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On Black Hole Solutions of Second Order Differential Equations with a Singular Nonlinearity in the Differential Operator

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1. Introduction

Consider the singular ordinary differential equation

$$(A) \quad (|y'|^{-\alpha})' + q(t)y^\beta = 0$$

where $\alpha > 0$, $\beta \in \mathbf{R}$ are constants and $q : [a, \infty) \rightarrow (0, \infty)$ is a continuous function.

By a solution of (A) we mean a positive function y defined on some interval of the form $J = [t_0, T)$, $a \leq t_0 < T \leq \infty$, which is continuously differentiable together with $|y'|^{-\alpha}$ and satisfies (A) at every point of J . If J is the maximal interval of existence of y , we say that $y(t)$ is *proper* or *singular* according to whether $T = \infty$ or $T < \infty$.

The main objective of this paper is to examine the structure of the set of solutions of (A), both proper and singular, and to show that nonlinear equations of the form (A) may have, along with proper and ‘usual’ blowing-up and extinct solutions, also singular solutions of a new type that we call *black hole solutions* because of their specific behavior at the right end-point of the maximal interval of existence.

More precisely, by a *black hole solution* we understand a singular solution y of Eq. (A) defined on some interval $[t_0, T)$, $T < \infty$, such that

$$(1) \quad \lim_{t \rightarrow T-0} y(t) < \infty \quad \text{and} \quad \lim_{t \rightarrow T-0} |y'(t)| = \infty.$$

Such a solution is said to have a ‘zero black hole’ at T if $\lim_{t \rightarrow T-0} y(t) = 0$ and is termed a singular solution with nonzero black hole otherwise.

A simple example of a singular equation of the form (A) (with $\beta = 0$) which really has black hole solutions is

$$(2) \quad (|y'|^{-\alpha})' + \left(\frac{\alpha-1}{\alpha}\right)^{-\alpha} = 0,$$

where $\alpha > 1$. For any given $T > a$ and $c \geq 0$, the function $y(t) = c +$

$(T-t)^{(\alpha-1)/\alpha}$ defined and positive on $[a, T)$ is a decreasing solution of (2) with a singularity of black hole type at T .

Similarly, for any $c > 0$ the function $y(t) = c - (T-t)^{(\alpha-1)/\alpha}$ provides an example of a 'local' increasing black hole solution of (2) which is defined and positive in some sufficiently small left neighborhood of T .

The qualitative behavior of the singular Emden-Fowler equation

$$(B) \quad (p(t)y')' + q(t)y^{-\beta} = 0$$

and its generalization

$$(C) \quad (p(t)|y'|^\alpha \operatorname{sgn} y')' + q(t)y^{-\beta} = 0$$

where $p, q : [a, \infty) \rightarrow (0, \infty)$ are continuous functions and $\alpha > 0$ and $\beta > 0$ are constants has been investigated in great detail by several authors; see, for example, the papers [1–15]. However, according to our knowledge, the singular equation (A) in our setting does not seem to have been the object of any systematic investigation, and the present work was motivated by this observation.

2. Classification of proper and singular solutions

We begin by classifying possible solutions of (A) according to their behavior as t approaches the right end-point T of the maximal interval of existence.

A routine analysis together with the assumption of positivity of the coefficient q shows that if $y(t)$ is a (positive) solution of (A) defined on $J = [t_0, T)$, then $y'(t)$ must be either strictly increasing or strictly decreasing on J .

Suppose first that $y'(t_0) > 0$. Then, $y'(t) \geq y'(t_0) > 0$, $t \in J$, so that $y(t)$ is strictly increasing on J and the equation (A) can be written as

$$((y'(t))^{-\alpha})' + q(t)(y(t))^\beta = 0, \quad t \in J.$$

In case $y(t)$ is a singular solution of (A) existing on $[t_0, T)$, $T < \infty$, the following two cases are possible: Either

$$(I) \quad \lim_{t \rightarrow T-0} y(t) = \infty, \quad \lim_{t \rightarrow T-0} y'(t) = \infty$$

or

$$(II) \quad \lim_{t \rightarrow T-0} y(t) < \infty, \quad \lim_{t \rightarrow T-0} y'(t) = \infty.$$

Usually a singular solution of (A) satisfying (I) is called a *blowing-up solution*. We note that a singular solution satisfying (II) is a new type of solution which has never appeared in any textbooks or research articles. We will call a solution of the type (II) an increasing *black hole solution* of (A).

As mentioned in the Introduction, our main purpose here is to show that

there does exist a class of equations of the type (A) which really admit black hole solutions.

If $y(t)$ is an increasing proper solution of (A), that is, if $T = \infty$, then either

$$(III) \quad \lim_{t \rightarrow \infty} y(t) = \infty, \quad \lim_{t \rightarrow \infty} y'(t) = \infty$$

or

$$(IV) \quad \lim_{t \rightarrow \infty} y(t) = \infty, \quad 0 < \lim_{t \rightarrow \infty} y'(t) < \infty.$$

We call $y(t)$ a *strongly increasing* proper solution if it satisfies (III) and a *weakly increasing* proper solution if (IV) holds.

It remains to examine the structure of the set of decreasing solutions of (A).

Thus, let $y(t)$ be a positive solution of (A) on J with $y'(t_0) < 0$. Then, $y'(t) \leq y'(t_0) < 0$, $t \in J$, and the equation (A) is written as

$$((-y'(t))^{-\alpha})' + q(t)(y(t))^\beta = 0, \quad t \in J.$$

Integrating the inequality $y'(t) \leq y'(t_0) < 0$, $t \in J$, we have

$$y(t) \leq y(t_0) + y'(t_0)(t - t_0), \quad t \in J,$$

which shows that the interval J cannot be infinite. It follows that any positive decreasing solution of (A) must necessarily be singular, in which case there are, in general, three possibilities: Either

$$(V) \quad \lim_{t \rightarrow T-0} y(t) > 0, \quad \lim_{t \rightarrow T-0} y'(t) = -\infty$$

or

$$(VI) \quad \lim_{t \rightarrow T-0} y(t) = 0, \quad \lim_{t \rightarrow T-0} y'(t) = -\infty$$

or

$$(VII) \quad \lim_{t \rightarrow T-0} y(t) = 0, \quad -\infty < \lim_{t \rightarrow T-0} y'(t) < 0.$$

A solution of the type (V) is referred to as a decreasing *black hole* solution, while a solution of the type (VI) is termed an *extinct* solution of the *second kind* (or a solution with a 'zero black hole' at T). Positive solutions of the type (VII) will be called *extinct* solutions of the *third kind*.

We remark that by *extinct* solutions of the *first kind* we understand singular solutions satisfying

$$\lim_{t \rightarrow T-0} y(t) = 0, \quad \lim_{t \rightarrow T-0} y'(t) = 0$$

where $T < \infty$ is the right end-point of the maximal interval of existence which may exist for equations of the form (A) with $\alpha < 0$. Clearly, equation (A) with $\alpha > 0$ and positive q cannot have extinct solutions of this kind.

3. Existence of black hole solutions

We begin this section by establishing 'local' existence of increasing black hole solutions of (A), that is, singular solutions satisfying

$$(3) \quad \lim_{t \rightarrow T-0} y(t) = c > 0 \quad \text{and} \quad \lim_{t \rightarrow T-0} y'(t) = \infty$$

at the right end-point T of the maximal interval of existence.

Theorem 1. *For any $T > a$ and any $c > 0$, Eq. (A) possesses an increasing black hole solution defined on some interval $[t_0, T)$, $a \leq t_0 < T$, and satisfying (3) at T if and only if $\alpha > 1$.*

Proof. (The “only if” part.) Let $y(t)$ be a positive increasing solution of (A) defined on $[t_0, T)$, $t_0 \geq a$, and such that (3) is satisfied. Rewriting (A) as

$$((y'(t))^{-\alpha})' + q(t)(y(t))^\beta = 0, \quad t \in [t_0, T),$$

and integrating it twice from t to T , we obtain

$$y'(t) = \left[\int_t^T q(r)(y(r))^\beta dr \right]^{-1/\alpha}$$

and

$$y(t) = c - \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [t_0, T),$$

where we have used that $(y'(t))^{-\alpha} \rightarrow 0$ as $t \rightarrow T-0$. Using the inequality $y(t_0) \leq y(t)$ if $\beta < 0$, and the inequality $y(t) \leq y(T-0) = c$ if $\beta \geq 0$, we have

$$\int_{t_0}^T \left[\int_s^T q(r) dr \right]^{-1/\alpha} ds < \infty$$

which is possible only if $\alpha > 1$.

(The “if” part.) Suppose that $\alpha > 1$. Let $T > a$ and $c > 0$ be given arbitrarily. Define $q_*(t, T) = \min_{s \in [t, T]} q(s)$, $a \leq t \leq T$, and choose a $t_0 \in [a, T)$ so large that

$$(4) \quad \frac{\alpha}{\alpha-1} [q_*(t_0, T)]^{-1/\alpha} (T-t_0)^{(\alpha-1)/\alpha} \leq \frac{1}{2} c^{1+(\beta/\alpha)}$$

if $\beta < 0$, and

$$(5) \quad \frac{\alpha}{\alpha-1} [q_*(t_0, T)]^{-1/\alpha} (T-t_0)^{(\alpha-1)/\alpha} \leq \left(\frac{c}{2} \right)^{1+(\beta/\alpha)}$$

if $\beta \geq 0$. Consider the set $Y \subset C[t_0, T]$ and the integral operator $F: Y \rightarrow C[t_0, T]$ given by

$$Y = \left\{ y \in C[t_0, T] : \frac{c}{2} \leq y(t) \leq c, t \in [t_0, T] \right\}$$

and

$$(Fy)(t) = c - \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [t_0, T].$$

It is easily checked that (i) $F(Y) \subset Y$, (ii) F is continuous, and (iii) $F(Y)$ is compact in $C[t_0, T]$. Thus, by the Schauder's fixed point theorem there exists a function $y \in Y$ such that $y = Fy$, that is

$$y(t) = c - \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [t_0, T],$$

and differentiation of this integral equation shows that $y(t)$, when restricted to $[t_0, T]$, is the desired increasing solution of (A) which satisfies (3).

An inspection of the proof of Theorem 1 shows that if $t_0 = a$ and $T > a$ are fixed and $c > 0$ is chosen so that (4) (resp. (5)) is satisfied if $\beta < 0$ and $\alpha + \beta \neq 0$ (resp. $\beta \geq 0$), then the desired black hole solution is guaranteed to exist on the entire given interval $[a, T]$ and the above theorem can be restated as the following 'global' existence result.

Theorem 2. Suppose that $\alpha + \beta \neq 0$. Then, for any $T > a$, Eq. (A) possesses an increasing black hole solution defined on $[a, T]$ and satisfying (3) for some $c > 0$ if and only if $\alpha > 1$.

Next we will show that the condition $\alpha > 1$ from Theorems 1 and 2 is a necessary and sufficient condition for equation (A) to have, for any $T > a$ and any $c > 0$, also a decreasing positive solution of black hole type.

Indeed, if $T > a$ and $c > 0$ are given arbitrarily and $t_0 \in [a, T]$ is chosen so large that the expression on the left hand side of (4) is less than $2^{\beta/\alpha} c^{1+(\beta/\alpha)}$ if $\beta < 0$, and less than $c^{1+(\beta/\alpha)}$ if $\beta \geq 0$, then the set $Y \subset C[t_0, T]$ and the operator $F : Y \rightarrow C[t_0, T]$ defined by

$$Y = \{y \in C[t_0, T] : c \leq y(t) \leq 2c, t \in [t_0, T]\}$$

and

$$(Fy)(t) = c + \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [t_0, T],$$

respectively, satisfy all the conditions of the Schauder fixed point theorem, so that there exists a function $y \in Y$ satisfying the integral equation

$$y(t) = c + \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [t_0, T]$$

which gives a decreasing black hole solution of (A) having at $t = T$ the singular behavior

$$(6) \quad \lim_{t \rightarrow T-0} y(t) = c > 0 \quad \text{and} \quad \lim_{t \rightarrow T-0} y'(t) = -\infty.$$

The necessity of the condition $\alpha > 1$ for the existence of decreasing black hole solutions can be proved similarly as in the proof of Theorem 1.

Thus, we have proved the following

Theorem 3. *A necessary and sufficient condition for Eq. (A) to have, for any given $T > a$ and $c > 0$, a decreasing black hole solution defined in some left neighborhood of T and satisfying (6) is $\alpha > 1$.*

Again, if we are interested in ‘global’ existence of decreasing black hole solutions rather than in a ‘local’ one, the above result can be restated as follows.

Theorem 4. *Suppose that $\alpha + \beta \neq 0$. Then, for any $T > a$, Eq. (A) has a ‘global’ decreasing black hole solution defined on $[a, T)$ and satisfying (6) for some $c > 0$ if and only if $\alpha > 1$.*

From Theorems 1 and 3 we know that regardless of positivity or negativity of the exponent β Eq. (A) always has decreasing and increasing black hole solutions iff $\alpha > 1$. It is natural to ask whether in this case (A) may have also other singular or proper solutions, that is, we are interested in the problem of coexistence of various kinds of proper and singular solutions appearing in the general classification scheme presented in Section 2.

First we will show that if $q \in C^1[a, \infty)$, $\alpha > 1$ and $\beta < -1$, then there are no other decreasing solutions of (A) except for black hole ones.

Indeed, let $y(t)$ be any decreasing solution of (A) on $J = [t_0, T)$. Define the function $W[y](t)$ by

$$(7) \quad W[y](t) = \frac{\alpha}{1-\alpha} (-y'(t))^{1-\alpha} + \frac{q(t)}{\beta+1} (y(t))^{\beta+1}, \quad t \in J,$$

and suppose that $\alpha > 1$ and $\beta < -1$. We easily have

$$\frac{d}{dt} W[y](t) = \frac{q'(t)}{\beta+1} (y(t))^{1+\beta}, \quad t \in J,$$

from which, noting that $W[y](t) < 0$, $t \in J$, and

$$-\frac{q'(t)}{\beta+1} (y(t))^{1+\beta} \leq -\frac{q'_+(t)}{\beta+1} (y(t))^{1+\beta} \leq \frac{q'_+(t)}{q(t)} |W[y](t)|, \quad t \in J,$$

we find that

$$\frac{d}{dt} |W[y](t)| \leq \frac{q'_+(t)}{q(t)} |W[y](t)|, \quad t \in J,$$

where $q'_+(t) = \max\{q'(t), 0\}$, which implies that

$$(8) \quad |W[y](t)| \leq |W[y](t_0)| \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J.$$

Thus,

$$(9) \quad -\frac{q(t)}{1+\beta} (y(t))^{\beta+1} \leq |W[y](t_0)| \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J,$$

which precludes the possibility $\lim_{t \rightarrow T-0} y(t) = 0$, showing that $y(t)$ must be a black hole solution.

Thus, we have proved the following

Theorem 5. *Suppose that $q \in C^1[a, \infty)$. If $\alpha > 1$ and $\beta < -1$, then the only decreasing solutions of (A) are black hole solutions.*

An analogous result holds also in the case of increasing singular solutions of (A).

Theorem 6. *Suppose that $q \in C^1[a, \infty)$ and $-\beta < 1 < \alpha$. Then all increasing singular solutions of (A) are black hole solutions.*

Proof. Let $y(t)$ be an increasing singular solution of (A) defined on $J = [t_0, T)$, $a \leq t_0 < T < \infty$. Suppose that $\alpha > 1$ and $\beta > -1$, and consider the function $V[y](t)$ defined by

$$(10) \quad V[y](t) = \frac{\alpha}{\alpha-1} (y'(t))^{1-\alpha} + \frac{q(t)}{\beta+1} (y(t))^{\beta+1}, \quad t \in J.$$

Differentiating (10) and using (A), we see that

$$(11) \quad \frac{d}{dt} V[y](t) = \frac{q'(t)}{\beta+1} (y(t))^{\beta+1}, \quad t \in J.$$

Since $V[y](t) > 0$ on J , we have

$$\frac{q'(t)}{\beta+1} (y(t))^{\beta+1} \leq \frac{q'_+(t)}{\beta+1} (y(t))^{\beta+1} \leq \frac{q'_+(t)}{q(t)} V[y](t), \quad t \in J,$$

and from (11) we obtain

$$(12) \quad \frac{d}{dt} V[y](t) \leq \frac{q'_+(t)}{q(t)} V[y](t), \quad t \in J.$$

Integration of (12) then yields

$$V[y](t) \leq V[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J,$$

which implies in particular that

$$\frac{q(t)}{\beta+1} (y(t))^{\beta+1} \leq V[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J.$$

The above inequality prevents $y(t)$ from blowing up at the right end-point of J , and so all increasing singular solutions must be black hole solutions.

In the next theorem we will show that in the case $\alpha > 1$ and $-1 < \beta < \alpha$ black hole solutions and extinct solutions of the second kind always coexist for (A).

Theorem 7. *Let $\alpha > 1$ and $-1 < \beta < \alpha$. Then, for any $T > a$, Eq. (A) has an extinct solution of the second kind defined on $[a, T]$.*

Proof. Let $T > a$ be given. Denote

$$q_* = \min_{t \in [a, T]} q(t), \quad q^* = \max_{t \in [a, T]} q(t)$$

and put

$$(13) \quad K = \left(\frac{\alpha + \beta}{\alpha - 1}\right) \left[\frac{\alpha(\beta + 1)}{\alpha + \beta}\right]^{1/\alpha}.$$

Consider first the case $-1 < \beta < 0$ and define

$$(14) \quad c_1 = \left(\frac{K^\alpha}{q^*}\right)^{1/(\alpha+\beta)} \quad \text{and} \quad c_2 = \left(\frac{K^\alpha}{q_*}\right)^{1/(\alpha+\beta)}.$$

Since $\alpha > 1$ and $-1 < \beta < 0$ imply $c_1 \leq c_2$, we can define the set $Y \subset C[a, T]$ and the mapping $F: Y \rightarrow C[a, T]$ by

$$Y = \{y \in C[a, T] : c_1(T-t)^{(\alpha-1)/(\alpha+\beta)} \leq y(t) \leq c_2(T-t)^{(\alpha-1)/(\alpha+\beta)}, t \in [a, T]\}$$

and

$$(Fy)(t) = \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [a, T].$$

It is not hard to verify that (i) $F(Y) \subset Y$, (ii) F is continuous, and (iii) $F(Y)$ is compact in $C[a, T]$. Therefore, there exists a fixed element $y \in Y$ of F by the Schauder fixed point theorem. This fixed element $y = y(t)$ satisfies the integral equation

$$y(t) = \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [a, T],$$

which implies that $y(t)$ is a solution of (A) on $[a, T)$ and satisfies $\lim_{t \rightarrow T-0} y(t) = 0$ and $\lim_{t \rightarrow T-0} y'(t) = -\infty$. This establishes the existence of the desired extinct solution of the second kind for (A).

Consider now the case $0 \leq \beta < \alpha$. Define

$$(15) \quad c_1 = (K_1^\alpha K_2^{-\beta})^{\alpha/(\alpha^2 - \beta^2)} \quad \text{and} \quad c_2 = (K_2^\alpha K_1^{-\beta})^{\alpha/(\alpha^2 - \beta^2)}$$

where $K_1 = (q^*)^{-1/\alpha} K$, $K_2 = (q_*)^{-1/\alpha} K$ and q_* , q^* and K are as above. Then $c_1 \leq c_2$ and we can consider the set $Y \subset C[a, T]$ and the mapping $F: Y \rightarrow C[a, T]$ given as in the first part of the proof concerning the case $-1 < \beta < 0$. The rest of the proof is similar to that of the preceding case and is omitted.

Example 1. Consider the equation

$$(16) \quad (|y'|^{-\alpha})' + \gamma y^\beta = 0$$

where $\alpha > 0$, $\beta \in \mathbf{R}$ and $\gamma > 0$ are constants. According to Theorem 7, if $\alpha > 1$ and $-1 < \beta < \alpha$, then, for any $T > a$, Eq. (16) has an extinct solution of the second kind defined on $[a, T)$. Indeed, for any $T > a$, the function

$$(17) \quad y(t) = c(T - t)^{(\alpha-1)/(\alpha+\beta)}$$

where

$$(18) \quad c = \left[\frac{\alpha(1+\beta)}{\gamma(\alpha+\beta)} \left| \frac{\alpha+\beta}{\alpha-1} \right|^\alpha \right]^{1/(\alpha+\beta)},$$

is the extinct solution of the second kind of the equation (16).

Motivated by this example, we believe that the following conjecture is true.

Conjecture 1. For any $T > a$, Eq. (A) has an extinct solution of the second kind defined on $[a, T)$ if and only if

$$(19) \quad 0 < \frac{\alpha-1}{\alpha+\beta} < 1.$$

In our next theorem we will show that there is a large class of equations of the form (A) which admit no decreasing black hole solutions and no extinct solutions of the second kind.

Theorem 8. Suppose that $q \in C^1[a, \infty)$. If $0 < \alpha < 1$ and $\beta > -1$, then the only possible decreasing solutions of Eq. (A) are extinct solutions of the third kind.

Proof. Let $y(t)$ be a positive decreasing solution of (A) on $J = [t_0, T)$. From Section 2 we know that this solution must necessarily be singular, that is, $T < \infty$.

Assume that $0 < \alpha < 1$, $\beta > -1$ and define the function $W[y](t)$ by (7). Then we have

$$(20) \quad \frac{d}{dt} W[y](t) \leq \frac{q'_+(t)}{q(t)} W[y](t), \quad t \in J,$$

which implies that

$$(21) \quad W[y](t) \leq W[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J.$$

From (7) and (21) it follows that

$$\frac{\alpha}{1-\alpha} (-y'(t))^{1-\alpha} \leq W[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J,$$

which precludes the possibility $\lim_{t \rightarrow T-0} y'(t) = -\infty$, showing that $y(t)$ must be an extinct solution of the third kind.

As our last theorem in this section shows, there is a class of equations of the form (A) for which the extinct solutions of the third kind (with arbitrarily prescribed extinction points) do really exist.

Theorem 9. *Suppose that $-1 < \beta < 0$ and $\alpha + \beta \neq 0$. Then, for any $T > a$, Eq. (A) has an extinct solution of the third kind with $[a, T]$ as its maximal interval of existence.*

Proof. Let $T > a$ be given arbitrarily. Put $q^* = \max_{t \in [a, T]} q(t)$ and choose $c > 0$ satisfying

$$\frac{1}{\beta+1} c^{\alpha+\beta} q^* T^{\beta+1} \leq 1 - 2^{-\alpha}.$$

Let Y denote the set

$$Y = \{y \in C[a, T] : c(T-t) \leq y(t) \leq 2c(T-t), t \in [a, T]\}$$

and define the integral operator $F : Y \rightarrow C[a, T]$ by

$$(Fy)(t) = \int_t^T \left[(2c)^{-\alpha} + \int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [a, T].$$

It is easy to verify that F is continuous and maps Y into a compact subset of Y . Therefore, by the Schauder fixed point theorem, F has a fixed element y in Y . Clearly, this fixed point $y = y(t)$ satisfies

$$y(t) = \int_t^T \left[(2c)^{-\alpha} + \int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [a, T]$$

and differentiation of this integral equation shows that $y(t)$, when restricted to $[a, T)$, is an extinct solution of the the third kind of (A) with extinction point at $t = T$. This completes the proof.

4. Existence of blowing-up solutions

Now we turn our attention to the set of increasing singular solutions of Eq. (A) which satisfy (I). Our first aim is to establish conditions guaranteeing the existence of such solutions with arbitrarily prescribed (finite) blow-up points.

Theorem 10. *Suppose that $0 < \alpha < 1$ and $|\beta| < \alpha$. Then, for any $T > a$, Eq. (A) has a positive solution defined on $[a, T)$ which blows up at T .*

Proof. Let $T > a$ be fixed arbitrarily. As in the proof of Theorem 7, denote $q_* = \min_{t \in [a, T]} q(t)$ and $q^* = \max_{t \in [a, T]} q(t)$ and put

$$K = \left(\frac{\alpha + \beta}{1 - \alpha} \right) \left[\frac{\alpha(1 + \beta)}{\alpha + \beta} \right]^{1/\alpha}.$$

With this K define the constants c_1 and c_2 by (14) or (15) according as $\beta < 0$ or $\beta \geq 0$.

Denote by $C[a, T)$ the locally convex space of all continuous functions on $[a, T)$ with the topology of uniform convergence on every compact subinterval of $[a, T)$, and consider a closed convex subset Y of $C[a, T)$ given by

$$(22) \quad Y = \{y \in C[a, T) : c_1(T - t)^{(\alpha-1)/(\alpha+\beta)} \leq y(t) \leq c_2(T - t)^{(\alpha-1)/(\alpha+\beta)}, t \in [a, T)\}.$$

Let $F : Y \rightarrow C[a, T)$ be the integral operator defined by

$$(23) \quad (Fy)(t) = c_1(T - a)^{(\alpha-1)/(\alpha+\beta)} + \int_a^t \left[\int_s^T q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \in [a, T).$$

An elementary computation shows that, for any $y \in Y$ and $t \in [a, T)$,

$$\begin{aligned} (Fy)(t) &\leq c_1(T - a)^{(\alpha-1)/(\alpha+\beta)} + (c_i^\beta q_*)^{-1/\alpha} K[(T - t)^{(\alpha-1)/(\alpha+\beta)} - (T - a)^{(\alpha-1)/(\alpha+\beta)}] \\ &\leq (c_i^\beta q_*)^{-1/\alpha} K(T - t)^{(\alpha-1)/(\alpha+\beta)} = c_2(T - t)^{(\alpha-1)/(\alpha+\beta)}, \end{aligned}$$

where $i = 1$ if $\beta \geq 0$ and $i = 2$ if $\beta < 0$, and

$$\begin{aligned} (Fy)(t) &\geq c_1(T - a)^{(\alpha-1)/(\alpha+\beta)} + (c_i^\beta q^*)^{-1/\alpha} K[(T - t)^{(\alpha-1)/(\alpha+\beta)} - (T - a)^{(\alpha-1)/(\alpha+\beta)}] \\ &\geq (c_i^\beta q^*)^{-1/\alpha} K(T - t)^{(\alpha-1)/(\alpha+\beta)} = c_1(T - t)^{(\alpha-1)/(\alpha+\beta)}, \end{aligned}$$

where $i = 1$ if $\beta < 0$ and $i = 2$ if $\beta \geq 0$, which implies that F maps Y into itself.

It can be shown in a routine manner that F is continuous on Y and $F(Y)$ is relatively compact in $C[a, T)$. From the Schauder-Tychonoff fixed point theorem it follows that F has a fixed element y in Y . Differentiating the integral equation $y = Fy$, we see that this fixed element $y = y(t)$ is a solution of (A) which is positive on $[a, T)$ and blows up as t approaches $T - 0$. This completes the proof.

From the results of the preceding section we know that there are no blowing-up solutions for Eq. (A) if $q \in C^1[a, \infty)$ and $-\beta < 1 < \alpha$, while Theorem 10 says that there always exists a blowing-up solution for Eq. (A) if $0 < \alpha < 1$ and $|\beta| < \alpha$. A question now naturally arises: What happens for blowing-up solutions in the remaining cases for α and β ? A partial answer to this question is given by the following theorem.

Theorem 11. *Suppose that $q \in C^1[a, \infty)$ and $0 < \alpha < 1$ and $\beta < -1$. Then there are no increasing singular solutions of Eq. (A).*

Proof. Let $y(t)$ be an increasing solution of (A) defined on $J \subset [a, \infty)$. Consider the function $V[y](t)$ defined by (10). In this case $V[y](t) < 0$, $t \in J$, and we have

$$(24) \quad \frac{d}{dt}|V[y](t)| = -\frac{q'(t)}{\beta+1}(y(t))^{\beta+1} \leq \frac{q'_+(t)}{q(t)}|V[y](t)|, \quad t \in J,$$

which implies that

$$(25) \quad |V[y](t)| \leq |V[y](t_0)| \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right)$$

for $t \in J$. In particular

$$(26) \quad \frac{\alpha}{1-\alpha}(y'(t))^{1-\alpha} \leq |V[y](t_0)| \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J,$$

which prevents $y'(t)$ from tending to $+\infty$ at a finite point of J . Therefore, $y(t)$ can be neither a blowing-up solution nor an increasing black hole solution.

We close this section with an example and a conjecture.

Example 2. Consider again the equation (16) with a constant coefficient γ . Theorem 10 ensures the existence of blowing-up solutions in the range $0 < \alpha < 1$, $|\beta| < \alpha$. As easily seen, if

$$(27) \quad \frac{\alpha-1}{\alpha+\beta} < 0,$$

which is obviously satisfied if the conditions of Theorem 10 hold, then the function $y(t)$ given by (17) provides an example of finite time blow-up for the equation (16).

The following conjecture has been motivated by the above example.

Conjecture 2. For any $T > a$, Eq. (A) has a solution defined on $[a, T)$ which blows up at T if and only if (27) is satisfied.

5. Existence of increasing proper solutions

Our final task is to study the problem of existence (and nonexistence) of proper solutions of the equation (A) which, as mentioned in Section 2, can be partitioned into two subclasses according to whether (III) or (IV) is satisfied.

One can characterize the existence of an increasing proper solution of type (IV) for the equation (A) as the following theorem shows.

Theorem 12. *The equation (A) has a weakly increasing proper solution if and only if*

$$(28) \quad \int_a^\infty t^\beta q(t) dt < \infty.$$

Proof. (The “only if” part.) Assume that (A) has a weakly increasing positive solution $y(t)$ on $[a, \infty)$. Integration of (A) on $[t, \infty)$, $t \geq a$, gives

$$(y'(\infty))^{-\alpha} - (y'(t))^{-\alpha} + \int_t^\infty q(s)(y(s))^\beta ds = 0,$$

or

$$(29) \quad y'(t) = \left[(y'(\infty))^{-\alpha} + \int_t^\infty q(s)(y(s))^\beta ds \right]^{-1/\alpha}, \quad t \geq a,$$

which implies in particular that

$$(30) \quad \int_t^\infty q(s)(y(s))^\beta ds < \infty, \quad t \geq a.$$

Since $y'(\infty) = \lim_{t \rightarrow \infty} y'(t) = \lim_{t \rightarrow \infty} [y(t)/t] = \text{const} > 0$ by (IV), there are positive constants c_1 , c_2 and $t_1 > a$ such that

$$(31) \quad c_1 t \leq y(t) \leq c_2 t, \quad t \geq t_1.$$

Combining (30) with (31) yields

$$c_i^\beta \int_{t_1}^\infty s^\beta q(s) ds < \infty,$$

where $i = 1$ if $\beta \geq 0$ and $i = 2$ if $\beta < 0$, which verifies the truth of (28).

We observe upon integration of (29) that any weakly increasing positive solution $y(t)$ defined on $[a, \infty)$ satisfies the integral equation

$$(32) \quad y(t) = y(a) + \int_a^t \left[(y'(\infty))^{-\alpha} + \int_s^\infty q(r)(y(r))^\beta ds \right]^{-1/\alpha} ds, \quad t \geq a.$$

(The “if” part.) Suppose that (28) holds. Let $\beta < 0$. Choose positive constants $y_0 > 0$ and $\eta_1 > 0$ so that

$$(33) \quad \int_a^\infty \left(y_0 + \frac{1}{2} \eta_1 (t-a) \right)^\beta q(t) dt \leq (2^\alpha - 1) \eta_1^{-\alpha},$$

which is clearly possible. Let Y denote the set

$$(34) \quad Y = \{ y \in C[a, \infty) : y_0 + \frac{1}{2} \eta_1 (t-a) \leq y(t) \leq y_0 + \eta_1 (t-a), t \geq a \}$$

and define the mapping $F : Y \rightarrow C[a, \infty)$ by

$$(35) \quad (Fy)(t) = y_0 + \int_a^t \left[\eta_1^{-\alpha} + \int_s^\infty q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \geq a.$$

The inequality (33) can be used to show that F maps Y into itself. It can be routinely verified that F is a continuous mapping and that $F(Y)$ is relatively compact in $C[a, \infty)$. Therefore the Schauder-Tychonoff fixed point theorem applies and there exists an element $y \in Y$ such that $y = Fy$, which satisfies

$$(36) \quad y(t) = y_0 + \int_a^t \left[\eta_1^{-\alpha} + \int_s^\infty q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \geq a.$$

(Note that (36) coincides with (32): $y_0 = y(a)$, $\eta_1 = y'(\infty)$.) Differentiating (36), we conclude that $y(t)$ is a positive increasing proper solution satisfying $y'(\infty) = \eta_1$. Hence $y(t)$ provides a desired weakly increasing solution of (A). This completes the proof in the case $\beta < 0$.

The proof in the case $\beta \geq 0$ is a slight variation of the procedure used above with

$$\int_a^\infty (y_0 + \eta_1 (t-a))^\beta q(t) dt \leq (2^\alpha - 1) \eta_1^{-\alpha},$$

replacing (33), and is omitted to avoid the repetition.

The next theorem concerns strongly increasing solutions of Eq. (A), that is, the proper solutions $y(t)$ satisfying $\lim_{t \rightarrow \infty} y(t) = \infty$ and $\lim_{t \rightarrow \infty} y'(t) = \infty$.

Theorem 13. *Suppose that there exist constants $q_1, q_2 > 0$ and $\sigma \in \mathbf{R}$ such that*

$$(37) \quad q_1 t^\sigma \leq q(t) \leq q_2 t^\sigma$$

for all large t . Moreover, let either (i) $\sigma \geq -1$ and $\sigma + 1 < -\beta < \alpha$; or (ii) $\sigma < -1$ and one of the following three cases holds:

- a) $0 < -\beta < \alpha$,
- b) $0 \leq \beta < \alpha \leq -\sigma - 1$,
- c) $0 \leq \beta < -\sigma - 1 < \alpha$.

Then Eq. (A) has a strongly increasing proper solution.

Proof. Let $t_1 \geq a$ be so large that (37) is satisfied for $t \geq t_1$. Put

$$L = \frac{\alpha + \beta}{\alpha - \sigma - 1} \left[-\frac{\alpha(\beta + \sigma + 1)}{\alpha + \beta} \right]^{1/\alpha}$$

and define

$$c_1 = (L^\alpha/q_2)^{1/(\alpha+\beta)}, \quad c_2 = (L^\alpha/q_1)^{1/(\alpha+\beta)},$$

if $\beta < 0$, that is, the case (i) or (ii)(a) holds, and

$$c_1 = (L_1^\alpha L_2^{-\beta})^{\alpha/(\alpha^2-\beta^2)}, \quad c_2 = (L_2^\alpha L_1^{-\beta})^{\alpha/(\alpha^2-\beta^2)},$$

where $L_1 = (q_2)^{-1/\alpha} L$ and $L_2 = (q_1)^{-1/\alpha} L$, if $\beta \geq 0$, (i.e., (ii)(b) or (ii)(c) is satisfied). Then $c_1 \leq c_2$ and we can define the set Y by

$$Y = \{y \in C[t_1, \infty) : c_1 t^{(\alpha-\sigma-1)/(\alpha+\beta)} \leq y(t) \leq c_2 t^{(\alpha-\sigma-1)/(\alpha+\beta)}, t \geq t_1\},$$

which is clearly a closed convex subset of the locally convex space $C[t_1, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$. Define the integral operator $F : Y \rightarrow C[t_1, \infty)$ by

$$(38) \quad (Fy)(t) = c_1 t_1^{(\alpha-\sigma-1)/(\alpha+\beta)} + \int_{t_1}^t \left[\int_s^\infty q(r)(y(r))^\beta dr \right]^{-1/\alpha} ds, \quad t \geq t_1.$$

From (37) and (38) we see that, for any $y \in Y$ and $t \geq t_1$,

$$\begin{aligned} (Fy)(t) &\leq c_1 t_1^{(\alpha-\sigma-1)/(\alpha+\beta)} + (c_i^\beta q_1)^{-1/\alpha} \int_{t_1}^t \left[\int_s^\infty r^\sigma r^{\beta(\alpha-\sigma-1)/(\alpha+\beta)} dr \right]^{-1/\alpha} ds \\ &= c_1 t_1^{(\alpha-\sigma-1)/(\alpha+\beta)} + (c_i^\beta q_1)^{-1/\alpha} L t^{(\alpha-\sigma-1)/(\alpha+\beta)} - (c_i^\beta q_1)^{-1/\alpha} L t_1^{(\alpha-\sigma-1)/(\alpha+\beta)} \\ &\leq (c_i^\beta q_1)^{-1/\alpha} L t^{(\alpha-\sigma-1)/(\alpha+\beta)} = c_2 t^{(\alpha-\sigma-1)/(\alpha+\beta)}, \end{aligned}$$

where $i = 2$ if $\beta < 0$ and $i = 1$ if $\beta \geq 0$, and

$$\begin{aligned} (Fy)(t) &\geq c_1 t_1^{(\alpha-\sigma-1)/(\alpha+\beta)} + (c_i^\beta q_2)^{-1/\alpha} \int_{t_1}^t \left[\int_s^\infty r^\sigma r^{\beta(\alpha-\sigma-1)/(\alpha+\beta)} dr \right]^{-1/\alpha} ds \\ &\geq (c_i^\beta q_2)^{-1/\alpha} L t^{(\alpha-\sigma-1)/(\alpha+\beta)} = c_1 t^{(\alpha-\sigma-1)/(\alpha+\beta)}, \end{aligned}$$

where $i = 1$ if $\beta < 0$ and $i = 2$ if $\beta \geq 0$, which implies that $Fy \in Y$, and hence F maps Y into itself.

We can prove routinely that F is continuous and $F(Y)$ is a relatively compact subset of $C[t_1, \infty)$. Therefore, the Schauder-Tychonoff fixed point theorem guarantees that F has a fixed $y \in Y$, giving rise to the desired strongly increasing solution $y(t)$ of (A).

Useful information about the structure of the set of proper solutions of (A) can be obtained by examining the Lyapunov-like function $V[y](t)$ defined by (10), as the following two theorems show.

Theorem 14. *Suppose that $q \in C^1[a, \infty)$, $\alpha < 1 < -\beta$ and that*

$$(39) \quad \int_a^\infty \frac{q'_+(t)}{q(t)} dt < \infty.$$

Then, all increasing solutions of (A) are weakly increasing proper solutions.

Proof. From Theorem 11 we know that the conditions $q \in C^1[a, \infty)$ and $\alpha < 1 < -\beta$ prevent (A) from having increasing singular solutions. Thus, if $y(t)$ is an increasing solutions of (A) defined on $J = [t_0, T)$, $t_0 \geq a$, then $T = \infty$, i.e. $y(t)$ must necessarily be a proper solution.

Consider the function $V[y](t)$ defined by (10). Then, as in the proof of Theorem 11, $V[y](t) < 0$ on J and we obtain (26) for all $t \in J$. If (39) is satisfied, then, in view of (26), $y'(t)$ cannot tend to ∞ as $t \rightarrow \infty$, which implies that all positive increasing proper solutions are necessarily weakly increasing proper solutions.

Theorem 15. *Suppose that $q \in C^1[a, \infty)$ and $-\beta < 1 < \alpha$. If, moreover, (39) and*

$$(40) \quad \limsup_{t \rightarrow \infty} t^{\beta+1} q(t) = \infty$$

hold, then there are no proper solutions of Eq. (A).

Proof. Let $y(t)$ be an increasing proper solution of (A) defined on $J = [t_0, \infty)$, $t_0 \geq a$. Consider again the function $V[y](t)$ defined by (10) which, in this case, is strictly positive on J and satisfies

$$(41) \quad V[y](t) \leq V[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right)$$

for $t \geq t_0$. Since an increasing proper solution $y(t)$ of (A) is necessarily unbounded and grows to infinity at least as fast as a constant multiple of t as $t \rightarrow \infty$, there is a constant $c > 0$ such that $y(t) \geq ct$ for $t \geq t_0$. We then have from (41)

$$\frac{c^{\beta+1}}{\beta+1} t^{\beta+1} q(t) \leq V[y](t) \leq V[y](t_0) \exp\left(\int_{t_0}^t \frac{q'_+(s)}{q(s)} ds\right), \quad t \geq t_0,$$

whence, using (39) and (40) we reach a contradiction in the limit as $t \rightarrow \infty$.

Example 3. The results obtained in this section will now be illustrated by the example

$$(42) \quad (|y'|^{-\alpha})' + \gamma t^\sigma y^\beta = 0, \quad t \geq a > 0,$$

where $\alpha > 0, \beta, \sigma \in \mathbf{R}$ and $\gamma > 0$ are constants.

Applying Theorem 12, we obtain that (42) has a weakly increasing proper solution if and only if $\beta < -\sigma - 1$.

For equation (42) the condition (39) is satisfied if and only if $\sigma \leq 0$ and Theorem 14 then shows that all increasing solutions of (42) are weakly increasing proper solutions if $\sigma \leq 0$ and $\alpha < 1 < -\beta$.

If, in addition to $\sigma \leq 0$, the conditions $-\beta < 1 < \alpha$ and $\beta + \sigma + 1 > 0$ hold, then, according to Theorem 15, there are no proper solutions of the equation (42).

Finally, we apply Theorem 13 to conclude that any of the conditions (i), (ii)(a)–(c) is sufficient for the existence of a strongly increasing proper solution of (42). Indeed, an elementary calculation shows that the function $y(t)$ given by

$$y(t) = ct^{(\alpha-\sigma-1)/(\alpha+\beta)}, \quad t \geq a > 0,$$

where

$$c = \left[-\frac{\gamma}{\alpha} \left(\frac{\alpha + \beta}{\beta + \sigma + 1} \right) \left| \frac{\alpha - \sigma - 1}{\alpha + \beta} \right|^\alpha \right]^{-1/(\alpha+\beta)},$$

is a strongly increasing proper solution of the equation (42) if

$$(43) \quad \frac{\alpha - \sigma - 1}{\alpha + \beta} > 1,$$

which is clearly satisfied if any of the four conditions of Theorem 13 holds.

We end with a conjecture motivated by the gap between (43) and the conditions of Theorem 13.

Conjecture 3. Suppose that there exist constants $q_1, q_2 > 0$ and $\sigma \in \mathbf{R}$ such that (37) holds for all large t . Then Eq. (A) has a strongly increasing proper solution if and only if (43) is satisfied.

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