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Shin, Jong Son
Naito, Toshiki
Van Minh, Nguyen

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Existence and Uniqueness of Periodic Solutions to Periodic Linear Functional Differential Equations with Finite Delay

By

Jong Son SHIN, Toshiki NAITO and Nguyen VAN MINH*

(Korea University, Japan, The University of Electro-Communications, Japan and University of Hanoi, Vietnam)

1. Introduction and preliminaries

Let R be a real line and E a Banach space with a norm $|\cdot|$. Let r be a given positive number. For a function $x : R \rightarrow E$, a function $x_t : [-r, 0] \rightarrow E$, $t \in R$, is defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. We deal with the linear functional differential equation with finite delay in the Banach space E of the form

$$(L) \quad \frac{dx(t)}{dt} = Ax(t) + L(t, x_t) + f(t).$$

Denote by $\mathcal{C} := C([-r, 0], E)$ the set of all continuous functions from $[-r, 0]$ to E with the supremum norm. We assume that Eq. (L) always satisfies the following hypothesis (H):

- (i) $A : \mathcal{D}(A) \subset E \rightarrow E$ is the infinitesimal generator of a C_0 -semigroup $T(t)$, $t \geq 0$, on E ;
- (ii) $L : R \times \mathcal{C} \rightarrow E$ is continuous and $L(t, \cdot) : \mathcal{C} \rightarrow E$ is linear;
- (iii) $f : R \rightarrow E$ is continuous.

Note that $\|L(t)\| := \|L(t, \cdot)\|$ is a locally bounded, lower semicontinuous function. Throughout this paper we assume that $\|L\|_\infty = \sup\{\|L(t)\| : t \geq 0\} < \infty$. If $L(t, \psi)$ and $f(t)$ in Eq. (L) are periodic functions with a period $\omega > 0$, we denote Eq. (L) by Eq. $(P_\omega L)$. If $f(t) \equiv 0$, we denote Eq. (L) and Eq. $(P_\omega L)$ by Eq. (L_0) and Eq. $(P_\omega L_0)$, respectively.

The purpose of this paper is to investigate criteria for the existence and the uniqueness of (mild) periodic solutions to Eq. $(P_\omega L)$ in the relation between the delay r and the period ω by materializing the theory of semi-Fredholm operators, which is used in [9]. In particular, we discuss the uniqueness of periodic solutions to Eq. $(P_\omega L)$ by using the manner of [9] and the standard way.

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First, we investigate criteria for the existence of periodic solutions to Eq. (P_ωL). In 1974, Chow and Hale [1] gave the following fixed point theorem for linear affine maps to show the existence of periodic solutions to Eq. (P_ωL) with $A = 0$ and $E = R^n$. Let X be a Banach space and T a bounded linear affine map defined by $Tx := Sx + z$ for $x \in X$, where S is a bounded linear operator on X and $z \in X$ is fixed.

Theorem 1.1. *If the range $\mathcal{R}(I - S)$, I being the identity, is closed and if there is an $x^0 \in X$ such that $\{x^0, Tx^0, T^2x^0, \dots\}$ is bounded in X , then T has a fixed point in X .*

The following trivial result is the case where the fixed point of T is unique.

Theorem 1.2. *If $1 \in \rho(S)$ (the resolvent set of S), then T has a unique fixed point in X .*

More recently, Shin and Naito [9] studied the existence and the uniqueness of periodic solutions to Eq. (P_ωL) with infinite delay, in which the semi-Fredholm operator theory plays an important role.

Denote by $\mathcal{F}_T$ the set of the fixed points of the affine map T on X given in the above theorem. Then $\mathcal{F}_T = y + N(I - S)$ for some $y \in \mathcal{F}_T$: it is an affine space, where $N(B)$ stands for the null space of linear operator B . The dimension of the affine space $\mathcal{F}_T$ is defined as the dimension of the null space of $I - S$; that is, $\dim \mathcal{F}_T = \dim N(I - S)$. Denote by $\Phi_+(X)$ the family of semi-Fredholm operators on X ; that is, the family of linear bounded operators F such that $\dim N(F)$ is finite and the range $\mathcal{R}(F)$ is closed. Notice that for a semi-Fredholm operator F on X , $\mathcal{R}(F)$ is closed in X if and only if there exists a constant $c > 0$ such that $\|x\| \leq c\|Fx\|$ for $x \in X$, where $\|x\| = \inf\{\|x + y\| : y \in N(F)\}$, cf. [9, Lemma 6.4]. Using such semi-Fredholm operators, Theorem 1.1 is refined as follows, see [9].

Theorem 1.3. *Assume that there is an $x^0 \in X$ such that $\{x^0, Tx^0, T^2x^0, \dots\}$ is bounded in X . If $I - S \in \Phi_+(X)$, then $\mathcal{F}_T \neq \emptyset$ and $\dim \mathcal{F}_T$ is finite.*

Remark. Let S be a bounded linear operator on X . Then the following statements hold true.

- 1) If S is an α -contraction operator on X , then $r_e(S) < 1$.
- 2) If $r_e(S) < 1$, then 1 is a normal point of S .
- 3) If 1 is a normal point of S , then $I - S \in \Phi_+(X)$.
- 4) If $I - S \in \Phi_+(X)$, then $\mathcal{R}(I - S)$ is closed.

In order to apply the above result to Eq. (P_ωL), the perturbation theory of semi-Fredholm operators is needed, cf. [3], [7], [9, Proposition 7.3]. Let $F, K : X \rightarrow X$ be bounded linear operators.

Theorem 1.4. *Assume that $F \in \Phi_+(X)$. If K is a compact operator on X , then $F + K \in \Phi_+(X)$.*

Theorem 1.5. *Assume that $F \in \Phi_+(X)$ (; and hence, $\dim N(F) = n < \infty$ and $\|x\| \leq c\|Fx\|$ for $x \in X$ and for some $c > 0$). If $\|K\| \leq 1/2c(1 + \sqrt{n})$, then $F + K \in \Phi_+(X)$, and $\dim N(F + K) \leq n$.*

Using those theorems we will show the existence of periodic solutions to Eq. (P_ωL) in the first half of the present paper.

Next, we investigate criteria for the uniqueness of periodic solutions to Eq. (P_ωL), which is considered in the latter half of the paper. We will employ two manners. The first manner (Proposition 5.5) is concerned with the following result, which is the case where $n = 0$ in the above theorem.

Proposition 1.6. *Assume that $I - F \in \Phi_+(X)$ and $\dim N(I - F) = 0$ (; and so, $|x| \leq c|(I - F)x|$ for $x \in X$ and for some $c > 0$). If $\|K\| \leq 1/2c$, then $I - F + K$ belongs to $\Phi_+(X)$, and $\dim N(I - F + K) = 0$.*

Notice that if $1 \in \rho(F)$ in Proposition 1.6, then $I - F \in \Phi_+(X)$ and $\dim N(I - F) = 0$. The second one (Theorem 6.4) is based on the following trivial result.

Proposition 1.7. *Assume that $1 \in \rho(F)$. If $1 \in \rho((I - F)^{-1}K)$, then $1 \in \rho(F + K)$.*

We here emphasize that in general, our results (Proposition 5.5 and Theorem 6.4) on the uniqueness of periodic solutions to Eq. (P_ωL) are independent of each other.

2. Estimates of $\|K(t)\|$

We denote by $u(t, \sigma, \phi)$ and $u(t, \sigma, \phi, f)$, respectively, the mild solutions of Eq. (L₀) and Eq. (L) through $(\sigma, \phi) \in R \times \mathcal{C}$. Refer to [8, 10, 12, 13, 14] for the existence and uniqueness of mild solutions to Eq. (L). Define the solution operator $U(t, \sigma)$, $t \geq \sigma$, on $\mathcal{C}$ for Eq. (L₀) by $U(t, \sigma)\phi = u_t(\sigma, \phi) := u(t + \cdot, \sigma, \phi)$. Then it may be seen that this is a family of bounded linear operators, and $\|U(t, \sigma)\|$ is bounded for t in every compact interval of $[\sigma, \infty)$. Put

$$z(t, \sigma, \phi) = \begin{cases} \int_{\sigma}^t T(t-s)L(s, u_s(\sigma, \phi))ds, & t \geq \sigma \\ 0, & \sigma - r \leq t \leq \sigma. \end{cases}$$

Then $z(t, \sigma, \phi)$ is continuous for $t \geq \sigma - r$. For $t \geq \sigma$ define an operator $K(t, \sigma)$ by $K(t, \sigma)\phi = z_t(\sigma, \phi)$ for $\phi \in \mathcal{C}$. Then it is a bounded linear operator on $\mathcal{C}$, and $U(t, \sigma)$ is decomposed as

$$(1) \quad U(t, \sigma) = \hat{T}(t - \sigma) + K(t, \sigma),$$

where

$$(\hat{T}(t)\phi)(\theta) = \begin{cases} T(t + \theta)\phi(0) & \text{for } t + \theta \geq 0 \\ \phi(t + \theta) & \text{for } t + \theta \leq 0. \end{cases}$$

Set $K(t) = K(t, 0)$ and let the C_0 -semigroup $T(t)$ on E in Hypothesis (H) satisfy the inequality $\|T(t)\| \leq M_w e^{wt}$, $t \geq 0$, where $M_w \geq 1$ and $w \in \mathbb{R}$. Then we have the following result, which can be found in [9].

1) $\hat{T}(t)$ is a C_0 -semigroup on $\mathcal{C}$; and hence,

$$(2) \quad |\hat{T}(t)\phi| \leq M_w |\phi| e^{\max\{0, w\}t} \quad \text{for } t \geq 0.$$

2) If $T(t)$ is compact for $t > 0$, or if $L(t, \cdot)$ is a compact operator for $t > 0$, then $K(t)$, $t > 0$, is a compact operator on $\mathcal{C}$.

3) The following inequality holds true:

$$(3) \quad \|K(t)\| \leq a \int_0^t e^{(a + \max\{0, w\})(t-s)} \|\hat{T}(s)\| ds$$

for $t \geq 0$, where $a = M_w \|L\|_\infty$.

Lemma 2.1. *Let a , r and w be positive constants, and let $f, u : [0, \infty) \rightarrow \mathbb{R}$ be nonnegative continuous functions. Suppose that $f(t)$ is a nondecreasing function in t and that $u(t)$ satisfies the inequality*

$$(4) \quad u(t) \leq a \sup_{\max\{0, t-r\} \leq \tau \leq t} \int_0^\tau e^{-w(\tau-s)} u(s) ds + f(t).$$

Then the following inequalities hold:

1)

$$u(t) \leq e^{at} f(t) \quad \text{for } t \geq 0.$$

2)

$$u(t) \leq wf(t)/(w - a) \quad \text{if } w > a \quad \text{and} \quad t \geq 0.$$

Proof. We prove the assertion 1). The inequality (4) becomes

$$u(t) \leq a \int_0^t u(s) ds + f(t).$$

By applying Gronwall's inequality to this inequality, we can obtain

$$\begin{aligned} u(t) &\leq \left[1 + a \int_0^t e^{a(t-s)} ds \right] f(t) \\ &= e^{at} f(t), \end{aligned}$$

which implies the assertion 1).

Next, we prove the assertion 2). Set $v(t) = \sup_{0 \leq s \leq t} u(s)$. Then from the inequality (4) we have

$$\begin{aligned} u(t) &\leq av(t) \sup_{\max\{0, t-r\} \leq \tau \leq t} \int_0^\tau e^{-w(\tau-s)} ds + f(t) \\ &\leq av(t) \sup_{\max\{0, t-r\} \leq \tau \leq t} (1 - e^{-w\tau})/w + f(t) \\ &\leq av(t)(1 - e^{-wt})/w + f(t) \\ &\leq av(t)/w + f(t), \end{aligned}$$

and hence,

$$v(t) \leq av(t)/w + f(t),$$

from which the described inequality follows, cf. [9, Lemma 3.6].

Using the above lemma, we have the following estimates on $\|K(t)\|$.

Proposition 2.2. *Suppose that $\|T(t)\| \leq M_w e^{-wt}$, $w > 0$, for $t \geq 0$. Then the following estimates hold true for $\|K(t)\|$.*

1)

$$\|K(t)\| \leq \min\{M_w^2 \|L\|_\infty (1 - e^{-wt}) e^{M_w \|L\|_\infty t}/w, M_w (e^{M_w \|L\|_\infty t} - 1)\} \quad \text{for } t \geq 0.$$

2)

$$\begin{aligned} \|K(t)\| &\leq M_w^2 \|L\|_\infty (1 - e^{-wt})/(w - M_w \|L\|_\infty) := k(t) \\ &\text{if } w > M_w \|L\|_\infty \text{ and } t \geq 0. \end{aligned}$$

Proof. Set $a = M_w \|L\|_\infty$. Using the decomposition (1) of the solution operator for Eq. (L₀), we have

$$\begin{aligned} |K(t)\phi| &\leq M_w \sup_{\max\{0, t-r\} \leq \tau \leq t} \int_0^\tau e^{-w(\tau-s)} |L(s, U(s, 0)\phi)| ds \\ &\leq a \sup_{\max\{0, t-r\} \leq \tau \leq t} \left\{ \int_0^\tau e^{-w(\tau-s)} |K(s)\phi| ds + \int_0^\tau e^{-w(\tau-s)} |\hat{T}(s)\phi| ds \right\}. \end{aligned}$$

Furthermore, it follows from the estimate (2) of $\hat{T}(t)$ that

$$\int_0^\tau e^{-w(\tau-s)} |\hat{T}(s)\phi| ds \leq \int_0^\tau e^{-w(\tau-s)} M_w |\phi| ds \leq M_w (1 - e^{-w\tau}) |\phi|/w.$$

If we set $u(t) = |K(t)\phi|$, then

$$u(t) \leq a \sup_{\max\{0, t-r\} \leq \tau \leq t} \int_0^\tau e^{-w(\tau-s)} u(s) ds + a M_w (1 - e^{-wt}) |\phi|/w.$$

Using Lemma 2.1 and the estimate (3), we have the estimates described in the proposition.

3. Existence of periodic solutions: bounded perturbations

In this section, we consider the existence of periodic solutions to Eq. $(P_\omega L)$ by using Theorem 1.3 and Theorem 1.5. The following two lemmas can be found in [9, Lemma 6.1 and Proposition 6.2].

Lemma 3.1. $\phi \in N(I - \hat{T}(\omega))$ if and only if $\phi(0) \in N(I - T(\omega))$ and $\phi(\theta) = T(n\omega + \theta)\phi(0)$, $-r \leq \theta \leq 0$, whenever $n\omega \geq r$; in particular, $\phi \in N(I - \hat{T}(\omega))$ is a restriction to $[-r, 0]$ of an ω -periodic function. Furthermore,

$$\dim N(I - \hat{T}(\omega)) = \dim N(I - T(\omega)).$$

Lemma 3.2. The functions $\phi, \psi \in \mathcal{C}$ satisfy the equation $(I - \hat{T}(\omega))\phi = \psi$ if and only if

- 1) $(I - T(\omega))\phi(0) = \psi(0)$.
- 2) $\phi(\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)\phi(0)$, $\theta \in I_k$, $k = 1, 2, \dots, p$, where $I_k = [-k\omega, -(k-1)\omega]$, $k = 1, 2, \dots, p-1$, $I_p = [-r, -(p-1)\omega]$ and p is the integer such that $(p-1)\omega < r \leq p\omega$.

Lemma 3.3. Suppose that $T \in \Phi_+(X)$ (; and hence, $\dim N(T) = n$ and $\|x\| \leq c|Tx|$ for $x \in X$ and for some $c > 0$). Then there exists a closed subspace Z of X such that $X = N(T) \oplus Z$, and $Q := T|_Z$, the restriction of T to Z , has a bounded inverse, and $\|Q^{-1}\| \leq (1 + \sqrt{n})c$.

Proof. Take a closed subspace Z such that $X = N \oplus Z$ and that $\|P_Z\| \leq \sqrt{n}$, where P_Z stands for the projection onto N along Z , cf. [6, B.4.9, pp. 29–30, 28.2.6, p. 386]. Take a $z \in Z$. It is represented by $z = (I - P_Z)x$ for some $x \in X$. Since $TP_Zx = 0$ and $|(I - P_Z)x| \leq (1 + \sqrt{n})|x|$, we have

$$|z| = |(I - P_Z)x| \leq c(1 + \sqrt{n})|Tx| = c(1 + \sqrt{n})|T(I - P_Z)x| = c(1 + \sqrt{n})|Tz|.$$

This implies the assertion, cf. [9, Proposition 7.3].

Let $I - T(\omega) \in \Phi_+(E)$, and put $N = N(I - T(\omega))$. Then there exists a closed subspace M of E satisfying the conditions in Lemma 3.3; and hence, $E = N \oplus M$. Let S_M be the restriction of $I - T(\omega)$ to M ; that is,

$$S_M := (I - T(\omega))|_M : M \rightarrow \mathcal{R}(I - T(\omega)).$$

From Lemma 3.3 we see that the inverse operator S_M^{-1} of S_M is continuous and

$$(5) \quad \|S_M^{-1}\| \leq c_0(1 + \sqrt{n}),$$

where $n = \dim N(I - T(\omega))$, and c_0 is a constant such that

$$(6) \quad |[x]| \leq c_0|(I - T(\omega))x|$$

for $x \in E$.

Put $\mathcal{D} = \{\psi \in \mathcal{C} : \psi(0) \in \mathcal{R}(I - T(\omega))\}$ and let $\psi \in \mathcal{D}$. Since $\mathcal{R}(S_M) = \mathcal{R}(I - T(\omega))$, $S_M^{-1}\psi(0)$ is well defined and $(I - T(\omega))S_M^{-1}\psi(0) = \psi(0)$. We define a function $(V\psi)(\cdot) : [-r, 0] \rightarrow E$ pointwise by

$$(7) \quad [V\psi](\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)S_M^{-1}\psi(0), \quad \theta \in I_k,$$

for $k = 1, 2, \dots, p$, and $[V\psi](0) = S_M^{-1}\psi(0)$. Notice that $\mathcal{D}(V) = \mathcal{D}$.

Lemma 3.4. *The operator V defined by (7) is a linear operator from $\mathcal{D}(V)$ to $\mathcal{C}$.*

Proof. It is sufficient to prove that $V\psi \in \mathcal{C}$ for $\psi \in \mathcal{D}(V)$. Clearly, $V\psi$ is continuous in each interval I_k . Thus we prove that $(V\psi)(-k\omega) = (V\psi)(-k\omega - 0)$ for $k = 1, 2, \dots, p - 1$. Notice that $S_M^{-1}\psi(0) = T(\omega)S_M^{-1}\psi(0) + \psi(0)$. From the definition (7) of V , we have,

$$\begin{aligned} (V\psi)(-k\omega) &= \sum_{j=0}^{k-1} \psi(-k\omega + j\omega) + T(0)S_M^{-1}\psi(0), \quad (-k\omega \in I_k) \\ &= \sum_{j=0}^{k-1} \psi(-k\omega + j\omega) + \psi(0) + T(\omega)S_M^{-1}\psi(0) \\ &= \sum_{j=0}^k \psi(-k\omega + j\omega) + T(-k\omega + (k+1)\omega)S_M^{-1}\psi(0) \\ &= \lim_{\theta \rightarrow -k\omega - 0} \left\{ \sum_{j=0}^k \psi(\theta + j\omega) + T(\theta + (k+1)\omega)S_M^{-1}\psi(0) \right\} \end{aligned}$$

as required.

Lemma 3.5.

$$\mathcal{R}(I - \hat{T}(\omega)) = \mathcal{D}(V).$$

Proof. Let $\psi \in \mathcal{R}(I - \hat{T}(\omega))$. Then there is a $\phi \in \mathcal{C}$ such that $[I - \hat{T}(\omega)]\phi = \psi$. From Lemma 3.2 we see that $(I - T(\omega))\phi(0) = \psi(0)$; and hence, $\psi(0) \in \mathcal{R}(I - T(\omega))$. This implies that $\psi \in \mathcal{D}(V)$.

Conversely, if $\psi \in \mathcal{D}(V)$, then $\psi(0) \in \mathcal{R}(I - T(\omega))$. Lemma 3.4 means that $V\psi \in \mathcal{C}$. Hence it follows from Lemma 3.2 that $[I - \hat{T}(\omega)]V\psi = \psi$. Therefore the proof is completed.

Proposition 3.6. $I - T(\omega) \in \Phi_+(E)$ if and only if $I - \hat{T}(\omega) \in \Phi_+(\mathcal{C})$.

Proof. If $\mathcal{R}(I - T(\omega))$ is closed in E , then $\mathcal{D}(V)$ is closed in $\mathcal{C}$. Indeed, if $\phi_n \in \mathcal{D}(V) \rightarrow \phi$ as $n \rightarrow \infty$, then $\phi_n(0) \rightarrow \phi(0)$ as $n \rightarrow \infty$. This implies that $\phi \in \mathcal{D}(V)$. From this fact and Lemma 3.5 it follows that $\mathcal{R}(I - \hat{T}(\omega))$ is closed in $\mathcal{C}$.

Conversely, we assume that $\mathcal{R}(I - \hat{T}(\omega))$ is closed in $\mathcal{C}$. Then it follows from Lemma 3.5 that $\mathcal{D}(V)$ is closed in $\mathcal{C}$. Let $x_n \in \mathcal{R}(I - T(\omega)) \rightarrow x$ as $n \rightarrow \infty$. Then there exist a sequence $\{\varphi_n\} \subset \mathcal{C}$ of constant functions and $\varphi \in \mathcal{C}$ such that $\varphi_n(0) = x_n$, $\varphi_n \in \mathcal{D}(V)$ and $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$ in $\mathcal{C}$. Hence we have $\varphi(0) = x$ and $\varphi \in \mathcal{D}(V)$. This implies that $x \in \mathcal{R}(I - T(\omega))$. The remainder follows from Lemma 3.1; and hence, the proof is complete.

Proposition 3.7. $1 \in \rho(T(\omega))$ if and only if $1 \in \rho(\hat{T}(\omega))$.

Proof. It is sufficient to prove that if $1 \in \rho(T(\omega))$, then $1 \in \rho(\hat{T}(\omega))$. Since $\mathcal{R}(I - T(\omega)) = E$, we have $\mathcal{D}(V) = \mathcal{C}$. Hence it follows from Lemma 3.5 that $\mathcal{R}(I - \hat{T}(\omega)) = \mathcal{D}(V) = \mathcal{C}$. Lemma 3.1 implies that $1 \in \rho(\hat{T}(\omega))$.

Lemma 3.8. Suppose that $I - T(\omega) \in \Phi_+(E)$, and take the quotient space $\mathcal{C}/N(I - \hat{T}(\omega))$. If there exists a positive constant c such that

$$(8) \quad |V\psi| \leq c|\psi| \quad \text{for all } \psi \in \mathcal{D}(V),$$

where V is defined by (7), then

$$\|\phi\| \leq c\|(I - \hat{T}(\omega))\phi\| \quad \text{for } \phi \in \mathcal{C}.$$

See [9, Theorem 6.5] for the proof.

From this lemma we can obtain the following result. Put $m_\omega := \sup\{\|T(t)\| : 0 \leq t \leq \omega\}$.

Theorem 3.9. *Assume that $I - T(\omega) \in \Phi_+(E)$ (; and hence, $\dim N(I - T(\omega)) = n < \infty$ and $\|x\| \leq c_0 \|(I - T(\omega))x\|$ for $x \in E$ and for a constant $c_0 > 0$). Then $I - \hat{T}(\omega) \in \Phi_+(\mathcal{C})$, $\dim N(I - \hat{T}(\omega)) = n$ and*

$$|V\psi| \leq (p + m_\omega c_0(1 + \sqrt{n}))|\psi| \quad \text{for } \psi \in \mathcal{D}(V),$$

as a result,

$$\|[\phi]\| \leq (p + m_\omega c_0(1 + \sqrt{n}))\|(I - \hat{T}(\omega))\phi\| \quad \text{for } \phi \in \mathcal{C},$$

where p is as in Lemma 3.2.

Proof. Take the closed subspace M such that the estimate (5) holds true. Suppose that $\psi \in \mathcal{D}(V)$. Then, for $\theta \in I_k$, $k = 1, 2, \dots, p$,

$$|V\psi(\theta)| \leq k|\psi| + m_\omega \|S_M^{-1}\| |\psi(0)| \leq (p + m_\omega \|S_M^{-1}\|)|\psi|.$$

This implies that

$$(9) \quad |V\psi| \leq (p + m_\omega \|S_M^{-1}\|)|\psi|.$$

Using this fact, the estimate (5) and Lemma 3.8 we have the conclusion of the theorem.

Denote by $\mathcal{S}_L(\omega)$ the set of all ω -peiodic solutions of Eq. $(P_\omega L)$. Using Theorem 3.9, Theorem 1.3 and Theorem 1.5 we can find the main result on the existence of periodic solutions.

Theorem 3.10. *Assume that $I - T(\omega) \in \Phi_+(E)$ (; and hence, $\dim N(I - T(\omega)) = n < \infty$ and $\|x\| \leq c_0 \|(I - T(\omega))x\|$ for $x \in E$ and for a constant $c_0 > 0$), and that Eq. $(P_\omega L)$ has a bounded solution. If*

$$\|K(\omega)\| \leq 1/2(1 + \sqrt{n})(p + m_\omega c_0(1 + \sqrt{n})),$$

then $\mathcal{S}_L(\omega) \neq \emptyset$ and $\dim \mathcal{S}_L(\omega) \leq n$.

4. Existence of periodic solutions: compact perturbations

In this section, we consider the existence of periodic solutions for Eq. $(P_\omega L)$ by using Theorem 1.3 and Theorem 1.4. In particular, in order to check the condition $I - \hat{T}(\omega) \in \Phi_+(\mathcal{C})$, we will compute the radius $r_e(\hat{T}(t))$ of the essential spectrum of $\hat{T}(t)$.

For a subset $D \subset E$ we denote by $T(\cdot)D$ the family of functions $T(t)x$ defined for $t \in [0, \infty)$ with a parameter $x \in D$, and by $T(\cdot)D| [a, b]$ its restriction to $[a, b]$. We note that $\alpha(\Omega(0)) \leq \alpha(\Omega)$ for a bounded set $\Omega \subset \mathcal{C}$, where $\Omega(0) = \{\phi(0) : \phi \in \Omega\}$. Put $\delta_T := \max\{1, \gamma_T\}$, where $\gamma_T = \limsup_{t \rightarrow 0} \|T(t)\|$.

Lemma 4.1.

1) Let $T(t)$ be a C_0 -semigroup on E . Then

$$\alpha(\hat{T}(t)) \leq \begin{cases} \sup_{0 \leq s \leq t} \|T(s)\|, & r \geq t \geq 0 \\ \sup_{t-r \leq s \leq t} \|T(s)\|, & t > r. \end{cases}$$

2) Let $T(t)$ be a C_0 -semigroup on E such that $T(t)x \in \mathcal{D}(A)$ for all $x \in E$ and $t > 0$. Then

$$\alpha(\hat{T}(t)) \leq \begin{cases} \max\{\delta_T, \sup_{0 < s \leq t} \alpha(T(s))\}, & r \geq t > 0 \\ \sup_{t-r \leq s \leq t} \alpha(T(s)), & t > r. \end{cases}$$

3) Let $T(t)$ be a C_0 -compact semigroup for $t > t_0$ on E .

(i) Case $t_0 \geq r$.

$$\alpha(\hat{T}(t)) \leq \begin{cases} \sup_{0 \leq s \leq t} \|T(s)\|, & r > t \geq 0 \\ \sup_{t-r \leq s \leq t} \|T(s)\| \delta_T, & t_0 \geq t \geq r \\ \sup_{t_0-r \leq s \leq t_0} \|T(s)\| \delta_T, & t_0 + r \geq t > r \\ 0, & t > t_0 + r. \end{cases}$$

(ii) Case $r > t_0$.

$$\alpha(\hat{T}(t)) \leq \begin{cases} \sup_{0 \leq s \leq t} \|T(s)\|, & t_0 \geq t \geq 0 \\ \sup_{0 \leq s \leq t_0} \|T(s)\| \delta_T, & r \geq t > t_0 \\ \sup_{t-r \leq s \leq t_0} \|T(s)\| \delta_T, & t_0 + r \geq t > r \\ 0, & t > t_0 + r. \end{cases}$$

Proof. For a bounded set $\Omega \subset \mathcal{C}$ and for $t \geq 0$, we have

$$\alpha(\hat{T}(t)\Omega) \leq \begin{cases} \max\{\alpha(T(\cdot)\Omega(0)| [0, t]), \alpha(\Omega| [t-r, 0])\}, & r \geq t > 0 \\ \alpha(T(\cdot)\Omega(0)| [t-r, t]), & t > r. \end{cases}$$

From this we can easily obtain the conclusion of the lemma. For example, we here will give the proof of the relation

$$\alpha(\hat{T}(t)) \leq \sup_{0 \leq s \leq t_0} \|T(s)\| \delta_T \quad \text{for } t \in (t_0, r]$$

in the assertion 3) (ii). For any $\varepsilon > 0$, $t - t_0 > \varepsilon > 0$ we have

$$\begin{aligned}
& \alpha(\hat{T}(t)\Omega) \\
& \leq \max\{\alpha(\Omega \mid [t-r, 0]), \alpha(T(\cdot)\Omega(0) \mid [0, t_0 + \varepsilon]), \alpha(T(\cdot)\Omega(0) \mid [t_0 + \varepsilon, t])\} \\
& \leq \max\{\alpha(\Omega), \alpha(T(\cdot)\Omega(0) \mid [0, t_0 + \varepsilon])\} \\
& \leq \max\left\{1, \sup_{0 \leq s \leq t_0 + \varepsilon} \|T(s)\|\right\} \alpha(\Omega) \\
& \leq \max\left\{\sup_{0 \leq s \leq t_0} \|T(s)\|, \sup_{t_0 \leq s \leq t_0 + \varepsilon} \|T(s)\|\right\} \alpha(\Omega) \\
& \leq \max\left\{\sup_{0 \leq s \leq t_0} \|T(s)\|, \|T(t_0)\| \sup_{0 \leq s \leq \varepsilon} \|T(s)\|\right\} \alpha(\Omega) \\
& \leq \max\left\{\sup_{0 \leq s \leq t_0} \|T(s)\|, \|T(t_0)\| \delta_T\right\} \alpha(\Omega) \\
& \leq \sup_{0 \leq s \leq t_0} \|T(s)\| \delta_T \alpha(\Omega).
\end{aligned}$$

This implies the described inequality, cf. [9, Lemma 4.2, Lemma 4.3].

Corollary 4.2. *If $T(t)$ is a C_0 -compact semigroup for $t > t_0$ on E , then $\hat{T}(t)$ is a C_0 -compact semigroup for $t > t_0 + r$ on $\mathcal{C}$.*

Now, we will find sufficient conditions that the radius of the essential spectrum of the solution operator $U(t, 0)$ of Eq. (PL₀) is less than 1. Suppose that a function $g : [0, \infty) \rightarrow [0, \infty)$ is locally bounded, and submultiplicative (that is, $g(t+s) \leq g(t)g(s)$ for $t, s \geq 0$). Then it is well known that $\lim_{t \rightarrow \infty} t^{-1} \log g(t) = \inf_{t > 0} t^{-1} \log g(t)$, which may be $-\infty$, but not be $+\infty$. This quantity is called the type number of the function $g(t)$. We denote respectively by $\tau, \tau^\nu, \hat{\tau}, \hat{\tau}^\nu$ the type numbers of the functions $\alpha(T(t)), \|T(t)\|, \alpha(\hat{T}(t)), \|\hat{T}(t)\|$, provided that $T(t)$ is a C_0 -semigroup on E . Then $\tau \leq \tau^\nu$, and $\hat{\tau} \leq \hat{\tau}^\nu$.

Notice that

$$r_e(\hat{T}(t)) = e^{\hat{\tau}t}, \quad t > 0,$$

by using the Nussbaum formula. In particular, if $T(t)$ is a C_0 -compact semigroup on E , or if $L(t, \cdot)$ is a compact operator for each $t \in \mathbb{R}$, then the radius of the essential spectrum of the solution operator $U(t, 0)$ of Eq. (L₀) becomes $r_e(U(t, 0)) = r_e(\hat{T}(t)) = \exp(\hat{\tau}t)$, $t > 0$. Thus, if $\hat{\tau} < 0$, then $I - \hat{T}(\omega) \in \Phi_+(\mathcal{C})$. Now we give several conditions that $\hat{\tau}$ is negative.

Lemma 4.3. *Let $\|T(t)\| \leq M_w e^{-wt}$, $w > 0$, for $t \geq 0$. Then*

$$(10) \quad \|\hat{T}(t)\| \leq \begin{cases} M_w e^{-w(t-r)} & \text{for } t > r \\ M_w & \text{for } t \leq r. \end{cases}$$

Theorem 4.4. 1) *Let $T(t)$ be a C_0 -semigroup on E . If τ^v is negative, then $\hat{\tau}^v < 0$.*

2) *Let $T(t)$ be a C_0 -semigroup on E such that $T(t)x \in \mathcal{D}(A)$ for all $x \in E$ and $t > 0$. If $\tau < 0$, then $\hat{\tau} < 0$.*

3) *If $T(t)$ is a C_0 -compact semigroup on E or a C_0 -compact semigroup for $t > t_0$ on E , then $\hat{\tau} < 0$.*

Proof. It is sufficient to prove only the assertion 1). Since τ^v is negative, there is a μ such that $-\tau^v > \mu > 0$ and $\|T(t)\| \leq M_\mu e^{-\mu t}$ for all $t \geq 0$. Using the estimate (10), we see that

$$\begin{aligned} \hat{\tau}^v &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\hat{T}(t)\| \\ &\leq \lim_{t \rightarrow \infty} \frac{1}{t} \log M_\mu e^{-\mu(t-r)} \\ &= -\mu < 0, \end{aligned}$$

from which we have the assertion 1). The remainder follows from Lemma 4.1, cf. [9, Theorem 4.8].

We are now in a position to give a criterion for the existence of periodic solution of Eq. $(P_\omega L)$.

Theorem 4.5. *Assume that at least one of the following conditions is satisfied:*

- 1) *$T(t)$ is a C_0 -compact semigroup on E .*
- 2) *$L(t, \cdot)$ is a compact operator for each $t \in \mathbb{R}$ and $\tau^v < 0$.*
- 3) *$T(t)$ is a C_0 -compact semigroup for $t > t_0$ on E and $L(t, \cdot)$ is a compact operator for each $t \in \mathbb{R}$.*
- 4) *$T(t)$ is a C_0 -semigroup on E such that $T(t)x \in \mathcal{D}(A)$ for all $x \in E$ and $t > 0$, $L(t, \cdot)$ is a compact operator for each $t \in \mathbb{R}$, and $\tau < 0$.*

Then 1 is a normal point of $U(\omega, 0)$; and hence, the following results hold true:

- i) *in the case where $1 \in \rho(U(\omega, 0))$, Eq. $(P_\omega L)$ has a unique ω -periodic solution.*
- ii) *in the case where 1 is a normal eigenvalue of $U(\omega, 0)$, if Eq. $(P_\omega L)$ has an bounded solution, then $\mathcal{S}_L(\omega) \neq \emptyset$ and $\dim \mathcal{S}_L(\omega)$ is finite.*

Proof. From Theorem 4.4, Theorem 1.2 and Theorem 1.3 we can easily obtain the conclusion of the theorem.

5. The uniqueness of periodic solutions (I)

We consider criteria for the uniqueness of ω -periodic solutions to Eq. $(P_\omega L)$. First, we state the general results. From Proposition 1.6 we have the following result.

Theorem 5.1. *Assume that $I - \hat{T}(\omega) \in \Phi_+(\mathcal{C})$ and $\dim N(I - \hat{T}(\omega)) = 0$ (; and hence, $|\phi| \leq c|(I - \hat{T}(\omega))\phi|$ for all $\phi \in \mathcal{C}$ and for some $c > 0$), and that the inequality $\|K(\omega)\| \leq 1/2c$ holds. If Eq. $(P_\omega L)$ has a bounded solution, then Eq. $(P_\omega L)$ has a unique ω -periodic solution.*

The following result is an immediate consequence of Theorem 3.10.

Theorem 5.2. *Assume that $I - T(\omega) \in \Phi_+(E)$ and $\dim N(I - T(\omega)) = 0$ (; and hence, $|x| \leq c_0|(I - T(\omega))x|$ for all $x \in E$ and for some $c_0 > 0$), and that the inequality $\|K(\omega)\| \leq 1/2(p + m_\omega c_0)$ holds. If Eq. $(P_\omega L)$ has a bounded solution, then Eq. $(P_\omega L)$ has a unique ω -periodic solution.*

Next, we consider the uniqueness of ω -periodic solutions to Eq. $(P_\omega L)$ for the special case where the type number τ^ν of the function $\|T(t)\|$ is negative. Then there exists a positive number $w > 0$ such that $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$. Hence $1 \in \rho(T(\omega))$, and so, $1 \in \rho(\hat{T}(\omega))$, because of Proposition 3.7. For each $\phi \in \mathcal{C}$ take the space $BC(\phi)$, the set of bounded, continuous functions $x : [0, \infty) \rightarrow E$ such that $x(0) = \phi(0)$. This is a complete metric space with the metric $d(x, y) = \|x - y\|_\infty := \sup\{|x(t) - y(t)| : t \geq 0\}$. Define an operator F_ϕ on $BC(\phi)$ by

$$(F_\phi x)(t) = T(t)\phi(0) + \int_0^t T(t-s)(L(s, \tilde{x}_s) + f(s))ds, \quad t \geq 0,$$

where $\tilde{x}(t) = \phi(t)$ for $-r \leq t \leq 0$ and $\tilde{x}(t) = x(t)$ for $t \geq 0$. Set

$$\|f\|_\infty = \sup\{|f(t)| : t \geq 0\}.$$

Then we have $\|F_\phi x\|_\infty \leq M_w |\phi(0)| + M_w w^{-1} [\|L\|_\infty \max\{\|x\|_\infty, |\phi|\} + \|f\|_\infty]$. Using this relation we have the following result, cf. [9, Proposition 8.1, Theorem 8.2].

Lemma 5.3. *Suppose that $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$. If $M_w \|L\|_\infty < w$, then every solution of Eq. $(P_\omega L)$ is bounded, and every ω -periodic solution $v(t)$ of Eq. $(P_\omega L)$ is estimated as*

$$(11) \quad \|v\|_\infty \leq \frac{M_w}{w - M_w \|L\|_\infty} \|f\|_\infty.$$

Theorem 5.4. *Suppose that $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$ (; and so, $|\phi| \leq c|(I - \hat{T}(\omega))\phi|$ for all $\phi \in \mathcal{C}$ and for some $c > 0$). If*

$$2ck(\omega) < 1 \quad \text{and} \quad w > M_w \|L\|_\infty,$$

then Eq. (P_ωL) has a unique ω -periodic solution, where $k(t)$ is as in Proposition 2.2.

Proof. From the assumption on $\|T(t)\|$ we have that $1 \in \rho(\hat{T}(\omega))$. On the other hand, using Proposition 2.2 and the assumption in this theorem, we have

$$\|K(\omega)\| \leq k(\omega) < \frac{1}{2c}.$$

Therefore, by combining Theorem 5.1 and Lemma 5.3 we can obtain that Eq. (P_ωL) has a unique ω -periodic solution. This proves the theorem.

Proposition 5.5. *Let $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$ and let p be a nonnegative integer such that $(p-1)\omega < r \leq p\omega$. Assume that $w\omega > \log M_w$. If*

$$(12) \quad \frac{2(p + M_w)k(\omega)}{(1 - M_w e^{-w\omega})} < 1 \quad \text{and} \quad w > M_w \|L\|_\infty,$$

then Eq. (P_ωL) has a unique ω -periodic solution.

Proof. Now we will compute the value of the constant c in Theorem 5.4. Since $w\omega > \log M_w$, we have $\|T(\omega)\| \leq M_w e^{-w\omega} < 1$, and hence,

$$(13) \quad \|S_M^{-1}\| = \|(I - T(\omega))^{-1}\| \leq 1/(1 - \|T(\omega)\|) \leq 1/(1 - M_w e^{-w\omega}).$$

Combining the relation (9) and (13) we get

$$\begin{aligned} c &\leq p + m_\omega \|S_M^{-1}\| \\ &\leq p + M_w / (1 - M_w e^{-w\omega}) \\ &\leq (p + M_w) / (1 - M_w e^{-w\omega}). \end{aligned}$$

Therefore it follows from the assumption that all conditions of Theorem 5.4 are satisfied, and the proof is complete.

Notice that the above result is also derived from Theorem 5.2.

Remark. From the above proof we can obtain the following condition

$$\frac{2(p - pM_w e^{-w\omega} + M_w)k(\omega)}{(1 - M_w e^{-w\omega})} < 1 \quad \text{and} \quad w > M_w \|L\|_\infty,$$

which is weaker than the condition (12). However, we do not use this condition, because it is sufficient to consider only the condition (12) in our discussion.

Remark. Let $\|T(t)\| \leq e^{-wt}$, $w > 0$, in the Proposition 5.5. Then (12) becomes

$$\frac{2(p+1)\|L\|_{\infty}}{w - \|L\|_{\infty}} < 1 \quad \text{and} \quad w > \|L\|_{\infty}.$$

6. The uniqueness of periodic solutions (II)

In this section we consider criteria for the uniqueness of periodic solutions to Eq. $(P_{\omega}L)$ by using Proposition 1.7.

Theorem 6.1. *Assume that $1 \in \rho(T(\omega))$. If*

$$(14) \quad 1 \in \rho((I - \hat{T}(\omega))^{-1}K(\omega)),$$

then Eq. $(P_{\omega}L)$ has a unique ω -periodic solution.

Proof. From Proposition 3.7 we have that if $1 \in \rho(T(\omega))$, then $1 \in \rho(\hat{T}(\omega))$. For the solution operator $U(t, 0)$ of Eq. $(P_{\omega}L_0)$, the periodic map $U(\omega, 0)$ is decomposed as $U(\omega, 0) = \hat{T}(\omega) + K(\omega)$. We here note that $1 \in \rho(U(\omega, 0))$ if and only if $1 \in \rho((I - \hat{T}(\omega))^{-1}K(\omega))$. Therefore, applying Proposition 1.7 to our situation, we can obtain the proof of the theorem.

We now get a sufficient condition for the condition (14) in the above theorem.

Lemma 6.2. *Assume that $\|\hat{T}(\omega)\| < 1$. If*

$$(15) \quad \|K(\omega)\| < 1 - \|\hat{T}(\omega)\|,$$

then the condition (14) holds.

Proof. From the condition (15) we have

$$\begin{aligned} \|((I - \hat{T}(\omega))^{-1}K(\omega))\| &\leq \|((I - \hat{T}(\omega))^{-1})\| \|K(\omega)\| \\ &\leq \frac{\|K(\omega)\|}{1 - \|\hat{T}(\omega)\|} < 1. \end{aligned}$$

This fact implies the condition (14).

Lemma 6.3. *Let $\|T(t)\| \leq M_w e^{-wt}$, $w > 0$. Assume that $\omega > r$ and $w(\omega - r) > \log M_w$. Then the inequality (15) in Lemma 6.2 can be replaced by the following inequality:*

$$(16) \quad \frac{\|K(\omega)\|}{1 - M_w e^{-w(\omega-r)}} < 1.$$

Proof. From assumptions and the estimate (10) of $\hat{T}(t)$ we have $\|\hat{T}(\omega)\| \leq M_w e^{-w(\omega-r)} < 1$. Hence, the inequality (15) follows from the inequality (16). This proves the lemma.

Combining Lemma 6.3 with Proposition 2.2, we can obtain the following result.

Theorem 6.4. *Suppose that $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$ and that $\omega > r$ and $w(\omega - r) > \log M_w$. If*

$$\frac{k(\omega)}{(1 - M_w e^{-w(\omega-r)})} < 1 \quad \text{and} \quad w > M_w \|L\|_\infty,$$

then Eq. (P_ωL) has a unique ω -periodic solution.

In general, Proposition 5.5 and Theorem 6.4 are independent of each other, which will be shown in Remark of the next section. Hence, summarizing those results, we have the following result.

Theorem 6.5. *Suppose that $\|T(t)\| \leq M_w e^{-wt}$ for $t \geq 0$ and that $\omega > r$, $w > M_w \|L\|_\infty e^{wr}$ and $w(\omega - r) > \log M_w$. If*

$$(17) \quad \min \left\{ \frac{2(p + M_w)}{1 - M_w e^{-w\omega}}, \frac{1}{1 - M_w e^{-w(\omega-r)}} \right\} k(\omega) < 1,$$

then Eq. (P_ωL) has a unique ω -periodic solution.

Remark. We note that Theorem 5.4, Proposition 5.5, Theorem 6.4 can be expressed by using the estimate 1) in place of the estimate 2) of $K(\omega)$ in Proposition 2.2. To avoid the complexity of the statement of the paper, we omitted them.

7. An example

In this section, we shall see, by means of a simple example, how the results of Proposition 5.5 and Theorem 6.4 can be used to prove the unique existence of a periodic solution of a partial differential-integral equation.

Denote by $E = C[-\infty, \infty]$, the space of all continuous real valued functions $u(x)$, defined on $(-\infty, \infty)$, satisfying the condition that $\lim_{x \rightarrow -\infty} u(x)$ and

$\lim_{x \rightarrow +\infty} u(x)$ exist, and take its norm as $\|u\| = \sup_{-\infty < x < +\infty} |u(x)|$. Then E is a Banach space.

We consider the initial value problem for the equation of the form

$$(18) \quad \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} - \alpha u(t, x) + b(t, x) \int_{t-r}^t e^{-c(t-s)} u(s, x) ds + f(t, x),$$

$$u(\theta, x) = \phi(\theta, x), \quad -r \leq \theta \leq 0, \quad \phi \in \mathcal{C}.$$

It is well known that the linear operators A and A_0 , defined by

$$Au = \frac{d^2 u}{dx^2} - \alpha u \quad \text{for } u \in \mathcal{D}(A), \quad A_0 u = \frac{d^2 u}{dx^2} \quad \text{for } u \in \mathcal{D}(A_0)$$

and

$$\mathcal{D}(A) = \mathcal{D}(A_0) = \left\{ u \in E : \frac{d}{dx} u, \frac{d^2}{dx^2} u \in E \right\},$$

are infinitesimal generators of C_0 -semigroups $T(t)$ and $T_0(t)$ on E , respectively, cf. [2, Chapter VIII]. Hence,

$$\|T_0(t)\| = 1, \quad T(t) = e^{-\alpha t} T_0(t), \quad \text{and} \quad \|T(t)\| = e^{-\alpha t}$$

for all $t \geq 0$.

Assume that

(C-1) $\alpha > 0$ and $c > 0$.

(C-2) $b(t, x)$ and $f(t, x) : R \times R \rightarrow R$ are continuous and ω -periodic functions in t such that $b(t, \cdot), f(t, \cdot) \in E, t \in R$.

Put $\|b(t)\| = \sup_{-\infty < x < \infty} |b(t, x)|$, $\|b\|_\infty = \sup_{0 \leq t \leq \omega} \|b(t)\|$. Similarly, we define $\|f(t)\|$ and $\|f\|_\infty$ for $f(t, x)$. Set

$$B(t, \phi)(x) = b(t, x) \int_{-r}^0 e^{c\theta} \phi(\theta, x) d\theta, \quad \phi \in \mathcal{C}.$$

Then we have

$$|B(t, \phi)(x)| \leq \|b(t)\| \int_{-r}^0 e^{c\theta} |\phi| d\theta \leq \frac{\|b(t)\|}{c} |\phi|$$

and hence, $\|B\|_\infty \leq \|b\|_\infty / c$. Therefore from Proposition 5.5 and the estimate (11) we have the following result.

Theorem 7.1. *Assume that the conditions (C-1) and (C-2) are satisfied. Let p be a positive integer such that $(p-1)\omega < r \leq p\omega$. If*

$$(19) \quad \frac{2(p+1)\|b\|_\infty}{c\alpha - \|b\|_\infty} < 1 \quad \text{and} \quad c\alpha > \|b\|_\infty,$$

then Eq. (18) has a unique ω -periodic solution v , and

$$(20) \quad \|v\|_{\infty} \leq \frac{c}{c\alpha - \|b\|_{\infty}} \|f\|_{\infty}.$$

The following theorem follows immediately from Theorem 6.4.

Theorem 7.2. *Assume that $\omega > r$ and the conditions (C-1) and (C-2) are satisfied. If*

$$(21) \quad \frac{\|b\|_{\infty}(1 - e^{-\alpha\omega})}{(1 - e^{-\alpha(\omega-r)})(c\alpha - \|b\|_{\infty})} < 1 \quad \text{and} \quad c\alpha > \|b\|_{\infty},$$

then Eq. (18) has a unique ω -periodic solution v with the estimate (20).

Remark. For simplicity, let $\omega > r$, $c = 1$ and $\alpha = 1 > \|b\|_{\infty}$. Then we compare the condition (19) in Theorem 7.1 with the condition (21) in Theorem 7.2. Since $\omega > r$, we have $p = 1$. Hence, using the condition (17) in Theorem 6.5 we can obtain the following fact: if

$$\|b\|_{\infty} < \max \left\{ \frac{1}{4 + 1}, \frac{1}{\frac{1 - e^{-\omega}}{1 - e^{-(\omega-r)}} + 1} \right\},$$

then Eq. (18) has a unique ω -periodic solution. However, we see that

1) there is a periodic ω close to the delay r such that

$$\frac{1 - e^{-\omega}}{1 - e^{-(\omega-r)}} > 4;$$

2) there is a sufficiently large periodic ω such that

$$\frac{1 - e^{-\omega}}{1 - e^{-(\omega-r)}} < 4.$$

Those facts show that the condition (19) in Theorem 7.1 and the condition (21) in Theorem 7.2 are independent of each other.

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nuna adreso:

Jong Son Shin

Department of Mathematics

Korea University

Ogawa, Kodaira, Tokyo 187-8560

Japan

E-mail: shinjs@tech.korea-u.ac.jp

Toshiki Naito

The University of Electro-Communications

Chofu, Tokyo 182-8585

Japan

E-mail: naito@e-one.uec.ac.jp

Nguyen Van Minh

Department of Mathematics

University of Hanoi

90 Nguyen Trai, Hanoi

Vietnam

E-mail: nvminh@netnam.ozg.vn

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