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Bounded Solutions of Quasilinear Hyperbolic Equations of Kirchhoff Type with Dissipative and Forcing Terms

By

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Introduction

Let X be a complex Hilbert space with the norm $\|\cdot\|$. Let A be a self-adjoint operator in X . Assume that A is positive, that is, $\min \sigma(A) = \sigma_0 > 0$, where $\sigma(A)$ denotes the spectrum of A . For $\alpha \geq 0$, let X_α be a Hilbert space defined by

$$X_\alpha = D(A^\alpha)$$

with norm $\|x\|_{X_\alpha} = \|A^\alpha x\|$.

First, we consider the initial value problem for the following abstract hyperbolic equation of Kirchhoff type with dissipation.

$$(0.1) \quad \frac{\partial^2 u}{\partial t^2}(t) + m(\|A^{1/2}u(t)\|^2)Au(t) + \gamma \frac{\partial u}{\partial t}(t) = f(t), \quad t \geq 0,$$

$$(0.2) \quad u(0) = \phi, \quad \frac{\partial u}{\partial t}(0) = \psi,$$

where γ is a positive constant,

$$m(\cdot) \in C^1([0, \infty)) \quad \text{and} \quad m(\lambda) > 0 \quad \text{for every } \lambda \geq 0.$$

We assume that

$$f \in C([0, \infty); X_{1/2}).$$

We give two examples which satisfy our assumptions.

Example 1. Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary, in which Poincaré's inequality holds. For example,

- (i) Ω is a bounded domain with smooth boundary, or
- (ii) $\Omega = (a, b) \times \mathbf{R}^{n-1}$ for $a, b \in \mathbf{R}$, or
- (iii) $\Omega = \mathbf{R}^n \setminus \bigcup_{J \in \mathbf{Z}^n} B_r(J)$, where $B_r(J)$ is a ball centered at J with radius r for fixed $r > 0$.

For such a domain Ω , we consider the initial boundary value problem

$$(0.3) \quad \frac{\partial^2 u}{\partial t^2} - m(\|\nabla u\|_{L^2}^2)Au + \gamma \frac{\partial u}{\partial t} = f(x, t), \quad x \in \Omega, \quad t > 0,$$

$$(0.4) \quad u(x, 0) = \phi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad x \in \Omega,$$

$$(0.5) \quad u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0.$$

To treat the above problem, we put $X = L^2(\Omega)$ and $A = -\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that $D(A^{1/2}) = H_0^1(\Omega)$ and $\|A^{1/2}u\| = \|\nabla u\|_{L^2}$, and that A is a positive self-adjoint operator.

Example 2. We consider the Cauchy problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - m(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2)(A - 1)u + \gamma \frac{\partial u}{\partial t} = f(x, t), & x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) = \phi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), & x \in \Omega. \end{cases}$$

Let $X = L^2(\mathbf{R}^n)$ and $A = -\Delta + I$ with domain $H^2(\mathbf{R}^n)$. It is well known that $D(A^{1/2}) = H^1(\mathbf{R}^n)$ and $\|A^{1/2}u\| = \sqrt{\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2}$, and that A is a positive self-adjoint operator.

In non-dissipative case, we do not know the unique global solvability of (0.3)–(0.4) for arbitrary small initial data in H^∞ , except for the special case $m(x)^2 = (a_1x + a_2)^{-2}$ with positive constants a_1 and a_2 . In this case, Pokhozhaev [9] proved the unique existence of global solutions of (0.3)–(0.5) with $\gamma = 0$ and Ω is a bounded domain in $\mathbf{R}^n$ for arbitrary initial data $(\phi, \psi) \in H^2(\Omega) \times H^1(\Omega)$.

In the case that $\gamma = 0$ and $m(x) = x + 1$, Greenberg and Hu [4] first showed the unique existence of a global solutions of (0.3)–(0.4) with $f \equiv 0$ and $\Omega = \mathbf{R}$ for small initial data in $H^2(\mathbf{R}) \times H^1(\mathbf{R})$ satisfying some decay conditions with respect to x . D’Ancona and Spagnolo [2] improved the result of [4] for arbitrary $\mathbf{R}^n$ and general form m . (See also, Yamazaki [12]). D’Ancona and Spagnolo [3] showed the global solvability for (0.3)–(0.4) with perturbation for initial data with compact support, and especially treat the case $\|f(t)\|_{H^1} = O(t^{-k})$ with some $k > 1$ as $t \rightarrow \infty$.

In case $\gamma > 0$, Brito [1] showed the unique existence of global solutions of (0.1)–(0.2) with $f \equiv 0$ for positive self-adjoint operators A , under the assumption that $\|A\phi\| + \|A^{1/2}\psi\|$ is sufficiently small. Independently, Yamada [11] showed the unique existence of a global solution of (0.3)–(0.4) for $\Omega = \mathbf{R}^n$, or (0.3)–(0.5) for a bounded domain Ω , under the assumption that $\|\phi\|_{H^2} + \|\psi\|_{H^1} + \int_0^\infty \|f(s)\|_{H^1} ds$ is sufficiently small for $f \in C([0, \infty); H^1)$. Ikehata and

Okazawa [6] proved a similar result for (0.1)–(0.2) under the assumption that $(\|A\phi\| + \|A^{1/2}\psi\| + \int_0^\infty \|A^{1/2}f(s)\|ds)$ ($\|A^{1/2}\phi\| + \|\psi\| + \int_0^\infty \|f(s)\|ds$) is sufficiently small, where A is a non-negative self-adjoint operator with compact resolvent and $f \in L^\infty([0, \infty); X)$. (See also Ikehata [5].) On the other hand, Tucsnak [10] considered the initial boundary value problem for (0.3)–(0.4) with $\gamma = 0$ and $f(t) \equiv 0$ on a bounded interval with a boundary condition $u(0, t) = 0$, $\partial_x u(1, t) = -\alpha \partial_t u(1, t)$ for $\alpha > 0$, which has the effect of dissipation, and showed the unique existence of global solutions provided $\|\phi\|_{H^2} + \|\psi\|_{H^1}$ is sufficiently small. Ono [8] generalized the result of [10] for $f \in \bigcap_{i=0,1} C^{1-i}([0, \infty); H^i)$ such that $\sup_{t \geq 0} \|f(t)\|_{H^1}$ is sufficiently small. In the papers [1], [11], [5] and [6], they first obtain the estimate of $\|A^{1/2}u(t)\| + \|\partial_t u(t)\|$ by assuming the boundedness of $\int_0^\infty \|f(s)\|ds$, (which means the decay of $\|f(t)\|$ as $t \rightarrow \infty$), and secondly obtain the estimate of $\|Au(t)\| + \|A^{1/2}\partial_t u(t)\|$ by assuming the smallness of $\int_0^\infty \|A^{1/2}f(s)\|ds$. In the papers [10] and [8], they adopted the procedure above, by using the boundary dissipation and the smallness of $\sup_{t \geq 0} \|f(t)\|_{H^1}$ instead of assuming the decay on $f(t)$ as $t \rightarrow \infty$.

In this paper, without assuming the decay of $\|f(t)\|$ as $t \rightarrow \infty$, we prove the estimate of $\|Au(t)\| + \|A^{1/2}\partial_t u(t)\|$, by using the fact that this quantity dominates $\|A^{1/2}u(t)\| + \|\partial_t u(t)\|$ if $\sigma_0 > 0$. Then, from the estimate above, we deduce the unique global solvability. In this procedure, the condition $\sigma_0 > 0$ is essential. In fact, the results of this paper do not hold in general if $\sigma_0 = 0$, as we will show by some examples given later. (See Remark 2.)

Theorem 1. *Let $\phi \in X_1$, $\psi \in X_{1/2}$. If $\|A\phi\| + \|A^{1/2}\psi\| + \sup_{t \geq 0} \|A^{1/2}f(t)\|$ is sufficiently small, then the initial value problem (0.1)–(0.2) has a unique global solution $u \in \bigcap_{i=0,1,2} C^i((0, \infty); X_{1-i/2})$.*

Furthermore, for every $\varepsilon > 0$, we can take $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that

$$(0.6) \quad \|A\phi\|^2 + \|A^{1/2}\psi\|^2 \leq \delta_1(\varepsilon)^2$$

and

$$(0.7) \quad \sup_{t \geq 0} \|A^{1/2}f(t)\| \leq \delta_2(\varepsilon)$$

imply

$$(0.8) \quad \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 \leq \varepsilon^2$$

for $t \geq 0$.

Remark 1. The constants $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ are defined by (1.3) in the proof.

Remark 2. Since we do not assume that $f \in L^1([0, \infty); X_{1/2})$, the assumption $\sigma_0 > 0$ is necessary for $\sup_{t \geq 0} \|u(t)\| < \infty$. In fact, if $0 \in \sigma(A)$,

then for every $0 < \theta < 1$ and $\varepsilon > 0$, we can construct examples f and $u \in C^\infty([0, \infty); D(A))$ such that $\sup_{t \geq 0} \|(I + A^{1/2})f(t)\| \leq \varepsilon$, $\|f(t)\| = O(t^{-\theta})$ as $t \rightarrow \infty$ and that u is the unique solution of the linear wave equation

$$(0.9) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(t) + Au(t) + \frac{\partial u}{\partial t}(t) = f(t), \\ u(0) = \frac{\partial u}{\partial t}(0) = 0, \end{cases}$$

and satisfies $\limsup_{t \rightarrow \infty} \|u(t)\| = \infty$. (The construction is given at the end of section 1.)

Next we consider the above equation for $t \in \mathbf{R}$:

$$(0.10) \quad \frac{\partial^2 u(t)}{\partial t^2} + m(\|A^{1/2}u(t)\|^2)Au(t) + \gamma \frac{\partial u(t)}{\partial t} = f(t),$$

where $m(\lambda)$ and γ are same as in the case of (0.1), and $f \in C(\mathbf{R}; X_{1/2})$.

Nakao [7] showed the unique existence of small bounded global solutions of semilinear hyperbolic equations with dissipation on bounded domains, under the assumption that $\sup_{t \in \mathbf{R}} \int_t^{t+1} \|f(s)\|_{L^2}^2 ds$ is small. He also showed that the small bounded solution is periodic or almost-periodic if f is periodic or almost-periodic. Ono [8] showed the unique existence of small bounded global solutions for (0.3) ($t \in \mathbf{R}, \gamma = 0$)–(0.4) with the dissipative boundary condition considered by Tucsnak [10] (see above), under the assumption that $\sup_{t \in \mathbf{R}} \|f(t)\|_{H^1}$ is small. He also showed that the small bounded solution is periodic if f is periodic. Using the method by Nakao [7] and Theorem 1, we obtain the following theorem:

Theorem 2. *Let $\delta_2(\cdot)$ be as in Theorem 1. There exists a positive number $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, if*

$$(0.11) \quad \sup_{t \in \mathbf{R}} \|A^{1/2}f(t)\| \leq \delta_2(\varepsilon),$$

then (0.10) has a unique global solution $u \in \bigcap_{i=0,1,2} C^i(\mathbf{R}; X_{1-i/2})$ satisfying

$$(0.12) \quad \sup_{t \in \mathbf{R}} (\|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2) \leq \varepsilon.$$

This solution is unique in the class $u \in \bigcap_{i=0,1,2} C^i(\mathbf{R}; X_{1-i/2})$ satisfying

$$(0.13) \quad \sup_{t \in \mathbf{R}} (\|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2) < \varepsilon_1^2.$$

Furthermore, the mapping $L : f \mapsto (u, \partial_t u)$ defined above is uniformly continuous on $\{f \in C(\mathbf{R}; X_{1/2}) \mid \sup_{t \in \mathbf{R}} \|A^{1/2}f(t)\| \leq \delta_2(\varepsilon)\}$ equipped with the topology of $C(\mathbf{R}; X)$ to $C(\mathbf{R}; X_{1/2} \times X)$.

In case f is time periodic, the next corollary follows immediately from the uniqueness in Theorem 2.

Corollary 1. *Let ε_1 and $\delta_2(\cdot)$ be as in Theorem 2. Let T be a positive number. If f is time-periodic (that is, periodic with respect to t), with period T and satisfies*

$$\sup_{0 \leq t \leq T} \|A^{1/2}f(t)\| \leq \delta_2(\varepsilon) \quad \text{for some } \varepsilon < \varepsilon_1,$$

then the solution $u(t)$ given in Theorem 2 is time-periodic with period T .

In case f is X -almost-periodic with respect to t , the next corollary follows immediately from the uniform continuity of the mapping L in Theorem 2. Here, for a Banach space Y with norm $\|\cdot\|_Y$, the function $f(t) \in C(\mathbf{R}; Y)$ is said to be Y -almost-periodic with respect to t , if for every $\varepsilon > 0$, there exists a positive number ℓ such that every interval of length ℓ contains at least one point τ for which

$$\sup_{t \in \mathbf{R}} \|f(t + \tau) - f(t)\|_Y \leq \varepsilon.$$

Corollary 2. *Let ε_1 and $\delta_2(\cdot)$ be as in Theorem 2. If f is X -almost-periodic with respect to t and satisfies (0.11) for some $\varepsilon < \varepsilon_1$, then the solution $(u(t), \partial_t u(t))$ given in Theorem 2 is $X_{1/2} \times X$ -almost-periodic.*

1. The global existence

In this section we prove Theorem 1 and construct examples mentioned in Remark 2.

By the assumption that $\min \sigma(A) = \sigma_0 > 0$, we have

$$(1.1) \quad \|x\| \leq \frac{1}{\sqrt{\sigma_0}} \|A^{1/2}x\| \equiv K \|A^{1/2}x\| \quad \text{for every } x \in D(A^{1/2}), \quad \left(K = \frac{1}{\sqrt{\sigma_0}}\right).$$

Yamada [11] showed the local solvability of (0.1)–(0.2), where $A = -\Delta$ with domain $D(A) = H^2(\mathbf{R}^n)$. He used the Friedrichs mollifier in order to obtain the estimate of $\|u(t)\|_{H^2} + \|\partial_t u(t)\|_{H^1}$. We invoke instead the spectral projection associated with the general non-negative self-adjoint operator A and apply the same method as in [11] to prove the following local existence result.

Theorem A ([11, Theorem I, Theorem 5.1]). *For every $T > 0$, there exists a positive constant $T_0(\leq T)$ such that the initial value problem (0.1)–(0.2) has a unique solution $u \in \bigcap_{i=0,1,2} C^i([0, T_0]; X_{1-i/2})$. The constant T_0 depends on $\|\phi\|_{X_1}$, $\|\psi\|_{X_{1/2}}$ and $\int_0^T \|f(s)\|_{X_{1/2}}^2 ds$.*

Let T be a positive number and let $u(t)$ be a local solution for (0.1)–(0.2) on $[0, T]$. Then we define two functions of $t \in [0, T]$:

$$F(t) = m(\|A^{1/2}u(t)\|^2)\|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2,$$

$$G(t) = F(t) + \gamma \operatorname{Re}(A^{1/2}u(t), A^{1/2}\partial_t u(t)) + \frac{\gamma^2}{2}\|A^{1/2}u(t)\|^2.$$

We define four functions on $[0, \infty)$:

$$m_0(r) = \min \left\{ \min_{0 \leq s \leq r} m(s^2), 1 \right\}, \quad m_1(r) = \left(\frac{3}{2} + \frac{\gamma^2 K^2}{m_0(r)} \right)^{-1},$$

$$m_2(r) = \max \left\{ \max_{0 \leq s \leq r} m(s^2), 1 \right\}, \quad m_3(r) = \max_{0 \leq s \leq r} |m'(s^2)|.$$

If $m_3(r) \not\equiv 0$, then $m_0(r)/m_3(r) \in (0, \infty]$ is a non-increasing function and r^2 is a strictly increasing function of r , respectively. Thus, there uniquely exists $\varepsilon_0 > 0$ such that

$$(1.2) \quad \frac{\gamma m_0(K\varepsilon_0)}{K m_3(K\varepsilon_0)} = 4\varepsilon_0^2.$$

We can now prove the following lemma.

Lemma 1. *Let ε_0 be the positive number defined by (1.2) in case $m_3(r) \not\equiv 0$, and an arbitrary fixed positive number in case $m_3(r) \equiv 0$. Let ε be a positive number such that $\varepsilon \leq \varepsilon_0$. Put*

$$(1.3) \quad \delta_1(\varepsilon) = \frac{\varepsilon}{2} \sqrt{\frac{m_0(K\varepsilon_0)m_1(K\varepsilon_0)}{2m_2(K\varepsilon_0)}}, \quad \delta_2(\varepsilon) = \frac{\varepsilon \gamma m_0(K\varepsilon_0)m_1(K\varepsilon_0)}{16(2 + K\gamma^2)}.$$

Assume (0.6) and

$$(1.4) \quad \sup_{0 \leq s \leq T} \|A^{1/2}f(s)\| \leq \delta_2(\varepsilon).$$

If

$$(1.5) \quad \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 \leq \varepsilon^2$$

for $0 \leq t \leq T$, then

$$(1.6) \quad \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 \leq \frac{\varepsilon^2}{2}$$

for $0 \leq t \leq T$.

Proof. We abbreviate $\mu_i = m_i(K\varepsilon_0)$, for $i = 0, 1, 2, 3$. By the assumption (1.5) together with (1.1), we have

$$(1.7) \quad \|A^{1/2}u(t)\| \leq K\|Au(t)\| \leq K\varepsilon \leq K\varepsilon_0$$

for $0 \leq t \leq T$. Thus,

$$(1.8) \quad \mu_0 \leq m(\|A^{1/2}u(t)\|^2) \leq \mu_2 \quad \text{and} \quad |m'(\|A^{1/2}u(t)\|^2)| \leq \mu_3$$

for $0 \leq t \leq T$. This implies

$$(1.9) \quad \frac{1}{\mu_2}F(t) \leq \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 \leq \frac{1}{\mu_0}F(t).$$

We prove that

$$(1.10) \quad \frac{1}{2}F(t) \leq G(t) \leq \frac{1}{\mu_1}F(t)$$

for $0 \leq t \leq T$. From the estimate

$$(1.11) \quad |(A^{1/2}u_t(t), A^{1/2}u(t))| \leq F(t)^{1/2}\|A^{1/2}u(t)\| \leq \frac{1}{2\gamma}F(t) + \frac{\gamma}{2}\|A^{1/2}u(t)\|^2,$$

it follows that

$$(1.12) \quad G(t) \geq \frac{1}{2}F(t).$$

On the other hand, by using (1.9) and (1.1), we have

$$(1.13) \quad \|A^{1/2}u(t)\|^2 \leq K^2\|Au(t)\|^2 \leq \frac{K^2}{\mu_0}F(t).$$

It follows from (1.11) and (1.13) that

$$(1.14) \quad G(t) \leq \frac{3}{2}F(t) + \gamma^2\|A^{1/2}u(t)\|^2 \leq \frac{1}{\mu_1}F(t).$$

Inequalities (1.12) and (1.14) imply (1.10).

For every positive number $N > \sigma_0 = \min \sigma(A)$, put

$$u_N(t) = E(N)u(t) \equiv \int_{\sigma_0}^N dE(\lambda)u(t), \quad \phi_N(t) = E(N)\phi(t),$$

$$\psi_N(t) = E(N)\psi(t) \quad \text{and} \quad f_N(t) = E(N)f(t),$$

where $A = \int_{\sigma_0}^{\infty} \lambda dE(\lambda)$ is the spectral resolution of A . Then

$$(1.15) \quad u_N \in C^2([0, T]; D(A^j)) \quad \text{and} \quad f_N \in C([0, T]; D(A^j)),$$

for every $j \geq 0$, and satisfies the following initial value problem;

$$(1.16) \quad \frac{\partial^2 u_N}{\partial t^2} + m(\|A^{1/2}u(t)\|^2)Au_N(t) + \gamma \frac{\partial u_N}{\partial t} = f_N(t) \quad \text{for } t \geq 0,$$

$$(1.17) \quad u_N(0) = \phi_N, \quad \frac{\partial u_N}{\partial t}(0) = \psi_N.$$

We also put

$$\begin{aligned} F_N(t) &= m(\|A^{1/2}u(t)\|^2)\|Au_N(t)\|^2 + \|A^{1/2}\partial_t u_N(t)\|^2, \\ G_N(t) &= F_N(t) + \gamma \operatorname{Re}(A^{1/2}u_N(t), A^{1/2}\partial_t u_N(t)) + \frac{\gamma^2}{2}\|A^{1/2}u_N(t)\|^2. \end{aligned}$$

Taking the inner product between (1.16) and $Au_N(t)$, we obtain

$$\begin{aligned} (1.18) \quad \frac{d}{dt} G_N(t) &= 2m'(\|A^{1/2}u(t)\|^2) \operatorname{Re}(A^{1/2}u(t), A^{1/2}\partial_t u(t))\|Au_N(t)\|^2 \\ &\quad + 2 \operatorname{Re}(A^{1/2}\partial_t u_N(t), A^{1/2}f_N(t)) \\ &\quad + \gamma^2 \operatorname{Re}(A^{1/2}u_N(t), A^{1/2}f_N(t)) - \gamma F_N(t). \end{aligned}$$

Integrating both sides of

$$\frac{d}{dt}(e^{\mu_1 \gamma t/2} G_N(t)) = e^{\mu_1 \gamma t/2} \left(\frac{\mu_1 \gamma}{2} G_N(t) + \frac{d}{dt} G_N(t) \right),$$

we have

$$e^{\mu_1 \gamma t/2} G_N(t) - G_N(0) = \int_0^t e^{\mu_1 \gamma s/2} \left(\frac{\mu_1 \gamma}{2} G_N(s) + \frac{d}{dt} G_N(s) \right) ds.$$

Inserting (1.18) in the right hand side of the above equality and letting $N \rightarrow \infty$, we obtain

$$\begin{aligned} (1.19) \quad e^{\mu_1 \gamma t/2} G(t) - G(0) &= \int_0^t e^{\mu_1 \gamma s/2} \left\{ \frac{\mu_1 \gamma}{2} G(s) \right. \\ &\quad + 2m'(\|A^{1/2}u(s)\|^2) \operatorname{Re}(A^{1/2}u(s), A^{1/2}\partial_t u(s))\|Au(s)\|^2 \\ &\quad + 2 \operatorname{Re}(A^{1/2}\partial_t u(s), A^{1/2}f(s)) \\ &\quad \left. + \gamma^2 \operatorname{Re}(A^{1/2}u(s), A^{1/2}f(s)) - \gamma F(s) \right\} ds. \end{aligned}$$

By (1.5), (1.7), (1.8), (1.9) and (1.2), we have

$$\begin{aligned} m'(\|A^{1/2}u(s)\|^2) \operatorname{Re}(A^{1/2}u(s), A^{1/2}\partial_t u(s)) \|Au(s)\|^2 \\ \leq \frac{\mu_3 K}{\mu_0} \varepsilon^2 F(s) \leq \frac{\gamma}{4} F(s). \end{aligned}$$

By using this inequality and (1.14), it follows from (1.19) that

$$\begin{aligned} e^{\mu_1 \gamma t/2} G(t) - G(0) \\ \leq \int_0^t e^{\mu_1 \gamma s/2} (2 \operatorname{Re}(A^{1/2}\partial_t u(s), A^{1/2}f(s)) + \gamma^2 \operatorname{Re}(A^{1/2}u(s), A^{1/2}f(s))) ds. \end{aligned}$$

This inequality together with (1.5) and (1.7) yields

$$e^{\mu_1 \gamma t/2} G(t) - G(0) \leq (2 + K\gamma^2)\varepsilon \sup_{0 \leq s \leq T} \|A^{1/2}f(s)\| \frac{2}{\mu_1 \gamma} (e^{\mu_1 \gamma t/2} - 1).$$

Thus we obtain

$$(1.20) \quad G(t) \leq G(0) + \frac{2(2 + K\gamma^2)\varepsilon}{\mu_1 \gamma} \sup_{0 \leq s \leq T} \|A^{1/2}f(s)\|$$

for every $0 \leq t \leq T$. From (1.20) and (1.10), it follows that

$$F(t) \leq \frac{2}{\mu_1} F(0) + \frac{4(2 + K\gamma^2)\varepsilon}{\mu_1 \gamma} \sup_{0 \leq s \leq T} \|A^{1/2}f(s)\|$$

for every $0 \leq t \leq T$. This, together with (1.9), yields

$$\begin{aligned} \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 &\leq \frac{1}{\mu_0} F(t) \\ &\leq \frac{2\mu_2}{\mu_0 \mu_1} (\|A\phi\|^2 + \|A^{1/2}\psi\|^2) + \frac{4(2 + K\gamma^2)\varepsilon}{\mu_0 \mu_1 \gamma} \sup_{0 \leq s \leq T} \|A^{1/2}f(s)\| \end{aligned}$$

for every $0 \leq t \leq T$. Hence, by using the assumptions (0.6), (1.3) and (1.4), we obtain inequality (1.6). $\square$

Proof of Theorem 1. Let ε_0 as in Lemma 1. It suffices to prove the conclusion for ε such that $0 < \varepsilon \leq \varepsilon_0$. Let δ_1 and δ_2 be as in Lemma 1, and assume that (0.6) and (0.7) hold. By Theorem A, there exists a unique solution u of (0.1)–(0.2) on $[0, T_0]$ for some positive T_0 .

We show that u satisfies the estimate (1.5) in Lemma 1 for $0 \leq t \leq T_0$. Assume that (1.5) does not hold for some $t \leq T_0$. Put

$$(1.21) \quad T^* = \inf_t \{0 \leq t \leq T_0 \mid \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 > \varepsilon^2\}.$$

Then, by the assumption (0.6) with $\delta_1 \leq \varepsilon$ and the continuity of the solution, it follows that $T^* > 0$ and (1.5) holds for $0 \leq t \leq T^*$. Thus, by Lemma 1 with $T = T^*$, we obtain

$$(1.22) \quad \|Au(t)\|^2 + \|A^{1/2}\partial_t u(t)\|^2 \leq \frac{\varepsilon^2}{2}$$

for $0 \leq t \leq T^*$. On the other hand, by the continuity, it follows from (1.21) that $\|Au(T^*)\|^2 + \|A^{1/2}\partial_t u(T^*)\|^2 = \varepsilon^2$, which contradicts (1.22). Thus we have an a-priori estimate (1.5). Hence, we obtain a global solution satisfying (0.8), by the usual way with the aid of Theorem A.

Construction of examples for Remark 2.

Case 1 (0 is an isolated point of $\sigma(A)$.) In this case 0 is in the point spectrum of A . Let ϕ_0 be an eigenvector of A corresponding to the first eigenvalue 0 such that $\|\phi_0\| = 1$. Let $g_1 \in C^\infty([0, \infty))$ be a function such that

$$g_1(t) \begin{cases} = 1 & \text{for } t \geq 2, \\ = 0 & \text{for } 0 \leq t \leq 1 \\ \in [0, 1] & \text{otherwise,} \end{cases}$$

and that $|g_1'(t)| \leq 2$ and $|g_1''(t)| \leq 3$. Put $u(t) = (\varepsilon/15)(t+1)^{1-\theta}g_1(t)\phi_0$ and $f(t) = u'(t) + u''(t)$. Then $u(t) \in C^\infty([0, \infty); D(A))$ is the unique solution of (0.9), and it is easy to see that $\|u(t)\| = \varepsilon(t+1)^{1-\theta}/15$ for $t \geq 2$, $\sup_{t \geq 0} \|(I + A^{1/2})f(t)\| = \sup_{t \geq 0} \|f(t)\| < \varepsilon$ and $\|f(t)\| = O(t^{-\theta})$ as $t \rightarrow \infty$.

Case 2 (0 is an accumulation point of $\sigma(A)$.) We construct a function $u(t)$ such that $\lim_{t \rightarrow \infty} \|A^\alpha u(t)\| = \infty$ for every $\alpha \in [0, 1 - \theta)$ and that $u(t)$ is the unique solution of (0.9) with $\|f(t)\| = O(t^{-\theta})$. Take $\{\lambda_j \in \sigma(A) | j = 0, 1, \dots\}$ as to satisfy the following;

$$\lambda_0 \leq 1, \quad \lambda_{j+1} < \frac{1}{4}\lambda_j, \quad E(\lambda_j) - E\left(\frac{\lambda_j}{2}\right) \neq \{0\}.$$

where E is the spectral family associated with the self-adjoint operator A . Take ϕ_j such that

$$\phi_j \in E(\lambda_j) - E\left(\frac{\lambda_j}{2}\right) \quad \text{and} \quad \|\phi_j\| = 1.$$

Let $g_2 \in C^\infty([0, \infty))$ be a function such that

$$g_2(t) \begin{cases} = 1 & \text{for } 2 \leq t \leq 3, \\ = 0 & \text{for } t \leq 1, \text{ or } t \geq 4, \\ \in [0, 1] & \text{otherwise,} \end{cases}$$

and that $|g_2'(t)| \leq 2$ and $|g_2''(t)| \leq 3$. Put

$$u(t) = \frac{\varepsilon}{12} \sum_{j=1}^{\infty} \lambda_j^{\theta-1} g_2(\lambda_j t) \phi_j,$$

and $f(t) = \partial_t^2 u(t) + Au(t) + \partial_t u(t)$. We note that $u(t)$ is expressed as

$$(1.23) \quad u(t) = \frac{\varepsilon}{12} \lambda_j^{\theta-1} g_2(\lambda_j t) \phi_j \quad \text{if } \frac{1}{\lambda_j} \leq t \leq \frac{4}{\lambda_j},$$

$$u(t) = 0 \quad \text{otherwise.}$$

Here we remark that for each $t \geq 0$, there exists at most one j satisfying (1.23). Hence, we have $u \in C^\infty([0, \infty); D(A))$, and u is the unique solution of (0.9). Moreover, we easily see that $\|A^{\beta+1}u(t)\| \leq \varepsilon/12$, $\|A^\beta \partial_t u(t)\| \leq \varepsilon/6$, $\|A^\beta \partial_t^2 u(t)\| \leq \varepsilon/4$, for every $\beta \geq 0$ and $\|Au(t)\| = O(t^{-\theta})$, $\|\partial_t u(t)\| = O(t^{-\theta})$ and $\|\partial_t^2 u(t)\| = O(t^{-\theta-1})$. Hence, $f \in C^\infty([0, \infty); D(A))$, $\|(I + A^{1/2})f(t)\| \leq \varepsilon$ and $\|f(t)\| = O(t^{-\theta})$ as $t \rightarrow \infty$.

On the other hand, for every j , we have

$$\left\| A^\alpha u \left(\frac{2}{\lambda_j} \right) \right\| = \frac{\varepsilon}{12} \|\lambda_j^{\theta-1} A^\alpha \phi_j\| \geq \frac{\varepsilon}{12} 2^{-\alpha} \lambda_j^{\alpha+\theta-1}.$$

Since $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$, the above inequality yields that $\limsup_{t \rightarrow \infty} \|A^\alpha u(t)\| \geq \lim_{j \rightarrow \infty} \|A^\alpha u(2/\lambda_j)\| = \infty$, for every $\alpha \in [0, 1 - \theta)$.

2. Bounded solution on R

In this section, we prove Theorem 2.

Following the method in the proof of (3.14) in Nakao [7], we prove the following lemma, where $\|w(t)\|_{H_0^1(\Omega)}^2 + \|\partial_t w(t)\|_{L^2(\Omega)}^2$ in [7] is replaced by H_u defined by (2.2) below.

Lemma 2. *Let ε_0 be as in Lemma 1. There exists a positive number $\varepsilon_1 (\leq \varepsilon_0)$ such that the following holds for $\varepsilon \in (0, \varepsilon_1)$. Let T_0 be an arbitrary fixed real number. Let u be the solution of*

$$(2.1) \quad \frac{\partial^2 u(t)}{\partial t^2} + m(\|A^{1/2}u(t)\|^2)Au(t) + \gamma \frac{\partial u(t)}{\partial t} = f(t), \quad t \geq T_0.$$

satisfying (1.5) for $t \geq T_0$, and let v be a solution of (2.1) with $f(t)$ replaced by $g(t)$ and satisfying (1.5) for $t \geq T_0$.

Put $w = u - v$, $h = f - g$ and

$$(2.2) \quad H_u(t) = m(\|A^{1/2}u(t)\|^2)\|A^{1/2}w(t)\|^2 + \|\partial_t w(t)\|^2.$$

Then the following estimate holds for $t \geq T$;

$$(2.3) \quad C_1 \max_{t \leq s \leq t+1} H_u(s) \leq H_u(t) - H_u(t+1) + C_2 \int_t^{t+1} \|h(s)\|^2 ds,$$

where C_1 and C_2 are positive constants which will be defined by (2.17) in the proof.

Remark 3. We can put $\varepsilon_1 = \min\{\varepsilon_0, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, where ε_2 , ε_3 and ε_4 are defined by (2.11), (2.14) and (2.16), respectively.

Proof. First, we assume that $\varepsilon \leq \varepsilon_0$. Since the assumption (1.3) and (1.4) are used only for the proof of (1.6) at the end of the proof of Lemma 1, estimates except (1.6) in the proof of Lemma 1 also hold for u and v . We also abbreviate $\mu_i = m_i(K\varepsilon_0)$, for $i = 0, 1, 2, 3$, as in the proof of Lemma 1.

We have

$$(2.4) \quad \begin{aligned} & \frac{\partial^2 w(t)}{\partial t^2} + m(\|A^{1/2}u(t)\|^2)Aw(t) + \gamma \frac{\partial w(t)}{\partial t} \\ &= (m(\|A^{1/2}u(t)\|^2) - m(\|A^{1/2}v(t)\|^2))Av(t) + h(t), \quad t \geq T_0. \end{aligned}$$

Taking the inner product between (2.4) and $\partial w(t)/\partial t$, we have

$$(2.5) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} H_u(t) + \gamma \|\partial_t w(t)\|^2 \\ &= m'(\|A^{1/2}u(t)\|^2) \operatorname{Re}(A^{1/2}u, A^{1/2}\partial_t u(t)) \|A^{1/2}w(t)\|^2 \\ & \quad + (m(\|A^{1/2}u(t)\|^2) - m(\|A^{1/2}v(t)\|^2)) \operatorname{Re}(Av(t), \partial_t w(t)) \\ & \quad + \operatorname{Re}(h(t), \partial_t w(t)). \end{aligned}$$

Since u and v satisfy (1.5) and (1.7), we have

$$(2.6) \quad \begin{aligned} & |m(\|A^{1/2}u(t)\|^2) - m(\|A^{1/2}v(t)\|^2)| \\ &= |m'(\theta\|A^{1/2}u(t)\|^2 + (1-\theta)\|A^{1/2}v(t)\|^2)| \|\|A^{1/2}u(t)\|^2 - \|A^{1/2}v(t)\|^2\| \\ &\leq \mu_3(\|A^{1/2}u(t)\| + \|A^{1/2}v(t)\|) \|A^{1/2}w(t)\| \leq 2K\mu_3\varepsilon \|A^{1/2}w(t)\|, \end{aligned}$$

for $t \geq T_0$. By using (1.5), (1.7), (1.8), (2.6) and the Schwarz's inequality, we have from (2.5) that

$$(2.7) \quad \begin{aligned} & \left| \frac{1}{2} \frac{d}{dt} H_u(t) + \gamma \|\partial_t w(t)\|^2 \right| \\ &\leq K\mu_3\varepsilon^2 \|A^{1/2}w(t)\|^2 + 2K\mu_3\varepsilon^2 \|A^{1/2}w(t)\| \|\partial_t w(t)\| \end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma}{4} \|\partial_t w(t)\|^2 + \frac{2}{\gamma} \|h(t)\|^2. \\
& \leq \frac{C_3 \varepsilon^2}{2} \|A^{1/2} w(t)\|^2 + \frac{\gamma}{2} \|\partial_t w(t)\|^2 + \frac{2}{\gamma} \|h(t)\|^2,
\end{aligned}$$

where

$$C_3 = 2 \left(K\mu_3 + \frac{4K^2\mu_3^2\varepsilon^2}{\gamma} \right).$$

Thus we have

$$\frac{d}{dt} H_u(t) + \gamma \|\partial_t w(t)\|^2 \leq C_3 \varepsilon^2 \|A^{1/2} w(t)\|^2 + \frac{4}{\gamma} \|h(t)\|^2,$$

for $t \geq T_0$. Integrating the above inequality over $[t, t+1]$, we obtain

$$\begin{aligned}
& \gamma \int_t^{t+1} \|\partial_t w(s)\|^2 ds \\
& \leq H_u(t) - H_u(t+1) + C_3 \varepsilon^2 \int_t^{t+1} \|A^{1/2} w(s)\|^2 ds + \frac{4}{\gamma} \int_t^{t+1} \|h(s)\|^2 ds \\
& \equiv \gamma B(t)^2,
\end{aligned}$$

and therefore,

$$(2.8) \quad \int_t^{t+1} \|\partial_t w(s)\|^2 ds \leq B(t)^2.$$

Thus, there exist $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$(2.9) \quad \|\partial_t w(t_i)\| \leq 2B(t) \quad (i = 1, 2).$$

Taking the inner product between $w(t)$ and (2.4), we have

$$\begin{aligned}
(2.10) \quad & \frac{d}{dt} \operatorname{Re}(w(t), \partial_t w(t)) - \|\partial_t w(t)\|^2 + m(\|A^{1/2} u(t)\|^2) \|A^{1/2} w(t)\|^2 \\
& + \gamma \operatorname{Re}(w(t), \partial_t w(t)) \\
& = (m(\|A^{1/2} u(t)\|^2) - m(\|A^{1/2} v(t)\|^2)) \operatorname{Re}(A^{1/2} w(t), A^{1/2} v(t)) \\
& + \operatorname{Re}(w(t), h(t)).
\end{aligned}$$

Assume moreover that

$$(2.11) \quad \varepsilon < \frac{\sqrt{\mu_0}}{2K\sqrt{\mu_3}} (\equiv \varepsilon_2),$$

and put

$$C_4 = \left(\frac{1}{2} - \frac{2K^2\mu_3\varepsilon^2}{\mu_0} \right)^{-1}, \quad C_5 = C_4 \left(1 + \frac{2K^2\gamma^2}{\mu_0} \right), \quad \text{and} \quad C_6 = \frac{C_4K^2}{\mu_0}.$$

Then by using (1.1), (2.6), (1.7) for u and v , (1.8) and Schwarz's inequality, we have from (2.10) that

$$\begin{aligned} m(\|A^{1/2}u(t)\|^2)\|A^{1/2}w(t)\|^2 \\ \leq -C_4 \frac{d}{dt} \operatorname{Re}(w(t), \partial_t w(t)) + C_5 \|\partial_t w(t)\|^2 + C_6 \|h(t)\|^2. \end{aligned}$$

Integrating the above inequality over $[t_1, t_2]$, and using (2.8) and (2.9), we obtain

$$\begin{aligned} (2.12) \quad & \int_{t_1}^{t_2} m(\|A^{1/2}u(s)\|^2)\|A^{1/2}w(s)\|^2 ds \\ & \leq 4C_4 B(t) \max_i \|w(t_i)\| + C_5 B(t)^2 + C_6 \int_{t_1}^{t_2} \|h(s)\|^2 ds. \end{aligned}$$

Estimates (2.8) and (2.12) together yield

$$\int_{t_1}^{t_2} H_u(s) ds \leq 4C_4 B(t) \max_i \|w(t_i)\| + (C_5 + 1)B(t)^2 + C_6 \int_{t_1}^{t_2} \|h(s)\|^2 ds.$$

Thus, there exists a $t^* \in [t_1, t_2]$ such that

$$(2.13) \quad H_u(t^*) \leq 8C_4 B(t) \max_i \|w(t_i)\| + 2(C_5 + 1)B(t)^2 + 2C_6 \int_{t_1}^{t_2} \|h(s)\|^2 ds.$$

From (2.7), it follows that

$$\left| \frac{d}{dt} H_u(t) \right| \leq 3\gamma \|\partial_t w(t)\|^2 + \frac{C_3\varepsilon^2}{\mu_0} H_u(t) + \frac{4}{\gamma} \|h(t)\|^2.$$

Hence, by using (2.8), (2.13), and that $\|w(t)\| \leq (K/\sqrt{\mu_0})H_u(t)^{1/2}$, we have

$$\begin{aligned} & \max_{t \leq s \leq t+1} H_u(s) \\ & \leq H_u(t^*) + 3\gamma B(t)^2 + \frac{C_3\varepsilon^2}{\mu_0} \max_{t \leq s \leq t+1} H_u(s) + \frac{4}{\gamma} \int_t^{t+1} \|h(s)\|^2 ds \\ & \leq \left(\frac{32K^2C_4^2}{\mu_0} + 2(C_5 + 1) + 3\gamma \right) B(t)^2 \\ & \quad + \left(\frac{C_3\varepsilon^2}{\mu_0} + \frac{1}{2} \right) \max_{t \leq s \leq t+1} H_u(s) + \left(2C_6 + \frac{4}{\gamma} \right) \int_t^{t+1} \|h(s)\|^2 ds. \end{aligned}$$

Thus, assuming furthermore that

$$(2.14) \quad \varepsilon < \sqrt{\frac{\mu_0}{2C_3}} (\equiv \varepsilon_3),$$

we obtain

$$\max_{t \leq s \leq t+1} H_u(s) \leq C_7 B(t)^2 + C_8 \int_t^{t+1} \|h(s)\|^2 ds,$$

where

$$C_7 = \left(\frac{1}{2} - \frac{C_3 \varepsilon^2}{\mu_0} \right)^{-1} \left(\frac{32K^2 C_4^2}{\mu_0} + 2(C_5 + 1) + 3\gamma \right),$$

$$C_8 = \left(\frac{1}{2} - \frac{C_3 \varepsilon^2}{\mu_0} \right)^{-1} \left(2C_6 + \frac{4}{\gamma} \right).$$

This implies that

$$(2.15) \quad \max_{t \leq s \leq t+1} H_u(s) \leq \frac{C_7}{\gamma} (H_u(t) - H_u(t+1)) + \frac{C_3 C_7 \varepsilon^2}{\gamma \mu_0} \max_{t \leq s \leq t+1} H_u(s) \\ + \left(\frac{4C_7}{\gamma^2} + C_8 \right) \int_t^{t+1} \|h(s)\|^2 ds.$$

Assuming moreover that

$$(2.16) \quad \varepsilon < \sqrt{\frac{\gamma \mu_0}{C_3 C_7}} (\equiv \varepsilon_4),$$

we obtain (2.3) from (2.15) by taking

$$(2.17) \quad C_1 = \frac{\gamma}{C_7} - \frac{C_3 \varepsilon^2}{\mu_0} \quad \text{and} \quad C_2 = \frac{4}{\gamma} + \frac{\gamma C_8}{C_7}.$$

This completes the proof. $\square$

Proof of Theorem 2. Let ε_0 and ε_1 be as in Lemma 2.

[Existence.] Let $\varepsilon \leq \varepsilon_1$, and assume that (0.11) holds. Consider the following equation;

$$\begin{cases} \frac{\partial^2 u_n(t)}{\partial t^2} + m(\|A^{1/2} u_n(t)\|^2) A u_n(t) + \gamma \frac{\partial u_n(t)}{\partial t} = f(t) & \text{for } t \geq -n, \\ u_n(-n) = 0, & \frac{\partial u_n}{\partial t}(-n) = 0. \end{cases}$$

By Theorem 1 with initial time $-n$ instead of 0, there exists a solution u_n satisfying estimate (1.5) for $t \geq -n$. Nakao [7] obtained a bounded solution as the limit of the sequences u_n , by using the compactness of the operator A^{-1} . Since A^{-1} is not necessarily compact under our assumption and since the definition of the solution is different from [7], we prove that the sequence converges to a solution by another method. Fix an arbitrary number t . Put

$$H_{n,\ell}(t) = m(\|A^{1/2}u_n(t)\|^2)\|A^{1/2}(u_n - u_\ell)(t)\|^2 + \|\partial_t(u_n - u_\ell)(t)\|^2,$$

for $\ell \geq n \geq \max\{-t, 0\}$. Since the solutions u_n and u_ℓ satisfy estimate (1.5) for $t \geq -n$, we obtain by Lemma 2 with $h \equiv 0$ that

$$(1 + C_1)^N H_{n,\ell}(t) \leq H_{n,\ell}(t - N) \leq m_1(K\varepsilon)\varepsilon^2,$$

for every non-negative integer N such that $N \leq t + n$. Hence

$$\|A^{1/2}(u_n - u_\ell)(t)\|^2 + \|\partial_t(u_n - u_\ell)(t)\|^2 \leq \frac{m_1(K\varepsilon)\varepsilon^2}{m_0(K\varepsilon)}(1 + C_1)^{-N}$$

for every non-negative integer N such that $N \leq t + n$. Taking N such that $t - 1 + n \leq N \leq t + n$, we obtain

$$\|A^{1/2}(u_n - u_\ell)(t)\|^2 + \|\partial_t(u_n - u_\ell)(t)\|^2 \leq \frac{m_1(K\varepsilon)\varepsilon^2}{m_0(K\varepsilon)}(1 + C_1)^{-n+1-t},$$

for every $\ell \geq n \geq \max\{1 - t, 0\}$. This implies that $\{(u_n(t), \partial_t u_n(t)) | n \geq \max\{1 - t, 0\}\}$ is a Cauchy sequence in $X_{1/2} \times H$. Since this fact holds for every fixed t , there exists a pair of functions $(u_\infty(t), v_\infty(t))$ such that

$$\lim_{n \rightarrow \infty} u_n(t) = u_\infty(t) \text{ in } X_{1/2}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \partial_t u_n(t) = v_\infty(t) \text{ in } H,$$

for each $t \in \mathbf{R}$. From this fact and the boundedness of $(u_n(t), \partial_t u_n(t))$ in $X_1 \times X_{1/2}$, it follows that $(u_n(t), \partial_t u_n(t)) \rightarrow (u_\infty(t), v_\infty(t))$ weakly in $X_1 \times X_{1/2}$, for each $t \in \mathbf{R}$. Thus, $(u_\infty(t), v_\infty(t)) \in X_1 \times X_{1/2}$ for every $t \in \mathbf{R}$. Therefore, for every $t_0 \in \mathbf{R}$, the local solvability implies the existence of a solution $\tilde{u}$ of (0.1) on $[t_0, t_0 + \tilde{\varepsilon}]$ ($\tilde{\varepsilon} > 0$) such that $\tilde{u}(t_0) = u_\infty(t_0)$, $\partial_t \tilde{u}(t_0) = v_\infty(t_0)$. From (2.5) with $u = \tilde{u}$, $v = u_n$ and $h \equiv 0$, we have

$$\frac{d}{dt} \tilde{H}_n(t) \leq C \tilde{H}_n(t),$$

where C is a constant and

$$\tilde{H}_n(t) = m(\|A^{1/2}\tilde{u}(t)\|^2)\|A^{1/2}(\tilde{u} - u_n)(t)\|^2 + \|\partial_t(\tilde{u} - u_n)(t)\|^2.$$

Therefore, we have

$$(2.18) \quad \tilde{H}_n(t) \leq e^{C(t-t_0)} \tilde{H}_n(t_0)$$

for $t_0 \leq t \leq t_0 + \tilde{\varepsilon}$. Since $\lim_{n \rightarrow \infty} \tilde{H}_n(t_0) = 0$, the inequality (2.18) implies that $\lim_{n \rightarrow \infty} \tilde{H}_n(t) = 0$ for every $t \in [t_0, t_0 + \tilde{\varepsilon})$, and therefore

$$u_\infty(t) = \tilde{u}(t) \quad \text{and} \quad v_\infty(t) = \partial_t \tilde{u}(t) \quad \text{for } t_0 \leq t \leq t_0 + \tilde{\varepsilon}.$$

Since t_0 is arbitrary, this fact implies that u_∞ is a solution of (0.10). From the fact that u_n satisfies the estimate (1.5) for $t \geq -n$, it follows that the limit u_∞ satisfies (0.12).

[Uniqueness and Continuity.] Let $f, g \in C(\mathbf{R}; X_{1/2})$ satisfy (0.1) for some $\varepsilon < \varepsilon_1$, and let u and v be solutions of (0.10) for f and g satisfying (0.13) respectively. Then, Lemma 2 yields

$$(1 + C_1)H_u(t) \leq H_u(t-1) + C_2 \sup_{t \in \mathbf{R}} \|f(t) - g(t)\|,$$

for every $t \in \mathbf{R}$. Repeating this estimate, we obtain

$$\begin{aligned} H_u(t) &\leq (1 + C_1)^{-N} H_u(t-N) + C_2 \left(\sum_{j=1}^N (1 + C_1)^{-j} \right) \sup_{t \in \mathbf{R}} \|f(t) - g(t)\| \\ &= (1 + C_1)^{-N} H_u(t-N) + \frac{C_2}{C_1} (1 - (1 + C_1)^{-N}) \sup_{t \in \mathbf{R}} \|f(t) - g(t)\| \end{aligned}$$

for every positive integer N . Since u and v satisfy (0.13), we have $\sup_{N \geq 0} H_u(t-N) < \infty$. Thus, the above inequality yields

$$H_u(t) \leq \frac{C_2}{C_1} \sup_{t \in \mathbf{R}} \|f(t) - g(t)\|,$$

and therefore

$$\|A^{1/2}(u-v)(t)\|^2 + \|\partial_t(u-v)\|^2 \leq \frac{C_2}{C_1 m_0(K\varepsilon)} \sup_{t \in \mathbf{R}} \|f(t) - g(t)\|.$$

This implies the uniqueness if we take $f = g$, and the uniform continuity of the mapping L . $\square$

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