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On the Spiral Structure of the (R, M) -Diagram for a Stellar Model of the Tolman-Oppenheimer-Volkoff Equation

By

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Dedicated to Professor Yoshinori Kametaka on his sixtieth birthday

1. Introduction and the main result

In general theory of relativity the spherically symmetric equilibrium of a gas is governed by the Tolman-Oppenheimer-Volkoff equation (TOV equation):

$$(1) \quad \begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \rho, \\ \frac{dP}{dr} &= -(\rho + P/c^2) \frac{G(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/c^2 r)}. \end{aligned}$$

For the derivation of this equation, see [4], [5], or Chapter 23 of [3]. Here ρ is the total mass density including the internal energy, m the mass in the sphere with radius r , P the pressure, and $r(>0)$ is a radial coordinate. The parameter G is the gravitational constant and c is the speed of light; they are positive constants.

In this article, the pressure P is supposed to be a given function of ρ satisfying the following assumption.

$$(A.0) \quad \begin{aligned} P = P(\rho) \text{ is a sufficiently smooth function of } \rho > 0, \quad P > 0, \\ 0 < dP/d\rho < c^2 \text{ for } \rho > 0 \text{ and } P \rightarrow 0 \text{ as } \rho \rightarrow 0. \end{aligned}$$

Given the central density $\rho_c > 0$ and the central pressure $P_c = P(\rho_c) > 0$, there is a unique (local) solution of (1) near $r = +0$ such that

$$(2) \quad m \rightarrow 0, \quad P \rightarrow P_c \quad \text{as} \quad r \rightarrow 0.$$

For a proof, see [2]. Let us denote by $m = m(r; \varepsilon)$, $P = P(r; \varepsilon)$ the solution for $P_c = 1/\varepsilon$.

If we continue the solution $P(r; \varepsilon)$ to the right as long as possible, there exists $R = R(\varepsilon) \leq +\infty$ such that $P(r; \varepsilon) \searrow 0$ as $r \nearrow R(\varepsilon)$. In order to guarantee the finiteness of $R(\varepsilon)$, we put the following assumption.

$$(A.1) \quad \text{As } \rho \searrow 0, \text{ we have}$$

$$\frac{\rho}{P} \frac{dP}{d\rho} = \gamma + O(\rho^{\gamma-1}),$$

where γ is a positive constant such that $4/3 < \gamma < 2$.

Under this assumption, we know that $R(\varepsilon)$ is finite for all ε . For a proof, see [2]. We put

$$M(\varepsilon) = \lim_{r \rightarrow R(\varepsilon)} m(r; \varepsilon) = \int_0^{R(\varepsilon)} 4\pi r^2 \rho(r; \varepsilon) dr.$$

We are interested in the asymptotic behavior of the radius $R(\varepsilon)$ and the total mass $M(\varepsilon)$ as $\varepsilon \searrow 0$. In order to study this, we put the following assumption:

(A2) The ratio $P/c^2\rho$ tends to a limit α_0 as $\rho \nearrow +\infty$ such that $0 < \alpha_0 < 1$ and there is a monotone non-decreasing function $\omega(x)$ such that

$$\left| \frac{P}{c^2\rho} - \alpha_0 \right| \leq \omega\left(\frac{1}{P}\right)$$

and

$$\int_0^1 \omega(x) \frac{dx}{x} < \infty$$

Then the main conclusion of this article is as follows.

Theorem 1. *As $\varepsilon \rightarrow 0$, we have*

$$(3) \quad (R(\varepsilon), M(\varepsilon))^T = (R_S, M_S)^T + \varepsilon^\mu B J(v \log \varepsilon) b + o(\varepsilon^\mu),$$

where R_S and M_S are positive constants, B is a non-singular matrix, $J(\theta)$ denotes the matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

b is a non-zero vector, and

$$\mu = \frac{1 + 3\alpha_0}{4(1 + \alpha_0)}, \quad v = \frac{\sqrt{7 + 42\alpha_0 - \alpha_0^2}}{4(1 + \alpha_0)}.$$

This theorem claims that $R(\varepsilon), M(\varepsilon)$ tends to a limit R_S, M_S spirally as $\varepsilon \rightarrow 0$. It is in [1] that this fact was first observed in an astrophysical context. Our purpose is to give a mathematically rigorous proof of this observation. Let us note astrophysical meanings of this fact. First the existence of the finite

limit M_S of the total mass as the central density tends to infinity implies the fact that the possible total mass of TOV-equilibria is bounded. This fact was first noticed by S. Chandrasekhar for the stellar model of white dwarfs. In other words, if we consider the stellar evolution of the initial mass beyond this bound, then we can claim that this star cannot tend to an equilibrium as the time passes and the star may turn to be a black hole. Secondly the fact that R, M tends to R_S, M_S spirally implies that there are infinitely many different equilibria with $M = M_S$. In other words, if we consider the stellar evolution of the initial mass M_S , then we have an infinitely many candidates of the limits of the stellar configuration as the time passes. Of course not all the equilibria can be stable. Actually astrophysicists claim some classification of stability of these equilibria. See e.g. [3], p. 48 or [7], Chapter 10. But mathematically rigorous justification of their analysis of nonlinear stability is beyond the scope of this article. This problem is interesting but still open so far as the author knows. Therefore the discussion of this article should be a preparation of the stability analysis. The conclusion of [2] on the linearized stability is restricted but its discussion could give some ideas concerning the difficulty of the stability problem. We should note that a similar behavior of the (M, R) -diagram was observed in [6] in the framework of the non-relativistic (Newtonian) theory.

As a concrete example of the function $P(\rho)$ satisfying the assumptions (A.0)(A.1)(A.2), we are keeping in mind the equation of state for neutron stars:

$$P = Kc^5 f(y), \quad \rho = Kc^3 g(y)$$

$$f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq, \quad g(y) = 3 \int_0^y q^2 \sqrt{1+q^2} dq.$$

Here K is a positive constant. For this equation of state, we see

$$\frac{dP}{d\rho} = c^2 \frac{y^2}{1+y^2},$$

thus the assumption (A.0) holds. As $\rho \rightarrow 0$, we see

$$P = \frac{K^{2/3}}{5} \rho^{5/3} (1 + O(\rho^{2/3})),$$

where the term $O(\rho^{2/3})$ is analytic in $\rho^{2/3}$, so (A.1) holds with $\gamma = 5/3$. As $\rho \rightarrow +\infty$, we see

$$P = \frac{c^2}{3} \rho (1 + O(\rho^{-1/2})),$$

so (A.2) holds with $\alpha_0 = 1/3$.

2. Qualitative analysis for the case of constant P/ρ

In this section we assume temporally that

$$P = c^2 \alpha \rho,$$

where α is a constant such that $0 < \alpha < 1$. The solution of (1)(2) satisfies

$$m = \frac{4\pi}{3} r^3 \rho_c + O(r^5), \quad P = P_c + O(r^2)$$

as $r \rightarrow +0$, where $\rho_c > 0$ and $P_c = c^2 \alpha \rho_c$.

We introduce the variables

$$(4) \quad v_1 = \frac{G}{c^2} \frac{m}{r}, \quad v_2 = 4\pi \frac{G}{c^4} r^2 P.$$

Then the equation (1) turns out to be

$$(5) \quad \begin{aligned} r \frac{dv_1}{dr} &= \frac{1}{\alpha} v_2 - v_1, \\ r \frac{dv_2}{dr} &= v_2 \left(2 - \frac{1 + \alpha}{\alpha} \frac{v_1 + v_2}{1 - 2v_1} \right). \end{aligned}$$

We are concerned with the solution $(v_1(r), v_2(r))$ such that

$$v_1 = \frac{4\pi G}{3 c^2} \rho_c r^2 + O(r^4), \quad v_2 = 4\pi \frac{G}{c^2} \alpha \rho_c r^2 + O(r^4)$$

as $r \rightarrow 0$.

Let us investigate the behavior of $(v_1(r), v_2(r))$ as $r \rightarrow +\infty$. Considering the equation (5) in the domain $\{v_1 < 1/2, |v_2| < +\infty\}$, we denote by $(0, R)$ the right maximal interval of existence. Fixing r_0 such that $0 < r_0 < R$, we introduce the variable s by

$$s = s(r) = \int_{r_0}^r \frac{dr}{(1 - 2v_1(r))r}$$

and the inverse function $r = r(s)$, $-\infty < s < S$. Putting $\tilde{v}_1(s) = v_1(r(s))$, $\tilde{v}_2(s) = v_2(r(s))$, we have a solution of the plane dynamical system

$$(6) \quad \begin{aligned} \frac{dv_1}{ds} &= \left(\frac{1}{\alpha} v_2 - v_1 \right) (1 - 2v_1), \\ \frac{dv_2}{ds} &= v_2 \left(2 - \frac{1 + 5\alpha}{\alpha} v_1 - \frac{1 + \alpha}{\alpha} v_2 \right). \end{aligned}$$

We note $\tilde{v}_1(s) \sim (4\pi G/c^2) \rho_c e^{2s}$, $\tilde{v}_2(s) \sim (4\pi G/c^2) \alpha \rho_c e^{2s}$ as $s \rightarrow -\infty$.

Now the domain $\{v_1 < 1/2, v_2 > 0\}$ is an invariant region to the system (6). Singular points in the closure of this region are:

$$O_1 = (0, 0), \quad O_2 = (1/2, 0), \quad Q = \left(\frac{2\alpha}{1 + 6\alpha + \alpha^2}, \frac{2\alpha^2}{1 + 6\alpha + \alpha^2} \right).$$

The singular point O_1 is a saddle whose separatrices are $L_1 = \{0 < v_1 < 1/2, v_2 = 0\}$, $L_2 = \{v_1 < 0, v_2 = 0\}$ and the orbit E of the solution $(\tilde{v}_1(s), \tilde{v}_2(s))$. The singular point O_2 is a saddle whose separatrices are L_1 and $L_3 = \{v_1 = 1/2, 0 < v_2\}$. The singular point Q is a stable focus whose eigenvalues are

$$-\frac{1 + 3\alpha}{2(1 + \alpha)} \pm \frac{\sqrt{7 + 42\alpha - \alpha^2}}{2(1 + \alpha)} \sqrt{-1}.$$

Let us consider the behavior of E . It is clear that E is confined into the region $\{0 < v_1 < 1/2, v_2 > 0\}$. Put $V(v_1, v_2) = v_2 - Kv_1 - K$, where K is a sufficiently large positive number such that $V(\tilde{v}_1(0), \tilde{v}_2(0)) < 0$. We are going to show that $V(\tilde{v}_1(s), \tilde{v}_2(s)) < 0$ for all $s \in [0, S)$. Otherwise, there would exist the smallest s_0 such that $V(\tilde{v}_1(s_0), \tilde{v}_2(s_0)) = 0$. At this time, we see

$$\begin{aligned} \frac{dV}{ds} &= v_2 \left(2 - \frac{1 + 5\alpha}{\alpha} v_1 - \frac{1 + \alpha}{\alpha} v_2 \right) - K \left(\frac{1}{\alpha} v_2 - v_1 \right) (1 - 2v_1) \\ &< \left(2v_2 - K \left(\frac{1}{\alpha} v_2 - v_1 \right) \right) (1 - 2v_1) = \left(\left(3 - \frac{K}{\alpha} \right) v_2 - K \right) (1 - 2v_1) < 0 \end{aligned}$$

provided that $K > 3\alpha$. Since $V < 0$, we see that E is confined into the bounded region $D = \{0 < v_1 < 1/2, 0 < v_2 < Kv_1 + K\}$. Therefore $S = \infty$. According to the Poincaré-Bendixson's theory, the ω -limit set of E , which is included in the closure of D , should be one of the following types: 1) a stable singular point; 2) a pathpolygon consisting of separatrices of saddle points; 3) a stable closed (periodic) orbit. First 2) is not the case, since the separatrix L_3 is unbounded. We are going to show that 3) is not the case. To do that, we introduce the variables

$$a = \log(1 - 2v_1), \quad b = \log v_2.$$

The equation turns out to be

$$\begin{aligned} \frac{da}{ds} &= 1 - e^a - \frac{2}{\alpha} e^b, \\ \frac{db}{ds} &= -\frac{1 + \alpha}{2\alpha} + \frac{1 + 5\alpha}{2\alpha} e^a - \frac{1 + \alpha}{\alpha} e^b. \end{aligned}$$

Thus

$$\frac{\partial}{\partial a} \frac{da}{ds} + \frac{\partial}{\partial b} \frac{db}{ds} = -e^a - \frac{1+\alpha}{\alpha} e^b < 0.$$

By the Bendixson's criterion, we know that there are no closed orbit in D . Therefore 3) is not the case, and 1) is the only possible case, that is, the orbit E tends to the focus Q . This implies that $R = \infty$ and $(v_1(r), v_2(r))$ tends to $(2\alpha/(1+6\alpha+\alpha^2), 2\alpha^2/(1+6\alpha+\alpha^2))$ as $r \rightarrow +\infty$.

3. Zone I (central core)

Let us go back to the TOV equation with a general $P(\rho)$ which satisfies (A.0)(A.1)(A.2). We consider the unique solution of (1)(2). Put

$$\frac{P}{c^2 \rho} = \alpha(x), \quad \frac{1}{P} = x = \frac{X}{P_c} = \varepsilon X.$$

We consider the solution for small ε (large P_c).

We can write the equation (1) as

$$(7) \quad \begin{aligned} r \frac{dX}{dr} &= \frac{1+\alpha}{\alpha} \frac{v_1+v_2}{1-2v_1} X, \\ r \frac{dv_1}{dr} &= \frac{1}{\alpha} v_2 - v_1, \\ r \frac{dv_2}{dr} &= \frac{v_2 \left(2 - \frac{1+5\alpha}{\alpha} v_1 - \frac{1+\alpha}{\alpha} v_2 \right)}{1-2v_1}, \end{aligned}$$

where

$$v_1 = \frac{G}{c^2} \frac{m}{r}, \quad v_2 = 4\pi \frac{G}{c^4} r^2 P,$$

and $\alpha = \alpha(x) = \alpha(\varepsilon X)$.

Since $dX/dr > 0$, $X(r)$ is monotone increasing and admits an inverse function $r(X)$, $1 < X < +\infty$, with $r(1+0) = 0$. Then $v_j = V_j(X; \varepsilon) = v_j(r(X); \varepsilon)$ is a solution of the equation

$$(8) \quad \begin{aligned} X \frac{dv_1}{dX} &= \frac{\alpha}{1+\alpha} \left(\frac{v_2}{\alpha} - v_1 \right) (1-2v_1)/(v_2+v_1), \\ X \frac{dv_2}{dX} &= \frac{\alpha}{1+\alpha} v_2 \left(2 - \frac{1+5\alpha}{\alpha} v_1 - \frac{1+\alpha}{\alpha} v_2 \right) / (v_2+v_1), \end{aligned}$$

where $\alpha = \alpha(\varepsilon X)$. The solution which we consider satisfies

$$\begin{aligned} v_1 &= \frac{2\alpha(\varepsilon)}{(1 + \alpha(\varepsilon))(1 + 3\alpha(\varepsilon))} (X - 1) + O((X - 1)^2), \\ v_2 &= 3\alpha(\varepsilon)v_1 + O((v_1)^2) \end{aligned}$$

as $X \searrow 1$.

Inversely, suppose $(V_1(X), V_2(X))$, $1 < X < X_0$ is a solution of (5) such that

$$\frac{1}{C}(1 - X) \leq V_j(X) \leq C(1 - X) < \frac{1}{2}.$$

Then by

$$\log r = \int_{X_1}^X \frac{\alpha}{1 + \alpha} \frac{1 - 2V_1(X)}{V_1(X) + V_2(X)} \frac{dX}{X} + \log r_1,$$

we get a monotone increasing function $r = r(X)$, $1 < X < X_0$, such that $r(1 + 0) = 0$, which has the inverse function $X = X(r)$, $0 < r < r_0$. Then $P(r) = 1/\varepsilon X$, $m = (c^2/G)rV_1(X(r))$ satisfies (2). On the other hand, if we put $\tilde{P} = (c^4/(4\pi G))r^{-2}V_2(X(r))$, then we get

$$\frac{r}{X} \frac{dX}{dr} = -\frac{r}{\tilde{P}} \frac{d\tilde{P}}{dr},$$

that is, there exists C such that $\tilde{P} = CP = C/X$. We can choose X_1 and r_1 so that $C = 1$. Then $(P(r), m(r))$ is a solution of (1). This is unique.

Now we approximate the solution $V(X, \varepsilon)$ by the solution $V^0(X)$ of (8) when $\alpha = \alpha(0) = \alpha_0$ constantly. Of course

$$V_1^0 = \frac{2\alpha_0}{(1 + \alpha_0)(1 + 3\alpha_0)} (X - 1) + O((X - 1)^2), \quad V_2^0 = 3\alpha_0 V_1^0 + O((V_1^0)^2)$$

as $X \searrow 1$. In order to fix the idea we assume

$$\frac{1}{2}\alpha_0 \leq \alpha(\varepsilon X) \leq \frac{3}{2}\alpha_0$$

for $\varepsilon \leq \varepsilon_0$ and $1 \leq X \leq X_0$. We shall show the following

Proposition 1. *For any $X_1 > 1$ there exists $\varepsilon_1 > 0$ such that*

$$(9) \quad |V(X; \varepsilon) - V^0(X)| \leq C(X - 1) \sup_{1 \leq \Xi \leq X} |\alpha(\varepsilon \Xi) - \alpha_0|,$$

for $0 < \varepsilon \leq \varepsilon_1$ and $1 \leq X \leq X_1$.

Proof. Put

$$W_j = V_j - V_j^0.$$

Then we get an equation

$$X \frac{dW}{dX} = \frac{1}{X-1} \mathcal{A}W + g(X, W),$$

where

$$\mathcal{A} = \frac{1}{1+3\alpha_0} \begin{pmatrix} -\frac{3(1+\alpha_0)}{2} & \frac{1+\alpha_0}{2\alpha_0} \\ -3\alpha_0 & 1 \end{pmatrix}$$

and $g(X, W)$ is continuous in $1 \leq X \leq X_0$ and $|W| \leq \kappa(X-1)$, where κ is an arbitrarily fixed positive number and

$$|g(X, W)| \leq M(|\alpha_1(X)| + |W(X)|),$$

where $\alpha_1(X) = \alpha(\varepsilon X) - \alpha_0$. So we are led to the integral equation

$$(10) \quad W(X) = \int_1^X U(X)U(\varepsilon)^{-1}g(\varepsilon, W(\varepsilon))\frac{d\varepsilon}{\varepsilon},$$

where $U(X)$ is the fundamental matrix of the linear equation

$$X \frac{dU}{dX} = \frac{1}{X-1} \mathcal{A}U.$$

Since the eigenvalues of $\mathcal{A}$ are 0 and $-1/2$, we have

$$|U(X)U(\varepsilon)^{-1}| \leq M' \left(\frac{X(\varepsilon-1)}{\varepsilon(X-1)} \right)^{1/2}.$$

Let us denote the right-hand side of the integral equation (10) by $TW(X)$, where $W(X)$ is a continuous function of $1 \leq X \leq a$ and $|W| \leq \theta\kappa(X-1)$ with a sufficiently small positive number θ specified later. Then we have

$$|TW(X)| \leq MM'a^{1/2} \left(\frac{2}{3} \|\alpha_1\|(X-1) + \frac{2}{5} \theta\kappa(X-1)^2 \right),$$

where $\|\alpha_1\| = \sup_{1 \leq \varepsilon \leq X_1} |\alpha(\varepsilon X) - \alpha_0|$. Then $|TW(X)| \leq \theta\kappa(X-1)$ provided that $\|\alpha_1\|$ and $a-1$ are small. Then there is a solution W such that $|W(X)| \leq C\|\alpha_1\|(X-1)$ provided that $(2/5)MM'a^{3/2}\kappa\theta < 1$. Since $V = V^0 + W$ is the unique solution, this completes the proof.

4. Zone II (transition layer)

Let us consider the approximation of $V(X; \varepsilon)$ as $x = \varepsilon X$ is large.

If $\alpha(x) = \alpha_0$ identically, then the system (8) has the singular point $v = b^S = (b_1, b_2)$, where

$$b_1 = \frac{2\alpha_0}{1 + 6\alpha_0 + \alpha_0^2}, \quad b_2 = \alpha_0 b_1.$$

As shown in Section 2 we know that $V^0(X)$ tends to b^S as $X \rightarrow +\infty$. By Proposition 1, we know that $V(X; \varepsilon)$ is near to $V^0(X)$ as ε is small. Therefore $V(X; \varepsilon)$ is near to b^S if $x/\varepsilon = X$ is sufficiently large and ε is sufficiently small.

We put

$$v = b^S + y,$$

and rewrite the system (8) as

$$(11) \quad x \frac{dy}{dx} = f_0(x) - A(x)y + f_2(x, y),$$

where $f_0(x)$, $A(x)$ are continuous functions of $0 \leq x \leq 1$ and $f_2(x, y)$ is a continuous function of $0 \leq x \leq 1$ and $|y| \leq \delta$ which is a rational function of y . In fact

$$f_{01}(x) = \frac{\alpha}{1 + \alpha} \left(\frac{b_2}{\alpha} - b_1 \right) (1 - 2b_1) / (b_1 + b_2),$$

$$f_{02}(x) = \frac{\alpha}{1 + \alpha} b_2 \left(2 - \frac{1 + 5\alpha}{\alpha} b_1 - \frac{1 + \alpha}{\alpha} b_2 \right) / (b_1 + b_2),$$

so

$$|f_0(x)| \leq C|\alpha(x) - \alpha_0|,$$

and

$$|A(x) - A(0)| \leq C|\alpha(x) - \alpha_0|$$

with

$$A(0) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\alpha_0} \\ \frac{\alpha_0(1 + 5\alpha_0)}{(1 + \alpha_0)^2} & \frac{\alpha_0}{1 + \alpha_0} \end{pmatrix}$$

The eigenvalues of $A(0)$ are $\mu \pm v\sqrt{-1}$ with μ, v of Theorem 1. Since $0 < \alpha_0 < 1$, we see $\mu > 0$ and $v > 0$. Then we shall apply the following

Theorem 2. *Let the equation*

$$(E) \quad x \frac{dy}{dx} = f_0(x) - A(x)y + f_2(x, y)$$

satisfy the following assumptions, (H0): $f_0(x)$ is continuous in $0 \leq x \leq a$ and

$$|f_0(x)| \leq C\omega(x),$$

where $\omega(x)$ is a monotone non-decreasing continuous function of $x \geq 0$ such that $\omega(x) \geq 0, \omega(0) = 0$ and

$$\int_0^a \omega(x) \frac{dx}{x} < +\infty;$$

(H1): $A(x)$ is continuous in $0 \leq x \leq a$ and

$$|A(x) - A(0)| \leq C\omega(x)$$

and the eigenvalues of $A(0)$ are $\mu \pm v\sqrt{-1}$ with $\mu > 0$ and $v > 0$; (H2): $f_2(x, y)$ is continuous in $0 \leq x \leq a$ and $|y| \leq \delta$ and admits a power series expansion in y of the form

$$f_2(x, y) = \sum_{j_1+j_2 \geq 2} f_{j_1 j_2}(x) y_1^{j_1} y_2^{j_2},$$

where $f_j(x)$ are continuous in $0 \leq x \leq a$ and

$$|f_j(x) - f_j(0)| \leq C_j \omega(x)$$

with

$$\sum (|f_j(0)| + C_j) \delta^{|j|} < +\infty.$$

Then there is a change of variables

$$(T) \quad y = p_0(x) + P_1(x)z + p_2(x, z)$$

which reduces (E) to

$$(F) \quad x \frac{dz}{dx} = -A(0)z.$$

Here $p_0(x), P_1(x)$ and $p_2(x, y)$ enjoy the following properties: (P0): $p_0(x)$ is continuous in $0 \leq x \leq a'$ and

$$|p_0(x)| \leq C' x^{-\mu} \int_0^x \omega(\xi) \xi^{\mu-1} d\xi;$$

(P1): $P_1(x)$ is continuous in $0 \leq x \leq a$ and

$$|P_1(x) - I| \leq C' \int_0^x \omega(\xi) \xi^{-1} d\xi,$$

where I is the identity matrix; (P2): $p_2(x)$ is continuous in $0 \leq x \leq a'$ and $|z| \leq \delta'$ and admits a power series expansion of the form

$$p_2(x, z) = \sum_{j_1+j_2 \geq 2} p_{j_1 j_2}(x) z_1^{j_1} z_2^{j_2},$$

where $p_J(x)$ are continuous in $0 \leq x \leq a'$ and

$$|p_J(x) - p_J(0)| \leq C_J' \left(\int_0^x \omega(\xi) \xi^{-1} d\xi + x^\mu \int_x^{a'} \omega(\xi) \xi^{-\mu-1} d\xi \right).$$

Moreover the change of variables

$$(T0) \quad y = z + p_2(0, z)$$

reduces

$$(E0) \quad x \frac{dy}{dx} = -A(0)y + f_2(x, y)$$

to (F).

A proof of this theorem will be given in Section 6. Applying this theorem to the equation (11) there is a change of variables of the form (T) which reduces (11) to the equation (F). We can rewrite

$$(12) \quad v = v^S(x) + P_1(x)z + p_2(x, z),$$

where $v = v^S(x) = b^S + p_0(x)$ is a particular solution of (11) such that $v^S(x) \rightarrow b^S$ as $x \rightarrow 0$. A solution $(P^S(r), m^S(r))$ of the TOV-equation (1) corresponds to this $v^S(x)$ and

$$(13) \quad P^S(r) \sim \frac{c^4}{4\pi G} b_2 r^{-2}, \quad m^S(r) \sim \frac{c^2}{G} b_1 r$$

as $r \rightarrow +0$. Of course, by (A.1), there is a finite R_S such that $P^S(r) \searrow 0$ as $r \nearrow R_S$.

Let us solve (12) as

$$(14) \quad z = P_1(x)^{-1}(v - v^S(x)) + q_2(x, v - v^S(x)).$$

Then the solution $v(x; \varepsilon)$ is characterized by the initial condition

$$v - v^S(x) = b(\varepsilon) \quad \text{at } x = \varepsilon X_2,$$

where

$$b(\varepsilon) = V(X_2; \varepsilon) - v^S(\varepsilon X_2).$$

Choose X_2 so large that

$$|V^0(X_2) - b^S| \leq \frac{\delta'}{8},$$

which is possible since $V^0(X) \rightarrow b^S$ as $X \rightarrow +\infty$. Taking $\varepsilon_2 > 0$ sufficiently small, we have, for $0 < \varepsilon \leq \varepsilon_2$,

$$|b(\varepsilon)| \leq |V(X_2; \varepsilon) - V^0(X_2)| + |v^S(\varepsilon X_2) - b^S| + |V^0(X_2) - b^S| \leq \delta',$$

since $v^S(x) \rightarrow b^S$ as $x \rightarrow 0$. Then we can substitute $b(\varepsilon)$ into (14) to define $c(\varepsilon)$ by

$$(X_2)^{-A(0)} c(\varepsilon) = P_1(\varepsilon X_2)^{-1} b(\varepsilon) + q_2(\varepsilon X_2, b(\varepsilon)).$$

Then the solution of (12)

$$z = (x/\varepsilon)^{-A(0)} c(\varepsilon)$$

gives the solution $v(x; \varepsilon)$, therefore we get

Proposition 2. *There are sufficiently large X_2 and sufficiently small ε_2 and x_2 such that the expression*

$$(15) \quad v(x; \varepsilon) = v^S(x) + P_1(x)(x/\varepsilon)^{-A(0)} c(\varepsilon) + p_2(x, (x/\varepsilon)^{-A(0)} c(\varepsilon))$$

holds as long as

$$0 \leq x \leq x_2, \quad X_2 \leq x/\varepsilon \quad 0 < \varepsilon \leq \varepsilon_2.$$

Here $c(\varepsilon)$ is continuous in $\varepsilon \geq 0$ and

$$c(0) = \lim_{x \rightarrow +\infty} x^{A(0)} (V^0(x) - b^S) \neq 0.$$

5. Zone III (surface)

Let us introduce the variable u by

$$u = \int_0^P \frac{dP}{\rho + P/c^2}.$$

Here we use (A.1). Then we can rewrite the TOV-equation (1), by taking u as the independent variable, as

$$(16) \quad \begin{aligned} \frac{dr}{du} &= -\frac{r^2(1 - 2Gm/c^2r)}{G(m + 4\pi Pr^3/c^2)}, \\ \frac{dm}{du} &= -\frac{4\pi r^4 \rho(1 - 2Gm/c^2r)}{G(m + 4\pi Pr^3/c^2)}. \end{aligned}$$

Here ρ and P are considered as given functions of $u \geq 0$ such that $\rho = P = 0$ as $u = 0$. The right-hand side of the equation (16) is continuous in $u \geq 0, r > 0$,

$m > 0$ and analytic in r and m . We note

$$(17) \quad \begin{aligned} r &= c^2(4\pi G)^{-1/2} P^{1/2}(v_2)^{1/2}, \\ m &= c^4 G^{-1}(4\pi G)^{-1/2} P^{1/2} v_1(v_2)^{1/2}. \end{aligned}$$

Let us denote by $(r_S(u), m_S(u))$ the solution of (16) which corresponds to the singular solution $P^S(r)$. This solution exists on $u \geq 0$. We denote by u_0 the value for x_2 of Proposition 2. Let $R_S = r_S(0)$, $M_S = m_S(0)$ be the radius and the total mass of the singular solution. We denote by $(r(u; \varepsilon), m(u; \varepsilon))$ the solution of (16) which is generated by $v(x; \varepsilon)$, say the solution of (1)(2). We are going to investigate the behavior of $R(\varepsilon) = r(0; \varepsilon)$ and $M(\varepsilon) = m(0; \varepsilon)$ as $\varepsilon \rightarrow 0$.

By Proposition 2 and (17), we see

$$(r(u_0; \varepsilon), m(u_0; \varepsilon))^T = (r_S(u_0), m_S(u_0))^T + Q_0 \varepsilon^{A(0)} c(\varepsilon) + q_0(\varepsilon^{A(0)} c(\varepsilon)),$$

where Q_0 is a non-singular matrix and $q_0(Z)$ is an analytic function of Z of the form $\sum_{|J| \geq 2} g_J Z^J$.

Then by the usual theory of regular perturbations, we have the following

Proposition 3. *There is a sufficiently small ε_3 such that*

$$(18) \quad (r(u; \varepsilon), m(u; \varepsilon))^T = (r_S(u), m_S(u))^T + Q(u) \varepsilon^{A(0)} c(\varepsilon) + q(u, \varepsilon^{A(0)} c(\varepsilon)),$$

for $0 \leq u \leq u_0$ and $0 < \varepsilon \leq \varepsilon_3$. Here $Q(u)$ is a non-singular matrix which is continuous in $0 \leq u \leq u_0$, and $q(u, Z) = O(Z^2)$ is continuous in $0 \leq u \leq u_0$ and $|Z| \leq \delta_3$ and analytic in Z .

Putting $u = 0$ in (18), we get the desired expression (3). This completes the proof of Theorem 1.

6. Proof of Theorem 2

In this section we give a brief sketch of a proof of Theorem 2. We use the following notations. $\mathcal{B}(a, \omega)$ is the set of continuous functions $\varphi(x)$ of $0 \leq x \leq a$ such that

$$\|\varphi\|_{a, \omega} = \inf\{C \|\varphi(x)\| \leq C\omega(x) \text{ for } 0 \leq x \leq a\} < +\infty.$$

$\mathcal{A}(a, b, \omega)$ is the set of continuous functions $\varphi(x, y)$ of $0 \leq x \leq a$ and $|y| = \max(|y_1|, |y_2|) \leq b$ such that

$$\begin{aligned} \varphi(x, y) &= \sum_{|J| \geq 2} \varphi_J(x) y^J, \\ \varphi_J(x) - \varphi_J(0) &\in \mathcal{B}(a, \omega), \\ \sum_{|J| \geq 2} (|\varphi_J(0)| + \|\varphi_J - \varphi_J(0)\|_{a, \omega}) b^{|J|} &< +\infty. \end{aligned}$$

We put

$$J_\mu \omega(x) = x^{-\mu} \int_0^x \omega(\xi) \xi^{\mu-1} d\xi,$$

$$J^\mu \omega(x) = x^\mu \int_x^\delta \omega(\xi) \xi^{-\mu-1} d\xi.$$

The proof of Theorem 2 is divided into three steps.

Lemma 1. *There is a solution $y = p_0(x)$ of (E) in $\mathcal{B}(a_1, J_\mu \omega)$.*

Proof. Putting

$$\hat{A}(x) = A(x) - A(0),$$

$$g(x, y) = f_0(x) - \hat{A}(x)y + f_2(x, y),$$

we solve the integral equation

$$(19) \quad \varphi(x) = T\varphi(x) = \int_0^x (x/\xi)^{-A(0)} g(\xi, \varphi(\xi)) \xi^{-1} d\xi.$$

We can assume

$$|f_0(x)| \leq M\omega(x), \quad |\hat{A}(x)| \leq M\omega(x), \quad |f(x, y)| \leq M|y|^2.$$

for $0 \leq x \leq a, |y| \leq b$, and

$$|t^{A(0)}| \leq Mt^\mu$$

for $0 \leq t \leq 1$. Consider the set Ω of continuous functions $\varphi(x)$ of $0 \leq x \leq a'(\leq a)$ such that $\|\varphi\|_{a'} = \sup_{0 \leq x \leq a'} |\varphi(x)| \leq \theta$. Then it is easy to see

$$|T\varphi(x)| \leq M \left(2MJ_\mu \omega(x) + \frac{1}{\mu} M\theta^2 \right).$$

Since $J_\mu \omega(0) = 0$, we can find sufficiently small a' and θ such that $\|T\varphi\|_{a'} \leq \theta$. Applying the fixed point theorem, we see that there is a solution φ of (19) in Ω . It is easy to see

$$\|\varphi\|_{a', J_\mu \omega} \leq \frac{(1+\theta)M^2}{1-M^2\theta/\mu},$$

where we note $J_\mu \omega \leq \omega/\mu$, since ω is non-decreasing. QED

Lemma 2. *If the equation*

$$(E2) \quad x \frac{dy}{dx} = -A(x)y$$

satisfy the assumption (H1), then there is a change of variables

$$(T2) \quad y = P_1(x)z$$

which reduces (E2) to (F) such that

$$P_1(x) - I \in \mathcal{B}(a_2, J_0\omega).$$

Proof. The equation which $P_1(x)$ should satisfy is

$$(20) \quad x \frac{dP_1}{dx} = -A(x)P_1 + P_1A(0).$$

We can assume that $A(0) = \text{diag}(\lambda_1, \lambda_2)$, where $\lambda_1 = \mu - v\sqrt{-1}$, $\lambda_2 = \mu + v\sqrt{-1}$.

We put

$$A(x) = \begin{pmatrix} \lambda_1 + a_{11}(x) & a_{12}(x) \\ a_{21}(x) & \lambda_2 + a_{22}(x) \end{pmatrix},$$

$$P_1(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}.$$

Then the equation (20) turns out to be

$$(21) \quad x \frac{dp_{11}}{dx} = -a_{11}p_{11} - a_{12}p_{21},$$

$$(22) \quad x \frac{dp_{21}}{dx} = -a_{21}p_{11} + (2v\sqrt{-1} + a_{22})p_{22},$$

$$(23) \quad x \frac{dp_{12}}{dx} = (2v\sqrt{-1} - a_{11})p_{12} - a_{12}p_{22},$$

$$(24) \quad x \frac{dp_{22}}{dx} = -a_{21}p_{12} - a_{22}p_{22}.$$

From (21)(22), the equation for $q = p_{21}/p_{11}$ is

$$(25) \quad x \frac{dq}{dx} = -a_{21} - 2v\sqrt{-1}q + a_{12}q^2.$$

Since $a_{21} \in \mathcal{B}(a, \omega)$, it is easy to see that (25) has a solution in $\mathcal{B}(a', J_0\omega)$.

Substituting this into (21), we get

$$x \frac{dp_{11}}{dx} - (-a_{11} - a_{12}q)p_{11},$$

which has the solution

$$p_{11}(x) = \exp\left(-\int_0^x (a_{11} + a_{12}q)d\xi/\xi\right).$$

Then $p_{11}(x) - 1 \in \mathcal{B}(a', J_\mu \omega)$, and $p_{21} = qp_{11} \in \mathcal{B}(a', J_0 \omega)$. The same argument can be applied to the couple of equations (23)(24). QED.

Lemma 3. *If the equation*

$$(E3) \quad x \frac{dy}{dx} = -A(0)y + g(x, y)$$

satisfies the assumption (H2), then there is a transformation

$$(T3) \quad y = z + p(x, z)$$

with $p \in \mathcal{A}(a_3, b_3, J^\mu \omega)$, which reduces the equation (E3) to (F).

Proof. The equation for $p(x, z)$ is

$$(G) \quad \left(x \frac{\partial}{\partial x} - \left(Az, \frac{\partial}{\partial z} \right) \right) p = -A(0)p + g(x, z + p).$$

We assume $A(0)$ is diagonal. A power series

$$p(x, z) = \sum_{|k| \geq 2} p_k(x) z^k$$

is substituted to (28). Then we have the equations

$$(G-k) \quad \left(x \frac{d}{dx} + A(0) - (k, \lambda) \right) p_k = h_k,$$

where

$$(k, \lambda) = k_1 \lambda_1 + k_2 \lambda_2, \\ h_k = H_k(g_i, p_j) \quad (2 \leq i \leq k, 2 \leq j \leq k-1)$$

We determine p_k by solving (G-k) under the boundary condition

$$(B-k) \quad p_{kl}(a') = -h_{kl}(0)/\lambda_{kl}, \quad l = 1, 2,$$

where a' is a fixed number, $0 < a' \leq a$ and $\lambda_{kl} = (k, \lambda) - \lambda_l$.

By the induction on k , we see that h_k and p_k belong to $C[0, a']$ and satisfy

$$p_{kl}(0) = h_{kl}(0)/\lambda_{kl}, \\ \|p_{kl}\|_{a'} \leq \|h_{kl}\|_{a'}/\mu(|k| - 1).$$

In fact the solution of (G-k)(B-k) is given by

$$p_{kl}(x) = -\frac{h_{kl}}{\lambda_{kl}} (x/a')^{\lambda_{kl}} - \int_x^{a'} (x/\xi)^{\lambda_{kl}} h_{kl}(\xi) \frac{d\xi}{\xi}.$$

Putting $M = 1 + 1/\mu$, we consider the equation

$$Y_l = Z_l + M \sum_{|j| \geq 2} \|g_{jl}\|_{a'} Y^j$$

which has a convergent power series solution

$$Y_l = Z_l + \sum_{|k| \geq 2} M_{kl} Z^k.$$

Then we see $\|p_{kl}\|_{a'} \leq M_{kl}$, that is the power series $\sum p_k(x) z^k$ is convergent. If we put

$$p^N(x, z) = \sum_{|k| \leq N} p_k(x) z^k$$

then p^N satisfies

$$(26) \quad x \frac{\partial}{\partial x} p^N = \left(-A(0) + \left(A(0)z, \frac{\partial}{\partial z} \right) \right) p^N + h^N,$$

where $h^N = \sum_{|k| \leq N} z^k$. As $N \rightarrow \infty$, the right-hand side of (28) uniformly tends to

$$\left(-A(0) + \left(A(0)z, \frac{\partial}{\partial z} \right) \right) p + g(x, p(x, z)).$$

Hence $x(\partial p / \partial x)$ exists and satisfies (G).

We shall show that $p \in \mathcal{A}(a', b', J^\mu \omega)$ if a' is small. To do that we put

$$\begin{aligned} \hat{p}_k(x) &= p_k(x) - p_k(0), & \hat{h}_k(x) &= h_k(x) - h_k(0), \\ \hat{g}_k(x) &= g_k(x) - g_k(0) \end{aligned}$$

Take a' such that

$$\|\omega\|_{a'} + \|J^\mu \omega\|_{a'} \leq 1.$$

By induction we see

$$\|\hat{p}_{kl}\|_{J^\mu \omega} \leq \left(\frac{1}{(|k| - 2)\mu + 1} + 1 \right) \|\hat{h}_{kl}\|_{\omega + J^\mu \omega},$$

$$\|\hat{h}_{kl}\|_{\omega + J^\mu \omega} \leq \hat{H}_k(|g_{il}(0)|, |p_j(0)|, \|\hat{g}_{il}\|_{\omega}, \|\hat{p}_j\|_{J^\mu \omega}),$$

where

$$\hat{H}_k(X_i, x_j, D_i, d_j) = H_k(X_i + D_i, x_j + d_j) - H_k(X_i, x_j).$$

We have used

$$\hat{p}_{kl}(x) = - \int_x^{a'} (x/\xi)^{\lambda_{kl}} \hat{h}_{kl}(\xi) \frac{d\xi}{\xi}.$$

We consider the equation

$$Y_l^0 = Z_l^0 + M \sum_{|j| \geq 2} |g_{jl}(0)| (Z^0)^j,$$

and its power series solution

$$Y_l^0 = Z_l^0 + \sum_{|k| \geq 2} M_{kl}^0 (Z^0)^k,$$

and the equation

$$Y_l^1 = Z_l^1 + M \sum_{|j| \geq 2} (|g_{jl}(0)| + \|\hat{g}_{jl}\|_{a', \omega}) (Y^1)^j$$

and its power series solution

$$Y_l^1 = Z_l^1 + \sum_{|k| \geq 2} M_{lk}^1 (Z^1)^k.$$

Then we see

$$\|\hat{p}_{kl}\|_{J^\mu \omega} \leq M_{kl}^1 - M_{kl}^0.$$

This completes the proof. QED.

It is easy to prove Theorem 2 by combining the above Lemmas 1, 2, 3. Here we note $\mathcal{B}(a, J_0(\omega + J_\mu \omega)) \subset \mathcal{B}(a, J_0 \omega)$ and $\mathcal{A}(a, b, J^\mu(J_0 \omega + \omega)) \subset \mathcal{A}(a, b, J^\mu \omega + J_0 \omega)$.

Remark. Even if $\alpha(x)$ is analytic in x^σ , σ being a positive index, the usual Briot-Bouquet theory cannot be applied because of the inconvenience of the small divisor, that is, $|j_0 \sigma - j_1 \lambda_1 - j_2 \lambda_2|$ can be infinitely small. Also we note that the transformation (T3) depends on the choice of a' , so it is not unique.

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