x = x+y$ and $y = x-y$ in (4), we have

$$2f(x+y)f(x-y) = f(2x) + f(2y) = f(0) + f(0) = 2f(0), \quad x, y \in G.$$

Hence,

$$(14) \quad f(x+y)f(x-y) = f(0), \quad x, y \in G.$$

Let $x, y \in G$ be fixed. Then, from (12), (v), (ii), (14) and (13), we deduce that

$$\begin{aligned}
& f(x+y) - f(x)f(y)^* \\
&= f(0)f(x+y) - f(0)f(x)f(y) \\
&= f(x-y)f(x-y)f(x+y) - f(x-y)f(x+y)f(x)f(y) \\
&= f(x-y)f(0) - f(x-y)\left(\frac{1}{2}f(2x+y) + \frac{1}{2}f(y)\right)f(y) \\
&= f(x-y)\left(f(0) - \frac{1}{2}(f(2x+y)f(y) + f(y)f(y))\right) \\
&= f(x-y)\left(f(0) - \frac{1}{2}\left(\frac{1}{2}f(2x) + \frac{1}{2}f(2x+2y) + f(0)\right)\right) \\
&= f(x-y)\left(f(0) - \frac{1}{2}\left(\frac{1}{2}f(0) + \frac{1}{2}f(0) + f(0)\right)\right) \\
&= f(x-y) \cdot 0 = 0,
\end{aligned}$$

which yields that the function f satisfies the following functional equation:

$$f(x+y) = f(x)f(y)^*, \quad x, y \in G.$$

Moreover, the last identity with $y = -y$ implies that

$$f(x-y) = f(x)f(-y)^*, \quad x, y \in G.$$

This condition combined with (i) shows that

$$f(x-y) = f(x)f(y)^*, \quad x, y \in G$$

and the proof is finished.

The main result of this part of our paper is the following

Theorem 1. *Let $(G, +)$ be an Abelian group and let f be a function defined on G with values in a $\mathcal{U}^*$ -algebra $\mathcal{A}$ satisfying condition (5). Then f satisfies Eq. (4) if and only if there exists a function $m : G \rightarrow \mathcal{A}$ such that*

$$(15) \quad f(x) = \frac{1}{2}(m(x) + m(-x)), \quad x \in G$$

and

$$(16) \quad m(x+y) = m(x)m(y)^*, \quad x, y \in G$$

or

$$(17) \quad m(x-y) = m(x)m(y)^* = m(y)^*m(x), \quad x, y \in G.$$

Proof. If f satisfies (12) or $f = 0$ then $m = f$ is a function having all required properties.

Assume that

$$(18) \quad f(a) \cdot f(a) \neq f(0),$$

for some $a \in G$. Then directly from Eq. (4) and the Lemma 1 it follows that

$$f(2a) + f(0) = 2f(a)f(a)^* = 2f(a)f(a)$$

and

$$(19) \quad f(2a) \neq f(0).$$

We define the function $g : G \rightarrow \mathcal{A}$ by

$$(20) \quad g(x) = f(x+a) - f(x-a), \quad x \in G.$$

Applying (i) we derive

$$\begin{aligned} g(x) + g(-x) &= f(x+a) - f(x-a) + f(-x+a) - f(-x-a) \\ &= f(x+a) - f(x-a) + f(-(-a+x)) - f(-(a+x)) \\ &= f(x+a) - f(x-a) + f(-a+x) - f(a+x) \\ &= f(x+a) - f(x-a) + f(x-a) - f(x+a) \\ &= 0, \quad x \in G, \end{aligned}$$

whence,

$$(21) \quad g(x) = -g(-x), \quad x \in G.$$

Now we shall prove two following conditions which describe the connections between f and g

$$(22) \quad f(x)g(y) + g(x)f(y) = g(x+y), \quad x, y \in G$$

and

$$(23) \quad g(x)g(y) = (f(2a) - f(0))(f(x+y) - f(x-y)), \quad x, y \in G.$$

In fact, for fixed $x, y \in G$, from Eq. (4) and (i) we have

$$\begin{aligned} &2(f(x)g(y) + g(x)f(y)) \\ &= 2f(x)f(y+a) - 2f(x)f(y-a) + 2f(x+a)f(y) - 2f(x-a)f(y) \\ &= f(x+y+a) + f(x-a-y) - f(x+y-a) - f(x+a-y) \\ &\quad + f(x+a+y) + f(x+a-y) - f(x-a+y) - f(x-a-y) \\ &= 2(f(x+y+a) - f(x+y-a)) \\ &= 2g(x+y). \end{aligned}$$

Further, from Eq. (4) and (v) we infer that

$$\begin{aligned}
 2g(x)g(y) &= 2(f(x+a) - f(x-a))(f(y+a) - f(y-a)) \\
 &= 2f(x+a)f(y+a) - 2f(x+a)f(y-a) \\
 &\quad - 2f(x-a)f(y+a) + f(x-a)f(y-a) \\
 &= f(x+y+2a) + f(x-y) - f(x+y) - f(x-y+2a) \\
 &\quad - f(x+y) - f(x-y-2a) + f(x+y-2a) + f(x-y) \\
 &= f(x+y+2a) + f(x+y-2a) - (f(x-y+2a) \\
 &\quad + f(x-y-2a)) - 2f(x+y) + 2f(x-y) \\
 &= 2f(x+y)f(2a) - 2f(x-y)f(2a) - 2f(x+y) + 2f(x-y) \\
 &= 2(f(2a) - f(0))(f(x+y) - f(x-y)).
 \end{aligned}$$

The last two identities combined with (i) and (21) imply that

$$(24) \quad -f(x)g(y) + g(x)f(y) = g(x-y), \quad x, y \in G$$

and

$$(25) \quad -g(x)g(y) = (f(2a) - f(0))(f(x-y) - f(x+y)), \quad x, y \in G.$$

Let us also observe that

$$(26) \quad f(0) \neq 0.$$

Indeed, if $f(0) = 0$, then from (v) we deduce that $f(x) = f(x)f(0) = 0$, for all $x \in G$ (then $m = f = 0$).

Now, applying successively (5), (4), (ii) and (i), we derive the Gram determinant of the elements $f(2a)$ and $f(0)$

$$\begin{aligned}
 0 &\leq G(f(2a), f(0)) \\
 &= (f(2a)|f(2a))(f(0)|f(0)) - (f(2a)|f(0))(f(0)|f(2a)) \\
 &= (f(2a)|f(2a))\|f(0)\|^2 - (f(2a)|f(0))(f(0)|f(2a)) \\
 &= (2f(a)^2 - f(0)|2f(a)^2 - f(0)) \\
 &\quad - (2f(a)^2 - f(0)|f(0))(f(0)|2f(a)^2 - f(0)) \\
 &= 4(f(a)^2|f(a)^2) - 2(f(a)^2|f(0)) - 2(f(0)|f(a)^2) + (f(0)|f(0)) \\
 &\quad - (2(f(a)^2|f(0)) - (f(0)|f(0)))(2(f(0)|f(a)^2) - (f(0)|f(0))) \\
 &= 4\|f(a)^2\|^2 - 2(f(a)|f(a)^*f(0)) - 2(f(a)^*f(0)|f(a)) + 1 \\
 &\quad - (2(f(a)|f(a)^*f(0)) - 1)(2(f(a)^*f(0)|f(a)) - 1)
 \end{aligned}$$

$$\begin{aligned}
&= 4\|f(a)^2\|^2 - 4(f(a)|f(a)) + 1 - (2(f(a)|f(a)) - 1)(2(f(a)|f(a)) - 1) \\
&= 4\|f(a)^2\|^2 - 4\|f(a)\|^2 + 1 - (2\|f(a)\|^2 - 1)(2\|f(a)\|^2 - 1) \\
&= 4\|f(a)^2\|^2 - 4\|f(a)\|^2 + 1 - 4\|f(a)\|^4 + 4\|f(a)\|^2 - 1 \\
&\leq 0, \quad x, y \in G.
\end{aligned}$$

Hence,

$$G(f(2a), f(0)) = 0,$$

which means that $f(2a)$ and $f(0) \neq 0$ are linearly dependent. Consequently, there exists $\beta \in \mathbf{C}$ such that

$$f(2a) = \beta f(0).$$

Therefore $f(2a) - f(0) = (\beta - 1)f(0)$ and by (19) we infer that $\beta - 1 \neq 0$. Moreover, condition (ii) yields that $\beta \in \mathbf{R}$. Let

$$(27) \quad 2\alpha^2 = \frac{1}{\beta - 1}.$$

Then $\alpha \in \mathbf{R}$ or $\alpha \in i\mathbf{R}$.

After these auxiliary calculation we define the function $m: G \rightarrow \mathcal{A}$ as follows

$$(28) \quad m(x) = f(x) + \alpha g(x), \quad x \in G.$$

Directly from (i) and (21) it follows that m satisfies (15).

Further, if $\alpha \in \mathbf{R}$, then using (ii), (22), (23) and (4) we obtain

$$\begin{aligned}
&m(x)m(y)^* \\
&= (f(x) + \alpha g(x))(f(y) + \alpha g(y))^* \\
&= (f(x) + \alpha g(x))(f(y) + \alpha g(y)) \\
&= f(x)f(y) + \alpha^2 g(x)g(y) + \alpha(f(x)g(y) + g(x)f(y)) \\
&= \frac{1}{2}(f(x+y) + f(x-y)) + \alpha^2(f(2a) - f(0))(f(x+y) - f(x-y)) \\
&\quad + \alpha g(x+y) \\
&= \frac{1}{2}(f(x+y) + f(x-y)) + \frac{1}{2}(f(x+y) - f(x-y)) + \alpha g(x+y) \\
&= m(x+y), \quad x, y \in G,
\end{aligned}$$

which ends the proof in this case.

If $\alpha \in i\mathbf{R}$, then from (iv) and the definitions of m and g we have

$$m(x)m(y)^* = m(y)^*m(x), \quad c, y \in G.$$

Moreover, repeating the arguments used in the proof of the last identity, with (22) and (23) replaced by (24) and (25) we get (17), which completes the proof of the first part of the theorem.

The proof of the converse statement relies on a direct verification that Eq. (4) holds and is therefore omitted.

3. A continuous solution

In the present section, $(G, +)$ stands for topological group. A form of the continuous solution f of Eq. (4) we shall describe in terms of representations of the group G . At this place we recall some definitions used in the sequel.

By a representation of a topological group G in a Hilbert space H we mean a map $\mathcal{R}$ from G into the space $\mathcal{L}(H)$ of all continuous linear operators defined on H onto H such that

$$\mathcal{R}(x+y) = \mathcal{R}(x)\mathcal{R}(y), \quad x, y \in G$$

and

$$\mathcal{R}(0) = I_H.$$

A representation $\mathcal{R}$ is said continuous if for each vector $u \in H$ the map

$$G \ni x \mapsto \mathcal{R}(x)u \in H$$

is continuous.

A vector $u_0 \in H$ is called a cyclic vector of a representation $\mathcal{R}$ if the space $\text{cl Lin}\{\mathcal{R}(x)u_0 : x \in G\}$ coincides with the whole of H . Here and later the symbol $\text{Lin } A$ designates the linear space spanned by a set $A \subset H$, whereas cl stands for the closure operation in the norm topology of H .

To find a form of continuous solutions of Eq. (4) we start with the following result of the type proved by Naimark in [10] (Chap. VI, Sect. 30, Theorem 1) for positive definite functions.

Lemma 3. *Let $(G, +)$ be a topological group and let $\mathcal{H}$ be a $\mathcal{H}^*$ -algebra. Suppose that $m : G \rightarrow \mathcal{H}$ is a continuous function. Then m satisfies the following functional equation*

$$(29) \quad m(x+y) = m(x)m(y), \quad x, y \in G$$

if and only if there exists a continuous representation $\mathcal{R}$ of a group G in the space $\mathcal{H}_0 = \text{cl Lin } m(G)$ with the cyclic vector $u_0 = m(0)$ such that

$$(30) \quad m(x) = \mathcal{R}(x)u_0, \quad x \in G$$

and

$$(31) \quad \mathcal{R}(x)u = \mathcal{R}(x)u_0u, \quad x \in G, \quad u \in \mathcal{H}_0.$$

Proof. Suppose that there exists a representation $\mathcal{R}$ and a vector u_0 such that formulae (30) and (31) hold. Then we have

$$\begin{aligned} m(x+y) &= \mathcal{R}(x+y)u_0 = \mathcal{R}(x)\mathcal{R}(y)u_0 = \mathcal{R}(x)m(y) = \mathcal{R}(x)u_0m(y) \\ &= m(x)m(y), \quad x, y \in G, \end{aligned}$$

which means that m defined by (30) satisfies Eq. (29).

Conversely, assume that m is a solution of Eq. (29). Then, for $x \in G$, we define $L(x) : \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$L(x)u := m(x)u, \quad u \in \mathcal{H}.$$

The operator $L(x)$ is linear and continuous. Moreover, the map

$$G \ni x \mapsto L(x)u \in \mathcal{H},$$

for all $u \in \mathcal{H}$, is continuous since the function m is also.

Let us observe that

$$\begin{aligned} L(x)u &= m(x)u = m(x+0)u = m(x)m(0)u \\ &= L(x)u_0u, \quad x \in G, \quad u \in \mathcal{H} \end{aligned}$$

and

$$\begin{aligned} L(x+y)u &= m(x+y)u = m(x)m(y)u = m(x)L(y)u \\ &= L(x)L(y)u, \quad x, y \in G, \quad u \in \mathcal{H}. \end{aligned}$$

Now we put $\mathcal{H}_0 := \text{cl Lin } m(G)$ and, for each fixed $x \in G$, we define an operator $\mathcal{R}(x)$ in $\mathcal{H}_0$ by

$$\mathcal{R}(x) := L(x)|_{\mathcal{H}_0}.$$

Then $\mathcal{R}(x)$, for $x \in G$, is a linear continuous operator on $\mathcal{H}_0$ and the map

$$G \ni x \mapsto \mathcal{R}(x)u \in \mathcal{H}$$

is continuous, for all $u \in \mathcal{H}_0$.

Moreover,

$$\mathcal{R}(x)(\text{Lin } m(G)) \subset \text{Lin } m(G).$$

Indeed, if $u \in \text{Lin } m(G)$ and u admits the expansion

$$(32) \quad u = \sum_{i=1}^n \alpha_i m(x_i),$$

for some $\alpha_i \in \mathbf{C}$ and $x_i \in G$, $i = 1, \dots, n$, $n \in \mathbf{N}$, then

$$\begin{aligned} \mathcal{R}(x)u &= L(x)u = m(x)u = \sum_{i=1}^n \alpha_i m(x)m(x_i) \\ &= \sum_{i=1}^n \alpha_i m(x+x_i) \in \text{Lin } m(G). \end{aligned}$$

Whence, the continuity of $L(x)$, $x \in G$, imply that

$$\mathcal{R}(x)(\mathcal{H}_0) \subset \mathcal{H}_0.$$

To the proof that $\mathcal{R}(x)$ is surjective, for every $x \in G$, first we observe that

$$\mathcal{R}(0) = \mathbf{I}_{\mathcal{H}_0}.$$

Indeed, for $u \in \text{Lin } m(G)$ of form (32), we have

$$\begin{aligned} L(0)u &= m(0) \sum_{i=1}^n \alpha_i m(x_i) = \sum_{i=1}^n \alpha_i m(0)m(x_i) = \sum_{i=1}^n \alpha_i m(0+x_i) \\ &= u. \end{aligned}$$

Therefore, $L(0)|_{\text{Lin } m(G)} = \mathbf{I}_{\mathcal{H}}|_{\text{Lin } m(G)}$, and so

$$\mathcal{R}(0) = L(0)|_{\mathcal{H}_0} = \mathbf{I}_{\mathcal{H}_0}.$$

Let $x \in G$ be fixed. For $u \in \mathcal{H}_0$ we put $v := m(-x)u = \mathcal{R}(-x)u \in \mathcal{H}_0$. Then

$$\mathcal{R}(x)v = m(x)v = m(x)m(-x)u = m(0)u = L(0)u = u$$

which means that $\mathcal{R}(x)$ is surjective.

Next, setting $u_0 := m(0)$, we infer that

$$m(x) = m(x+0) = m(x)m(0) = m(x)u_0 = \mathcal{R}(x)u_0, \quad x \in G,$$

which yields (30).

Moreover,

$$\begin{aligned} \mathcal{R}(x)u &= m(x)u = m(x+0)u = m(x)m(0)u \\ &= \mathcal{R}(x)u_0u, \end{aligned}$$

for all $x \in G$ and $u \in \mathcal{H}_0$.

The fact that u_0 is a cyclic vector of the representation $\mathcal{R}$ follows directly from (29) and from the definition of $\mathcal{H}_0$, which guarantee that

$$\begin{aligned} \text{cl Lin}\{\mathcal{R}(x)u_0 : x \in G\} \\ &= \text{cl Lin}\{m(x)u_0 : x \in G\} = \text{cl Lin}\{m(x)m(0) : x \in G\} \\ &= \text{cl Lin}\{m(x+0) : x \in G\} = \text{cl Lin } m(G) \\ &= \mathcal{H}_0. \end{aligned}$$

Thus the proof is finished.

Now we return to the case of Abelian group G and to study of Eq. (4).

Theorem 2. *Let $(G, +)$ be a topological Abelian group and let $\mathcal{H}$ be a $\mathcal{H}^*$ -algebra. Suppose that $f : G \rightarrow \mathcal{H}$ is a continuous function satisfying condition (5). Then f satisfies Eq. (4) if and only if there exists a continuous representation $\mathcal{R}$ of a group G in the space $\mathcal{H}_0 = \text{cl Lin } f(G)$ with the cyclic vector $u_0 = f(0)$ satisfying*

$$(33) \quad \mathcal{R}(x)\mathcal{R}(y)^* = \mathcal{R}(y)\mathcal{R}(x)^*, \quad x, y \in G,$$

$$(34) \quad \mathcal{R}(x)u = \mathcal{R}(x)u_0u, \quad x \in G, \quad u \in \mathcal{H}_0.$$

and

$$(35) \quad \mathcal{R}(x)^* = \mathcal{R}(x), \quad x \in G \quad \text{or} \quad \mathcal{R}(x)^* = \mathcal{R}(x)^{-1}, \quad x \in G$$

such that

$$(36) \quad f(x) = \frac{1}{2}(\mathcal{R}(x)u_0 + \mathcal{R}(-x)u_0), \quad x \in G.$$

Proof. From Theorem 1 we have that the function f is form (15), where $m : G \rightarrow \mathcal{H}$ satisfies (16) or (17). Moreover, m is continuous (see the proof of Theorem 1).

If m satisfies (16), then putting $y = 0$ we get

$$(37) \quad m(x) = m(x)m(0)^* = (m(0)m(x))^* = m(x)^*, \quad x \in G.$$

Hence m satisfies Eq. (29) and from Lemma 3 we have the existence a continuous representation $\mathcal{R}$ of a group G in the space $\mathcal{H}_0 = \text{cl Lin } m(G) = \text{cl Lin } f(G)$ with the cyclic vector $u_0 = m(0) = f(0)$ satisfying (30) and (31).

Moreover, by (37) and (29), we have

$$\begin{aligned} m(x)m(y)^* &= m(x+y) = m(x+y)^* = (m(x)m(y))^* \\ &= m(y)m(x)^*, \quad x, y \in G. \end{aligned}$$

Therefore, $\mathcal{R}$ satisfies (33), and by (10) and (37), we infer that

$$\begin{aligned}(\mathcal{R}(x)u|v) &= (m(x)u|v) = (u|m(x)^*v) = (u|m(x)v) \\ &= (u|\mathcal{R}(x)v),\end{aligned}$$

for all $x \in G$ and $u, v \in \mathcal{H}_0$. As a result we obtain that

$$\mathcal{R}(x)^* = \mathcal{R}(x), \quad x \in G,$$

which ends the proof in this case.

Similarly, if m satisfies (17), then

$$\begin{aligned}m(x)^* &= m(x-0)^* = (m(x)m(0)^*)^* = m(0)m(x)^* = m(0-x) \\ &= m(-x), \quad x \in G.\end{aligned}$$

Whence m satisfies equation (29) and we can use Lemma 3 to describe a form of m . By (17), the representation $\mathcal{R}$ satisfies (33). Moreover, $\mathcal{R}$ is a unitary representation of G , i.e.

$$\mathcal{R}(x)^* = \mathcal{R}(x)^{-1}, \quad x \in G.$$

Indeed, from (10) and (17) we get

$$\begin{aligned}\|\mathcal{R}(x)u\|^2 &= (\mathcal{R}(x)u|\mathcal{R}(x)u) = (m(x)u|m(x)u) = (m(x)^*m(x)u|u) \\ &= (m(x-x)u|u) = (m(0)u|u) = (\mathcal{R}(0)u|u) \\ &= (u|u) = \|u\|^2,\end{aligned}$$

for all $x \in G$ and $u \in \mathcal{H}_0$, which completes the proof of this part of our theorem.

To verify the converse statement of Theorem 2 one only needs to perform a simple calculation (see the proof of Lemma 3) which we omit here.

We can also study measurable solutions of Eq. (4). Some remarks to this problem are included in Jarai [7].

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