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Local Solvability and Smoothing Effects of Nonlinear Schrödinger Equations with Magnetic Fields

By

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1. Introduction

In this paper we consider the following nonlinear Schrödinger equation with a potential in a magnetic field

$$i\partial_t u = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u + V(t,x)u + F(u) \tag{1}$$

$$= H(t)u + F(u), \qquad (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n,$$

where $u = u(t,x)$ is a complex-valued function, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $A(t,x) = (A_1(t,x), A_2(t,x), \ldots, A_n(t,x))$ is a vector potential, $V(t,x)$ is a scalar potential and $F(u) = F \circ u$ is a local nonlinear operator given by a complex-valued function F and $\boldsymbol{C}$.

For the local Cauchy problem of equation (1), we construct weak solutions u and show the local smoothing effects for H^1-solutions. In Yajima [16, 17], he studied the corresponding linear equation

$$i\partial_t u = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u + V(t,x)u \tag{2}$$

$$= H(t)u, \qquad (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n,$$

and he proved that equation (2) generates a unique unitary propagator in $L^2(\boldsymbol{R}^n)$ under some assumptions for A and V. In this paper, under suitable conditions on the nonlinear term F, we construct local weak solutions of equation (1), using Yajima's results and the constructive method for solutions given in Kato [4, 5], Yajima [15, 17], etc.

We denote $\partial_x = (\partial_1, \ldots, \partial_n)$, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial_x^\alpha = (\partial_1^{\alpha_n}, \ldots, \partial_n^{\alpha_n})$ and we often denote ∂_x by ∂, and use ∂ as ∂^α if $|\alpha| = 1$.

We need the following assumptions;

Assumption (A) For $j = 1, \ldots, n$, $A_j(t,x)$ is a real-valued function of $(t,x) \in \boldsymbol{R}^1 \times \boldsymbol{R}^n$ such that $\partial_x^\alpha A_j(t,x)$ is C^1 function of t for each x and for any multi-index α. For $|\alpha| \geq 1$, we have, with some $\varepsilon > 0$

$$|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha(1+|x|)^{-1-\varepsilon}, \qquad j,k=1,\dots,n, \tag{3}$$

$$|\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \le C_\alpha, \qquad (t,x)\in \boldsymbol{R}^1\times\boldsymbol{R}^n, \tag{4}$$

where $B_{jk}(t,x)=\partial_j A_k(t,x)-\partial_k A_j(t,x)$.

Assumption (V) $V(t,x)$ is a real-valued function of $(t,x)\in\boldsymbol{R}^1\times\boldsymbol{R}^n$ such that $\partial_x^\alpha V(t,x)$ is a continuous function of t for each x and for every α. For $|\alpha|\ge 2$ we have

$$|\partial_x^\alpha V(t,x)| \le C_\alpha, \qquad (t,x)\in\boldsymbol{R}^1\times\boldsymbol{R}^n. \tag{5}$$

Assumption (F1) $F\in C^1(\boldsymbol{C};\boldsymbol{C})$, with $F(0)=0$.

Assumption (F2) For $\zeta=\xi+i\eta$, $|DF(\zeta)|\equiv\max\{|\partial_\zeta F|,|\partial_{\bar\zeta}F|\}\le M|\zeta|^{p-1}$, for $|\zeta|\ge 1$, $1\le p<\infty$, where $\partial_\zeta=\frac{1}{2}(\partial_\xi-i\partial_\eta)$, $\partial_{\bar\zeta}=\frac{1}{2}(\partial_\xi+i\partial_\eta)$.

Under these assumption, we can construct a weak solution for an initial data which is a L^2-function. For any interval $I=[0,T]\subset\boldsymbol{R}$ and any Banach space X, we denote by $L^q(I;X)$ the space of measurable functions u from I to X such that $\|u(\cdot)\|_X\in L^q$ and by $C(I;X)$ the space of strong continuous functions from I to X. We define the following function space

$$\mathscr{X}(I)\equiv\left(\bigcap L^s(I;L^q)\right)\cap C(I;L^2), \tag{6}$$

where $\bigcap$ is the intersection in (q,s) satisfying $1/q+2/ns=1/2$ and $1/2-1/n<1/q\le 1/2$.

Theorem 1. *Assume* (A), (V), (F1) *and* (F2) *with* $1<p<1+4/n$. *Then for each* $\phi\in L^2$, *there exists a unique solution* $u\in\mathscr{X}(I)$ *of equation* (1) *with* $u(0)=\phi$ *where* $T>0$ *depends only on* $\|\phi\|_2$.

Furthermore, the local solution $u\in\mathscr{X}(I)$ *depends on* $\phi\in L^2$ *continuously. More precisely, let* $u\in\mathscr{X}(I_0)$ *be a solution of the equation* (1) *with* $u(0)=\phi$ *on an interval* $I_0=[0,T_0]$. *If* $\phi_n\to\phi$ *in* L^2, *then the solution* u_n *of the equation* (1) *with* $u_n(0)=\phi_n$ *exists on* I_0 *for sufficiently large* n, *and* $u_n\to u$ *in* $\mathscr{X}(I_0)$.

Next we construct H^1-solution and consider the local smoothing effects. We set $\Sigma(k)=\{f\in L^2\mid\sum_{|\alpha+\beta|\le k}\|x^\alpha\partial_x^\beta f\|_2^2=\|f\|_{\Sigma(k)}^2<\infty\}$, $k=0,1,\dots,$ and let $\Sigma(-k)$ be the dual space of $\Sigma(k)$, where $x^\alpha u=x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}u$ for a multi-index $\alpha=(\alpha_1,\alpha_2\dots,\alpha_n)$. If $|\alpha|=1$, we write $x^\alpha u=xu$. We define

$$\mathscr{Y}(I)\equiv\{u\in\mathscr{X}(I)\mid\partial u\in\mathscr{X}(I),xu\in\mathscr{X}(I)\}\subset C(I;\Sigma(1)). \tag{7}$$

Then we have the following theorems;

Theorem 2. *Assume* (A), (V), (F1) *and, if* $n \geq 3$, (F2) *with* $1 < p < 1 + 4/(n-2)$. *Then for each* $\phi \in \Sigma(1)$, *there exists a unique solution* $u \in C(I; \Sigma(1))$ *of equation* (1) *with* $u(0) = \phi$ *where* $T > 0$ *depends only on* $\|\phi\|_{\Sigma(1)}$. *Moreover* $u \in \overline{\mathscr{Y}}(I)$.

Furthermore, $u \in C(I; \Sigma(1))$ *depends on* $\phi \in \Sigma(1)$ *continuously. More precisely, let* $u \in C(I_0; \Sigma(1))$ *be a solution of equation* (1) *with* $u(0) = \phi$. *If* $\phi_n \to \phi$ *in* $\Sigma(1)$, *then the solution* u_n *of equation* (1) *with* $u_n(0) = \phi_n$ *exists on* I_0 *for sufficiently large* n, *and* $u_n \to u$ *in* $C(I_0; \Sigma(1))$.

Theorem 3. *Let* u *be a solution to equation* (1) *in Theorem* 2. *Suppose* $\mu > 1/2$. *Then, in the case* $1 \leq n \leq 6$, *the following estimate holds;*

$$\int_I \|\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}u\|_2^2 dt < \infty, \tag{8}$$

where $\|\cdot\|_2$ *is the* L^2*-norm,* $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle D\rangle = (I-\Delta)^{1/2}$. *In the case* $n \geq 7$, *if* $p < 1 + 2/(n-4)$, *then the above inequality holds.*

Many authors studied nonlinear Schrödinger equations with power nonlinearity. In the case of $A = V = 0$, under some inequality conditions with the constants p and n, Ginibre and Velo [3], Tsutsumi [13, 14] constructs the local and global solution. In particular, Kato [4, 5] proved the local well-posedness in L^2, H^1 and H^2 and he also proved the global well-posedness in L^2, H^1 and H^2 with additional assumptions on F. In the case of $A = 0$ and $V \neq 0$, Kato [5], Oh [8] proved the existence of local and global solutions. When $A \neq 0$, $V \neq 0$, which are independent of t and $F(u) = \lambda|u|^{p-1}u$, Bouard [2] constructed the local and global solutions.

There are various results on the estimate as in Theorem 3 of smoothing effects. In Sjölin [10], he obtained the following inequality for the free Schrödinger equation; For some $C > 0$, depending on $\phi \in C_0^\infty(\boldsymbol{R}^{n+1})$,

$$\int_{\boldsymbol{R}}\int_{\boldsymbol{R}^n} |\phi(t,x)(1-\Delta)^{1/4}e^{it\Delta}f|^2 dxdt \leq C\|f\|_{L^2}^2, \qquad \forall f \in L^2. \tag{9}$$

This inequality manifests that the free Schrödinger propagator $e^{it\Delta}$ has the smoothing effect which can improve the differentiability locally in time and space.

Later the similar properties for e^{-itH}, where $H = -\Delta + V$ was a self-adjoint operator with various asumptions on scalar potentials $V = V(x)$ or $V(t,x)$, were studied by many authors (Ben-Artzi [1], Ruiz and Vega [9], etc.). In particular, Kato and Yajima [6] obtained the inequality replacing ϕ in (9) by $(1+|x|^2)^{-(1/4)-\varepsilon}$, $\varepsilon > 0$, and Yajima [16] obtained Proposition 5.1 (see later in Section 5) where he proved the smoothing effect for equation (2).

This type of smoothing properties for nonlinear Schrödinger equations in the case of $A = V = 0$ were studied in Nakamura [7], Sjölin [11, 12].

Our plan in this paper is as follows. In Section 2, we prepare the results in Yajima [17], the existence of the propagator of equation (2) and its action on some function spaces, to construct a local solution. In Section 3, we prove Theorem 1, the well-posedness in L^2 of equation (1). In Section 4, we prove Theorem 2, the well-posedness in $\Sigma(1)$ of equation (1). Here the commutators $[\partial, H(t)]$ and $[x, H(t)]$ play important roles. In Section 5, we prove Theorem 3 and show that the solutions of equation (1) improve the differentiability of initial data.

We end this section with several notations given. We abbreviate $L^p(\boldsymbol{R}^n)$ and $H^k(\boldsymbol{R}^n)$ to L^p and H^k, respectively. We denote L^2-inner product by $(\ ,\ )$, L^q-norm by $\|\ \|_q$. And, for $I = [0, T]$, put $L^{q,s} = L^s(I; L^q)$, where $1 < q < \infty$, $1 \leq s \leq \infty$, with its norm denoted by

$$\|f\|_{q,s} \equiv \left(\int_I \|f(t)\|_q^s dt\right)^{1/s}.$$

Set $r = 4(p+1)/n(p-1)$, i.e. $1/(p+1) + 2/nr = 1/2$. And set $r' = r/(r-1)$, i.e. $1/r + 1/r' = 1$. $\mathscr{S} = \mathscr{S}(\boldsymbol{R}^n)$ means the space of rapidly decreasing functions. We denote various constants by C, M, etc. They may differ from line to line.

2. Preliminaries

To construct solutions of equation (1), we introduce some results for equation (2) in Yajima [17].

Proposition 2.1. *Let Assumption* (A) *and* (V) *be satisfied. Then there exists a unique propagator* $\{U(t,s) \mid t,s \in \boldsymbol{R}^1\}$ *for equation* (2) *with the following properties* (a) (b):

(a) *For every* $t \neq s$, $U(t,s)$ *maps* $\mathscr{S}(\boldsymbol{R}^n)$ *into* $\mathscr{S}(\boldsymbol{R}^n)$ *continuously and extends to a unitary operator in* $L^2(\boldsymbol{R}^n)$ *and satisfies* $U(t,r)U(r,s) = U(t,s)$.

(b) *For* $f \in \Sigma(2)$, $U(t,s)f$ *is a* $\Sigma(2)$*-valued continuous and* L^2*-valued* C^1 *function of* (t,s), *and the following equations hold:*

$$i\partial_t U(t,s)f = H(t)U(t,s)f,$$

$$i\partial_s U(t,s)f = -U(t,s)H(s)f.$$

Proposition 2.2. *Let Assumption* (A) *and* (V) *be satisfied. Then there exists a positive number* T *such that for* $0 < |t-s| \leq T$, $U(t,s)$ *can be written in the form of an oscillatory integral operator;*

$$U(t,s)f(x) = (2\pi i(t-s))^{-n/2} \int e^{iS(t,s,x,y)} e(t,s,x,y) f(y) dy, \tag{10}$$

and if we set $U(t,t)=1$, *the identify operator, then* $\{U(t,s)\,|\,|t-s|\le T$, *for* $t,s\in \boldsymbol{R}^1\}$ *is strongly continuous in* $L^2(\boldsymbol{R}^n)$, *Here* $S(t,s,x,y)$, $e(t,s,x,y)$ *are uniquely given functions, respectively, which satisfy the following properties;*

(i) $S(t,s,x,y)$ *is* C^1 *in* (t,s,x,y) *and* C^∞ *in* (x,y) *and satisfies Hamiltonion-Jacobi equations*

$$(\partial_t S)(t,s,x,y)+\frac{1}{2}((\partial_x S)(t,s,x,y)-A(t,x))^2+V(t,x)=0,$$

$$(\partial_s S)(t,s,x,y)-\frac{1}{2}((\partial_y S)(t,s,x,y)-A(s,y))^2+V(s,y)=0.$$

Moreover

$$|\partial_x^\alpha\partial_y^\beta\{S(t,s,x,y)-(x-y)^2/2(t-s)\}|\le C_{\alpha\beta},\qquad |\alpha+\beta|\ge 2.$$

(ii) *For any* α *and* β, $\partial_x^\alpha\partial_y^\beta e(t,s,x,y)$ *is* C^1 *in* (t,s,x,y) *and satisfies*

$$|\partial_x^\alpha\partial_y^\beta e(t,s,x,y)|\le C_{\alpha\beta},\qquad 0<|t-s|<T,\ x,y\in\boldsymbol{R}^n.$$

The following lemma indicates that L^p-L^q estimate for the propagator $U(t,s)$ holds.

Lemma 2.3 (Yajima [17]). *Let* $T>0$ *be sufficiently small and let* $0<|t-s|<T$. *Then for* $2\le q\le\infty$,

$$\|U(t,s)f\|_q\le C_0|t-s|^{-n(1/2-1/n)}\|f\|_{q'} \tag{11}$$

where q' *is the index conjugate to* q, *i.e.,* $1/q+1/q'=1$, *and the constant* C_0 *does not depend on* t, s, *and* $f\in\mathscr{S}(\boldsymbol{R}^n)$.

Proof. In the case of $q=2$, since $U(t,s)$ is a unitary operator in $L^2(\boldsymbol{R}^n)$, $\|U(t,s)f\|_2=\|f\|_2$, $f\in\mathscr{S}$. In the case of $q=\infty$, by Proposition 2.2, $\|U(t,s)f\|_\infty\le C_0'|t-s|^{-n/2}\int_{\boldsymbol{R}^n}|e(t,s,x,y)|\,|f(y)|dy\le C_0''|t-s|^{-n/2}\|f\|_1$, $f\in\mathscr{S}$. Hence, using interpolation in L^p-space, $\|U(t,s)f\|_q\le C_0|t-s|^{-n(1/2-1/q)}\|f\|_{q'}$, $f\in\mathscr{S}$. By approximation procedure, Lemma is proved. □

We introduce the following spaces as in Kato [4, 5].

$$X=X(I)=L^{2,\infty}\cap L^{p+1,r},$$

$$|||u|||_X=\|u\|_{2,\infty}\vee\|u\|_{p+1,r},$$

$$\overline{X}=\overline{X}(I)=C(I;L^2)\cap L^{p+1,r},$$

$$X'=X'(I)=L^{2,1}+L^{1+1/p,r'},$$

$$|||f|||_{X'}=\inf\{\|f_1\|_{2,1}+\|f_2\|_{1+1/p,r'}\,|\,f=f_1+f_2\}.$$

Set $s=0$, for simplicity, we define two linear operators Γ and G by

$$(\Gamma\phi)(t) = U(t,0)\phi, \qquad t \in I, \tag{12}$$

$$(Gf)(t) = \int_0^t U(t,\tau)f(\tau)d\tau, \qquad t \in I. \tag{13}$$

The linear operators have the following properties in Yajima [16, 17];

Lemma 2.4. *Γ is a bounded operator from L^2 into $C(I;L^2)\cap AC(I;\Sigma(-2))$, which satisfies*

$$i\partial_t\Gamma\phi = H(t)\Gamma\phi, \tag{14}$$

in $\Sigma(-2)$ at almost every $t\in\boldsymbol{R}$, where AC denotes the class of absolutely continuous functions.

Lemma 2.5. *Let $f\in L^1(I;L^2)$. Then $Gf\in C(I;L^2)\cap AC(I;\Sigma(-2))$, which satisfies*

$$i\partial_t Gf = H(t)Gf + if, \tag{15}$$

in $\Sigma(-2)$ at almost every $t\in\boldsymbol{R}$.

Lemma 2.6. *Γ is bounded from L^2 to X, which satisfies*

$$|||\Gamma\phi|||_X \le C\|\phi\|_2, \qquad \phi\in L^2. \tag{16}$$

G is bounded from X' to X, which satisfies

$$|||Gf|||_X \le C'|||f|||_{X'}, \qquad f\in X', \tag{17}$$

where C, C' are independent of T. Moreover, $\Gamma\phi\in\overline{\mathscr{X}}(I)$, $Gf\in\overline{\mathscr{X}}(I)$, for any $\phi\in L^2$, $f\in X'$ on any interval I.

3. The proof of Theorem 1

We introduce the results for nonlinear term F in Kato [5].

Lemma 3.1. *Assume that F satisfies* (F1) *and* (F2) *with $1<p<1+4/n$. Then $F\in C^1(X;X')$ with*

$$|||F(u)|||_{X'} \le M_1T|||u|||_X + M_pT^\theta|||u|||_X^p, \qquad u\in X, \tag{18}$$

$$|||F'(u)v|||_{X'} \le (M_1T + M_pT^\theta|||u|||_X^{p-1})|||v|||_X, \qquad u,v\in X, \tag{19}$$

where M_1 and M_p are some constants, $\theta = 1/r - 1/r' = 1-2/r > 0$.

For Banach spaces X, Y, we say $F \in C^1(X; Y)$ if $F \in C(X; Y)$ and has a Gâteaux derivative $F'(u) \in L(X; Y)$ that depends on u strongly continuously. $L(X; Y)$ is the space of bounded linear operators from X to Y.

Remark 3.2. We mention the outline of proof of Lemma 3.1. Choose $\Psi \in C_0^\infty(\boldsymbol{C}; \boldsymbol{C})$ so that $\Psi = 1$ in a neighborhood of the origin. Set $F_1 = \Psi F$ and $F_p = (1 - \Psi)F$. Then we have

$$F = F_1 + F_p, \qquad F_1, F_p \in C^1(\boldsymbol{C}; \boldsymbol{C}), \qquad F_1(0) = F_p(0) = 0,$$

$$|F_1(\zeta)| \le M_1|\zeta|, \qquad |F_1'(\zeta)| \le M_1,$$

$$|F_p(\zeta)| \le M_p|\zeta|^p, \qquad |F_p'(\zeta)| \le M_p|\zeta|^{p-1},$$

for some $M_1 > 0$ and $M_p > 0$. Then we have the following estimates

$$\|F_1(u)\|_{2,1} \le M_1 T\|u\|_{2,\infty},$$

$$\|F_1'(u)v\|_2 \le M_1 T\|v\|_2,$$

$$\|F_p(u)\|_{1+1/p,r'} \le M_p T^\theta \|u\|_{p+1,r}^p,$$

$$\|F_p'(u)v\|_{1+1/p,r'} \le M_p T^\theta \|u\|_{p+1,r}^{p-1}\|v\|_{p+1,r}.$$

Combining the above estimates, we obtain the inequalities (18) and (19).

We introduce the following integral equation

$$u = \Phi(u) \equiv \Gamma\phi - iGF(u). \tag{20}$$

Lemma 3.3. *Let* $u \in L^\infty(I; L^2 \cap L^{p+1})$, *where* $1 < p < \infty$ *if* $n = 1, 2$, *or* $1 < p < (n+2)/(n-2)$ *if* $n \ge 3$. *Then equation* (1) *and equation* (20) *are equivalent.*

Proof. If $u \in L^\infty(I; L^2 \cap L^{p+1})$, then $H(\cdot)v \in L^\infty(I; \Sigma(-2))$. Since $F(u) \in X' = L^1(I; L^2) + L^{r'}(I; L^{1+1/p}) \subset L^\infty(I; \Sigma(-1))$, we have $H(\cdot)u + f \in L^\infty(I; \Sigma(-2))$. Hence $\partial_t u \in L^\infty(I; \Sigma(-2))$. Consequently $u \in C(I; \Sigma(-2))$, and $u(0) \in \Sigma(-2)$ exists. Since $U(t, s)$ is a continuous unitary group on $\Sigma(-2)$, we obtain equation (20). The converse is obvious by Lemma 2.4 and 2.5. □

We prove the theorem by solving equation (20).

Let $E[\bar{E}]$ be the closed ball in $X[\bar{X}]$ with radius R and center at the origin. It is our plan to apply the contraction map theorem on E to the function Φ.

Lemma 3.4. *Φ maps E into $\bar{E}$ if R is sufficiently large and T is sufficiently small, both depending only on $\|\phi\|_2$.*

Proof. Using the estimates in Lemma 2.6, 3.1, we have

$$|||\Phi(u)|||_X \leq C\|\phi\|_2 + C'(M_1TR + M_2T^\theta R^p).$$

If we choose R and T so that $C\|\phi\|_2 + C'(M_1TR + M_2T^\theta R^p) \leq R$, then $\Phi(u) \in E$. Actually, Since $U(t,s)$ is strongly continuous in $L^2(\boldsymbol{R}^n)$ and both Γ and G map into $\bar{X}$, we have $\Phi(u) \in \bar{E}$. □

Lemma 3.5. *Φ is a contraction map in the X-metric.*

Proof. By (19), we have

$$\begin{aligned}|||\Phi'(u)v|||_X &\leq C'|||F'(u)v|||_{X'} \\ &\leq C'(M_1T + M_pT^\theta R^{p-1})|||v|||_X, \qquad u \in E, v \in X.\end{aligned}$$

By choosing R, T as in Lemma 3.4, the coefficient of $|||v|||_X$ is smaller than 1. Hence Φ is a contraction map. □

Proof of Theorem 1. Since E is a closed subset of the Banach space X, it follows that Φ has a unique fixed point $u \in E$. Actually we have $u \in \bar{X}$ since Φ maps into $\bar{E}$. Moreover, since $GF(u) \in \overline{\mathscr{X}}(I)$ by $F(u) \in X'$, we have $u \in \overline{\mathscr{X}} \subset L^\infty(I; L^2 \cap L^{p+1})$. Then we conclude by Lemma 3.3 that u is a solution of equation (1).

To prove uniqueness, let $u, v \in \overline{\mathscr{X}}(I)$ be two solutions of equation (1) with $u(0) = v(0) = \phi$. By Lemma 3.3, they satisfy that $u = \Phi(u) = \Gamma\phi - iGF(u)$, $v = \Phi(v) = \Gamma\phi - iGF(v)$, respectively. Then, since $u, v \in E$ and T is sufficiently small, we have

$$\begin{aligned}|||u - v|||_X = |||\Phi(u) - \Phi(v)|||_X &\leq |||\Phi'(u)(u - v)|||_X \\ &\leq \rho|||u - v|||_X,\end{aligned}$$

where $\rho < 1$ is the same constant as in Lemma 3.5. Thus we obtain $u = v$, that is, equation (1) has a unique solution.

To prove continuous dependence of solutions on initial data, let $u \in \overline{\mathscr{X}}(I_0)$, where $I_0 = [0, T_0]$, be a solution of equation (1) with $u(0) = \phi$. Let $\phi_n \to \phi$ in L^2, then, for sufficiently large n, with the same E and T_0, $\Phi_n(u) = \Gamma\phi_n - iGF(u)$ are contraction maps of E into itself, with a contraction ratio $\rho < 1$. Thus each Φ_n has a unique fixed point $u_n \in E$, which implies, in particular, that the solutions u_n exist on a common interval $I_1 = [0, T_1] \subset I_0$. Hence

$$\begin{aligned}|||u_n - u|||_X = |||\Phi_n(u_n) - \Phi(u)|||_X &\leq |||\Phi_n(u_n) - \Phi_n(u)|||_X + |||\Phi_n(u) - \Phi(u)|||_X \\ &\leq \rho|||u_n - u|||_X + |||\Gamma(\phi_n - \phi)|||_X \\ &\leq \rho|||u_n - u|||_X + C\|\phi_n - \phi\|_2.\end{aligned}$$

Since $\phi_n \to \phi$ in L^2, it follows that $u_n \to u$ in $\overline{\mathscr{X}}(I_1)$. Thus, by the continuation argument for the sequence of initial states $u_n(T_1)$, it follows that $u_n \to u$ in $\overline{\mathscr{X}}(I_0)$.

We completed the proof of Theorem 1. □

4. The proof of Theorem 2

Before constructing H^1-solution, we introduce the following spaces;

$$Y = \{u \in X \mid \partial u \in X, xu \in X\},$$
$$|||u|||_Y = |||u|||_X \vee |||\partial u|||_X \vee |||xu|||_X,$$
$$\overline{Y} = \{u \in \overline{X} \mid \partial u \in \overline{X}, xu \in \overline{X}\},$$
$$Y' = \{f \in X' \mid \partial f \in X', xf \in X'\},$$
$$|||f|||_{Y'} = |||f|||_{X'} \vee |||\partial f|||_{X'} \vee |||xf|||_{X'}.$$

Lemma 4.1. *Let $\phi \in \mathscr{S}(\boldsymbol{R}^n)$ and $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$. If $v = \Gamma\phi - iGf$, then*

$$\partial_k v = \Gamma(\partial_k \phi) - iG\left(\frac{i}{2}(\partial_k(\operatorname{div} A))v + i(\partial_k A)\cdot \nabla v + (A\cdot \partial_k A + \partial_k V)v + \partial_k f\right), \tag{21}$$

$$x_k v = \Gamma(x_k\phi) - iG((\partial_k - iA_k)v + x_k f), \tag{22}$$

for $k = 1, \dots, n$, where $\operatorname{div} A = \partial_1 A + \partial_2 A + \cdots + \partial_n A$, $\nabla v = (\partial_1 v, \partial_2 v, \dots, \partial_n v)$.

Proof. To prove (21), we note that v satisfies the differential equation $i\partial_t v = H(t)v + f$. Then application of ∂_k gives

$$i\partial_t \partial_k v = H(t)\partial_k v + [\partial_k, H(t)]v + \partial_k f, \qquad \text{for } k = 1, \dots, n. \tag{23}$$

Here [,] denotes the commutator. Since $v \in \mathscr{S}(\boldsymbol{R}^{n+1})$, for $k = 1, \dots, n$,

$$[\partial_k, H(t)]v = \partial_k\left(\frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 v + V(t,x)v\right) - \left(\frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 \partial_k v + V(t,x)\partial_k v\right)$$
$$= \frac{i}{2}(\partial_k(\operatorname{div} A))v + i(\partial_k A)\cdot \nabla v + (A\cdot \partial_k A + \partial_k V)v.$$

Noting that $\partial_k v(0) = \partial_k \phi$, we obtain (21) by (23) for $k = 1, \dots, n$.

Again, application of the multiplication operator x_k gives

$$i\partial_t(x_k v) = H(t)(x_k v) + [x_k, H(t)]v + x_k f, \qquad \text{for } k = 1, \dots, n. \tag{24}$$

As above, for $k = 1, \dots, n$,

$$\begin{aligned}[x_k, H(t)]v &= x_k\left(\frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 v + V(t,x)v\right) \\ &\quad - \left(\frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 (x_k v) + V(t,x)(x_k v)\right) \\ &= (\partial_k - iA_k)v.\end{aligned}$$

Since $x_k v(0) = x_k \phi$, we obtain (22) by (24) for $k = 1, \dots, n$. □

Lemma 4.2. *Let $T > 0$ be sufficiently small and let $0 < |t| < T$. Then*

$$\Gamma : \Sigma(1) \to Y, \tag{25}$$

$$\text{with } |||\Gamma\phi|||_Y \le c\|\phi\|_{\Sigma(1)}, \qquad \phi \in \Sigma(1),$$

$$G : Y' \to Y, \tag{26}$$

$$\text{with } |||Gf|||_Y \le c'|||f|||_{Y'}, \qquad f \in Y'.$$

Proof. First we begin with the assumptions of Lemma 4.1. By the boundness of the operator Γ and G in X, we obtain from (21), (22)

$$\begin{aligned}|||\partial_k v|||_X &\le C\|\partial_k\phi\|_2 + C'|||\partial_k(\operatorname{div} A)v|||_{X'} + C'|||(\partial_k A)\cdot\nabla v|||_{X'} \\ &\quad + C'|||(A\cdot\partial_k A)v|||_{X'} + C'|||(\partial_k V)v|||_{X'} + C'|||\partial_k f|||_{X'}, \\ |||x_k v|||_X &\le C\|x_k\phi\|_2 + C'|||\partial_k v|||_{X'} + C'|||A_k v|||_{X'} + C'|||x_k f|||_{X'}.\end{aligned}$$

In view of the mean value theorem, Assumptions (A) and (V) imply that, for fixed $t \in I = [0, T]$, there exist constants $c_j, j = 1, 2, 3, 4$, depending on T, $|A(t,x)| \le c_1 + c_2|x|$, $|\partial_k V(t,x)| \le c_3 + c_4|x|$. And by definition of X, X', $|||v|||_{X'} \le \|v\|_{2,1} \le T\|v\|_{2,\infty} \le T|||v|||_X$, and similarly for others. By using these facts, we have

$$\begin{aligned}|||\partial_k v|||_X &\le C\|\partial_k\phi\|_2 + C''T(|||v|||_X + |||\partial v|||_X + |||xv|||_X) + C'|||\partial_k f|||_{X'} \\ &\le C\|\partial_k\phi\|_2 + C'|||\partial_k f|||_{X'} + C''T|||v|||_Y, \\ |||x_k v|||_X &\le C\|x_k\phi\|_2 + C''T(|||\partial_k v|||_X + |||xv|||_X) + C'|||x_k f|||_{X'} \\ &\le C\|x_k\phi\|_2 + C'|||x_k f|||_{X'} + C''T|||v|||_Y,\end{aligned}$$

where $C'' > 0$ depends on T. While, it is obvious that $|||v|||_X \le C\|\phi\|_2 + C'|||f|||_{X'}$. Since T is sufficiently small, we can solve the above system, obtaining

$$|||v|||_Y \le c\|\phi\|_{\Sigma(1)} + c'|||f|||_{Y'}, \tag{27}$$

with constants c, c'.

By a standard approximation procedure, (27) holds for $\phi \in \Sigma(1)$ and $f \in Y'$. □

We assume $n \ge 3$.

Lemma 4.3. *Assume* (F1) *and* (F2) *with* $1 < p < 1 + 4/(n-2)$. *If* $u \in Y$, *then* $F(u) \in Y'$ *with*

$$|||F(u)|||_{Y'} \le M_1 T|||u|||_Y + M_p' T^\theta |||u|||_Y^p, \tag{28}$$

where $\theta = 1 - 2/r > 0$, $M_p' = C_1' M_p$, *for some* $C_1' > 0$ *and* M_1, M_p *are as in Remark* 3.2.

Proof. We use the decomposition of F in Remark 3.2, that is, $F = F_1 + F_p$, and estimate the X'-norm of $F_j(u)$, $\partial_k(F_j(u))$, $x_k F_j(u)$, for $j = 1, p$, and $k = 1, \dots, n$. We note that $\partial_k(F(u)) = F'(u)\partial_k u$, for $k = 1, \dots, n$. (see Section 4 in Kato [5].)

$$\|F_1(u)\|_{2,1} \le M_1\|u\|_{2,1} \le M_1 T\|u\|_{2,\infty} \le M_1 T|||u|||_Y,$$

$$\|\partial_k(F_1(u))\|_{2,1} \le M_1\|\partial_k u\|_{2,1} \le M_1 T\|\partial_k u\|_{2,\infty} \le M_1 T|||u|||_Y,$$

$$\|x_k(F_1(u))\|_{2,1} \le M_1\|x_k u\|_{2,1} \le M_1 T\|x_k u\|_{2,\infty} \le M_1 T|||u|||_Y,$$

for $k = 1, \dots, n$. By $1 < p < 1 + 4/(n-2)$, Sobolev's embedding theorem implies

$$\|u\|_{p+1,\infty} \le C_1\|u\|_{L^\infty(I;H^1)},$$

for some $C_1 > 0$, which is independent of T. Hence we have, using Hölder inequality, Remark 3.2 and the above estimate, we have the following sequences of estimates;

$$\begin{aligned}\|F_p(u)\|_{1+1/p,r'} &\le M_p\|u\|_{p+1,\infty}^{p-1}\|u\|_{p+1,r'} \\ &\le C_1 M_p T^{1-2/r}\|u\|_{L^\infty(I;H^1)}^{p-1}\|u\|_{p+1,r} \\ &\le C_1' M_p T^{1-2/r}|||u|||_Y^p,\end{aligned}$$

$$\begin{aligned}
\|\partial_k(F_p(u))\|_{1+1/p,r'} &\le \|F_p'(u)\partial_k u\|_{1+1/p,r'} \\
&\le M_p\|u\|_{p+1,\infty}^{p-1}\|\partial_k u\|_{p+1,r'} \\
&\le C_1 M_p T^{1-2/r}\|u\|_{L^\infty(I;H^1)}^{p-1}\|\partial_k u\|_{p+1,r} \\
&\le C_1' M_p T^{1-2/r}|||u|||_Y^p, \\
\|x_k(F_p(u))\|_{1+1/p,r'} &\le M_p\|x_k|u|^p\|_{1+1/p,r'} \\
&\le M_p\|u\|_{p+1,\infty}^{p-1}\|x_k u\|_{p+1,r'} \\
&\le C_1 M_p T^{1-2/r}\|u\|_{L^\infty(I;H^1)}^{p-1}\|x_k u\|_{p+1,r} \\
&\le C_1' M_p T^{1-2/r}|||u|||_Y^p,
\end{aligned}$$

for $k = 1, \dots, n$ and for some $C' > 0$, which is independent of T. Thus we obtain (28). □

Let $E[\bar{E}]$ be the closed ball in $Y[\bar{Y}]$ with radius R and center at the origin. As in Section 3, we apply the contraction map theorem on E to the function Φ defined by (20).

Lemma 4.4. *Φ maps E into $E \cap \overline{\mathscr{Y}}(I) \subset \bar{E}$ if R is sufficiently large and T is sufficiently small, both depending only on $\|\phi\|_{\Sigma(1)}$.*

Proof. By Lemma 4.2 and 4.3, we have

$$|||\Phi(u)|||_Y \le c\|\phi\|_{\Sigma(1)} + c'(M_1 TR + M_p' T^\theta R^p), \qquad \text{for } u \in E.$$

If we choose R and T so that $c\|\phi\|_{\Sigma(1)} + c'(M_1 TR + M_p' T^\theta R^p) \le R$, then $\Phi(u) \in E$. Actually, Since $U(t,s)$ is strongly continuous in $L^2(\boldsymbol{R}^n)$ and, both Γ and G, which commute with ∂, map into $\bar{Y}$, we have $\Phi(u) \in \bar{E}$. □

Lemma 4.5. *Φ is a contraction map in the X-metric.*

Proof. By the similar argument used in the proof of Lemma 3.5, we have

$$\begin{aligned}
|||\Phi'(u)v|||_X &\le C'|||F_1'(u)v|||_{X'} + C'|||F_p'(u)v|||_{X'}^p \\
&\le C'\|F_1'(u)v\|_{2,\infty} + C'\|F_p'(u)v\|_{1+1/p,r'}^p \\
&\le C'(M_1 T + M_p T^\theta R^{p-1})|||v|||_X, \qquad u \in E, v \in Y.
\end{aligned}$$

By choosing the same constants R, T as in Lemma 4.4, the coefficient of $|||v|||_X$ is smaller than 1. Hence Φ is a contraction map. □

Lemma 4.6. *E is a complete metric space in the X-metric.*

Proof. It suffices to show that E is closed in X. Let $u_n \in E$ be a Cauchy sequence in the X-metric. Then $\lim u_n = u \in X$ exists. We have to show that $u \in E$.

Since $\|\partial u_n\|_{p+1,r} \le |||u_n|||_Y \le R$ and $L^{p+1,r}$ is reflexive, we have $\partial u \in L^{p+1,r}$ with $\|\partial u\|_{p+1,r} \le R$. Similarly we have $xu \in L^{p+1,r}$ with $\|xu\|_{p+1,r} \le R$. For $L^{2,\infty}$, which is not reflexive, we have $\|\partial u_n(t)\|_2 \le \|\partial u_n\|_{2,\infty} \le |||\partial u_n|||_X \le |||u_n|||_Y \le R$ for a.e. $t \in I$. Since $\|u_n(t) - u(t)\|_2 \le |||u_n - u|||_X \to 0$ for a.e. $t \in I$, it follows that $\|\partial u(t)\|_2 \le R$ for a.e. $t \in I$, so that $\|\partial u\|_{2,\infty} \le R$. Thus $\partial u \in L^{2,\infty} \cap L^{p+1,r} = X$ with $|||\partial u|||_X \le R$. Similarly we have $xu \in X$ with $|||xu|||_X \le R$. Hence $u \in Y$ with $|||u|||_Y \le R$. □

Proof of Theorem 2. Since E is a closed subset of the Banach space Y, it follows that Φ has a unique fixed point $u \in E$. Actually we have $u \in E \cap \overline{\mathscr{Y}}(I) \subset \bar{E}$ by Proposition 4.5. And, since G commutes with ∂ and by $F(u) \in Y'$, we have $GF(u) \in \overline{\mathscr{Y}}(I)$. Hence we obtain $u \in \overline{\mathscr{Y}}(I)$ and, as before, u is a unique solution of equation (1).

To prove continuous dependence of solutions on initial data, as in the proof of Theorem 1, let $u \in C(I_0; \Sigma(1))$ be a solution of equation (1) with $u(0) = \phi$ and $\phi_n \to \phi$ in $\Sigma(1)$. Set $\Phi_n(u) = \Gamma\phi_n - iGF(u)$, as before, we obtain that each Φ_n has a unique fix point $u_n \in E$, and $u_n \to u$ in X-metric. By (21) and (22), we have

$$
\begin{aligned}
&|||\partial_k u_n - \partial_k u|||_X \\
&\quad = \left|\left|\left| \Gamma(\partial_k(\phi_n - \phi)) - iG\left(\frac{i}{2}(\partial_k(\operatorname{div} A))(u_n - u) + i(\partial_k A)\cdot \nabla(u_n - u) \right.\right.\right. \\
&\qquad \left.\left.\left. + (A\cdot\partial_k A + \partial_k V)(u_n - u) + (\partial_k(F(u_n)) - \partial_k(F(u)))\right)\right|\right|\right|_X, \\
&\quad \le C\|\partial(\phi_n - \phi)\|_2 \\
&\qquad + C'\left|\left|\left| \frac{i}{2}(\partial_k(\operatorname{div} A))(u_n - u) + i(\partial_k A)\cdot\nabla(u_n - u) + (A\cdot\partial_k A + \partial_k V)(u_n - u)\right|\right|\right|_{X'} \\
&\qquad + C'|||\partial_k(F(u_n)) - \partial_k(F(u))|||_{X'} \\
&\quad \le C\|\partial(\phi_n - \phi)\|_2 + C''T|||u_n - u|||_X + C''T|||\nabla(u_n - u)|||_X + C''T|||x(u_n - u)|||_X \\
&\qquad + C'|||\partial_k(F(u_n)) - \partial_k(F(u))|||_{X'},
\end{aligned}
$$

$$\begin{aligned}
|||x_k u_n - x_k u|||_X &= |||\Gamma(x_k(\phi_n - \phi) - iG((\partial_k - iA_k)(u_n - u) \\
&\quad + x_k(F(u_n) - F(u)))|||_X \\
&\leq C\|x_k(\phi_n - \phi)\|_2 + C'|||\partial(u_n - u)|||_{X'} \\
&\quad + C'|||A_k(u_n - u)|||_{X'} + C'|||x_k(F(u_n) - F(u))|||_{X'} \\
&\leq C\|x_k(\phi_n - \phi)\|_2 + C'T|||\partial(u_n - u)|||_X \\
&\quad + C''T|||u_n - u|||_X + C''T|||x(u_n - u)|||_X \\
&\quad + C'|||x_k(F(u_n) - F(u))|||_{X'}.
\end{aligned}$$

Hence we have

$$|||u_n - u|||_Y \leq C\|\phi_n - \phi\|_{\Sigma(1)} + C''T|||u_n - u|||_Y + C'|||F(u_n) - F(u)|||_{Y'}.$$

If we choose sufficiently small $T = T_1$, then we have

$$|||u_n - u|||_Y \leq c\|\phi_n - \phi\|_{\Sigma(1)} + c'|||F(u_n) - F(u)|||_{Y'}.$$

It suffices to show that $|||F(u_n) - F(u)|||_{Y'} \to 0$ as $u_n \to u$ in X.

$$\begin{aligned}
|||\partial_k(F(u_n)) - \partial_k(F(u))|||_{X'} &= |||F'(u_n)\partial_k u_n - F'(u)\partial_k u|||_{X'} \\
&\leq |||F'(u_n)(\partial_k u_n - \partial_k u)|||_{X'} + |||(F'(u_n) - F'(u))\partial_k u|||_{X'},
\end{aligned}$$

for $k = 1, \ldots, n$. By Lemma 4.5, the operator norm of $F'(u_n)$ on X is smaller than 1, and it is easy to see that $|||(F'(u_n) - F'(u))\partial_k u|||_{X'} \to 0$ for $k = 1, \ldots, n$ (see Section 6 in Kato [5]). Then we have $|||\partial(F(u_n)) - \partial(F(u))|||_{X'} \to 0$. By Assumption (F1) and (F2), it is obvious that $|||F(u_n) - F(u)|||_{X'}$, $|||x_k(F(u_n)) - x_k(F(u))|||_{X'} \to 0$. Hence, if $\phi_n \to \phi$ in $\Sigma(1)$, then $u_n \to u$ in $C(I_1; \Sigma(1))$, for $I_1 = [0, T_1]$. Thus, by the continuation argument, it follows that $u_n \to u$ in $C(I_0; \Sigma(1))$.

When $n = 1, 2$, we set $F_1 = F$, $F_p = 0$ and $X = L^{2,\infty}$ in the proof above. Then $Y \subset L^\infty(I \times \boldsymbol{R}^n)$, and we obtain $|||F(u)|||_{Y'} \leq M_1 T|||u|||_Y$ instead of Lemma 4.3. We proved Theorem 2. □

5. The proof of Theorem 3

Before the proof of Theorem 3, we introduced results of Yajima [16].

Proposition 5.1. *Suppose that* (A) *and* (V) *be satisfied for equation* (2). *Let* $T > 0$ *be sufficiently small,* $\mu > 1/2$ *and* $\rho > 0$. *Then there exists a constant* $C_{\rho\mu} > 0$ *such that for* $s \in \boldsymbol{R}^1$

$$\int_{s-T}^{s+T} \|\langle x\rangle^{-\rho-\mu}\langle D\rangle^{\rho} U(t,s)f\|_2^2 dt \leq C_{\rho\mu}\|\langle D\rangle^{\rho-1/2} f\|_2^2, \qquad f \in \mathscr{S}(\boldsymbol{R}^n). \tag{29}$$

Proof of Theorem 3. We adapt Proposition 5.1 to $u = \Gamma\phi - iGF(u)$. For $\mu > 1/2$,

$$\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}u = \langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}\Gamma\phi - i\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}GF(u).$$

We have

$$\int_I \|\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}\Gamma\phi\|_2^2 dt \le C_{3/2,\rho}\|\langle D\rangle\phi\|_2^2 \le C'_{3/2,\rho}\|\phi\|_{H^1}^2 \le C''_{3/2,\rho}\|\phi\|_{\Sigma(1)}^2,$$

And, if $F(u) \in L^1(I;H^1)$, then, by duality, we have

$$\begin{aligned}
&\int_I \left(\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}\int_0^t U(t,\tau)F(u)(\tau)d\tau, \psi(t,x)\right)dt \\
&\quad\le \int_I\int_0^t \|\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}U(t,\tau)F(u)(\tau)\|_2\|\psi(t)\|_2 d\tau dt \\
&\quad\le \int_I\int_I \|\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}U(t,\tau)F(u)(\tau)\|_2\|\psi(t)\|_2 d\tau dt \\
&\quad\le \int_I \|\langle x\rangle^{-\mu-3/2}\langle D\rangle^{3/2}U(t,\tau)F(u)(\tau)\|_{L^2(I\times\boldsymbol{R}^n)}\|\psi\|_{L^2(I\times\boldsymbol{R}^n)}d\tau \\
&\quad\le \int_I C_{3/2,\mu}\|F(u)(\tau)\|_{H^1}\|\psi\|_{L^2(I\times\boldsymbol{R}^n)}d\tau \\
&\quad\le C_{3/2,\mu}\|\psi\|_{L^2(I\times\boldsymbol{R}^n)}\int_I \|F(u)(\tau)\|_{H^1}d\tau \\
&\quad\le C_{3/2,\mu}\|\psi\|_{L^2(I\times\boldsymbol{R}^n)}\|F(u)\|_{L^1(I;H^1)}
\end{aligned}$$

Hence it suffice to prove $F(u) \in L^1(I;H^1)$ for $u \in Y$.

Next lemma is the result for the regularity of nonlinear term F.

Lemma 5.2 (Sjölin [12]). *Assume* (F1) *and, if* $n \ge 3$, (F2) *with* $1 < p < 1 + 4/(n-2)$. *Suppose* $u \in \overline{\mathcal{Y}}_0(I) \equiv \{u \in \overline{\mathcal{X}}(I) \mid \partial u \in \overline{\mathcal{X}}(I)\}$ *with* $T < \infty$, *where* $\overline{\mathcal{X}}(I)$ *is defined in* (6). *If* $1 \le n \le 6$, *then* $F(u) \in L^1(I;H^1)$. *If* $n \ge 7$, *under the additional assumption* $p < 1 + 2/(n-4)$, $F(u) \in L^1(I;H^1)$.

Proof. In the case of $n = 1, 2$, we proved that $F(u) \in L^1(I;L^2)$ in Section 4. Assume $n \ge 3$. To prove this lemma, we decompose $F = F_1 + F_p$ as in Remark 3.2. In Section 4, we proved that $F_1(u) \in L^1(I;L^2)$ under lemma's assumption. It remains the estimate of F_p.

If $n = 3$ with $1 < p < 2/(n-2) = 2$, by Sobolev's embedding theorem, for some $C_2 > 0$,

$$\|u\|_{2(p+1),\infty} \le C_2\|u\|_{L^\infty(I;H^1)}.$$

Then we have, in view of Remark 3.2, for some $C_2' > 0$,

$$\begin{aligned}
\|F_p(u)\|_{2,1} &\leq M_p T^{1-1/r} \|u\|^{p-1}_{2(p+1),\infty} \|u\|_{p+1,r} \\
&\leq C_2 M_p T^{1-1/r} \|u\|^{p-1}_{L^\infty(I;H^1)} |||u|||_X \\
&\leq C_2' M_p T^{1-1/r} |||u|||^p_Y, \\
\|\partial(F_p(u))\|_{2,1} &\leq \|F_p'(u)\partial u\|_{2,1} \\
&\leq M_p T^{1-1/r} \|u\|^{p-1}_{2(p+1),\infty} \|\partial u\|_{p+1,r} \\
&\leq C_2 M_p T^{1-1/r} \|u\|^{p-1}_{L^\infty(I;H^1)} |||\partial u|||_X \\
&\leq C_2' M_p T^{1-1/r} |||u|||^p_Y.
\end{aligned}$$

We note that $Y \subset L^\infty(I;H^1)$. Hence we have that if $n = 3$ with $1 < p < 2/(n-2) = 2$, then $F_p(u) \in L^1(I;L^2)$.

In the case of $n = 3$ with $p > 2/(n-2) = 2$ or $n \geq 4$, it suffices to find $(p_1, r_1), (p_2, r_2)$ which satisfy

$$1/2 - 2/n < 1/p_1 \leq 1/2 - 1/n, \tag{30}$$

$$1/2 - 1/n < 1/p_2 \leq 1/2, \tag{31}$$

and

$$1/p_1 + 2/nr_1 = 1/2 - 1/n, \tag{32}$$

$$1/p_2 + 2/nr_2 = 1/2, \tag{33}$$

$$(p-1)/p_1 + 1/p_2 = 1/2, \tag{34}$$

$$(p-1)/r_1 + 1/r_2 < 1. \tag{35}$$

Then, by (31), (33) and the definition of $\overline{\mathscr{X}}(I)$, we have $\overline{\mathscr{X}}(I) \subset L^{p_2,r_2}$, and, set $1/p_3 = 1/p_1 + 1/n$, then, from (30) and (32), it follows $\overline{\mathscr{X}}(I) \subset L^{p_3,r_1}$. Hence, by Sobolev's embedding theorem, we have $\overline{\mathscr{Y}}_0(I) \subset L^{p_1,r_1}$. And, if (34) and (35) hold, then we have, by Hölder inequality,

$$\|F_p(u)\|_{2,1} \leq M_p T^{1-1/r_3} \|u\|^{p-1}_{p_1,r_1} \|u\|_{p_2,r_2}, \tag{36}$$

$$\begin{aligned}
\|\partial(F_p(u))\|_{2,1} &\leq \|F_p'(u)\partial u\|_{2,1} \\
&\leq M_p T^{1-1/r_3} \|u\|^{p-1}_{p_1,r_1} \|\partial u\|_{p_2,r_2},
\end{aligned} \tag{37}$$

where $(p-1)/r_1 + 1/r_2 = 1/r_3 < 1$. Hence the right hand side of (36) and (37) converge, respectively.

Solving (33) and (34), we obtain that $1/p_1 = 2/nr_2(p-1)$. By $1/2 - 2/n < 1/p_1 \leq 1/2 - 1/n$, we have $1/2 - 2/n < 2/nr_2(p-1) \leq 1/2 - 1/n$, i.e. $(n-4)(p-1)/4 < 1/r_2 < (n-2)(p-1)/4$. (33) and (31) imply that $2 < r_2 < \infty$, thus we need the condition that $p < 1 + 2/(n-4)$.

We note $u \in \overline{\mathscr{Y}}(I) \subset \overline{\mathscr{Y}}_0(I)$. The proof of Theorem 3 is completed. □

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