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Large Time Asymptotics of Small Solutions to Generalized Kolmogorov-Petrovskii-Piskunov Equation

By

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1. Introduction

It is well known that the classical Kolmogorov-Petrovskii-Piskunov (KPP) equation describes such phenomenas as combustion (physics), evolution of dominant genes and propagation of nerve pulses (biology), propagation of concentration waves (chemical kinetics), evolution of the set of Duffing oscillators (plasma) and many others (see Kolmogorov et al. [4], Vasilev et al. [9]).

In this paper we shall study the Cauchy problem for the generalized KPP equation

$$(1.1) \quad \begin{cases} u_t + au^3 + \mathbf{K}(u) = 0, & x \in \mathbf{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

Here $u(x, t)$ is a real valued function, a is a real number and $\mathbf{K}$ is a linear pseudo-differential operator defined by

$$\mathbf{K}(\varphi) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{ip \cdot x} K(p) \hat{\varphi}(p) dp,$$

where

$$\hat{\varphi}(p) = \int_{\mathbf{R}^N} e^{-ip \cdot x} \varphi(x) dx$$

is the Fourier transform of the function $\varphi(x)$ and $K(p)$ is the symbol of the operator $\mathbf{K}$.

The aim of this paper is to establish the asymptotic behavior as $t \rightarrow \infty$ of small solutions to the Cauchy problem (1.1).

2. Statement of the result

In order to state the assumptions on $K(p)$ and $u_0(x)$, we introduce the following notation.

$$m(p) = \min\{1, |p|\}, \quad M(p) = \max\{1, |p|\}.$$

Assumptions on $K(p)$

$$(A1) \quad K(p) = \lambda|p|^\delta + O(|p|^\beta) \quad \text{for } p \in \mathbf{R}^N, \quad |p| \leq 1,$$

where $0 < \delta < 2N$, $\beta > \delta$ and $\lambda > 0$.

$$(A2) \quad \operatorname{Re} K(p) \geq 2\kappa(p) \quad \text{for } p \in \mathbf{R}^N,$$

where $\kappa(p) = c_1 m(p)^\delta$ with $c_1 > 0$.

$$(A3) \quad |K(p_1) - K(p_2)| \leq c_2 |p_1 - p_2|^{\sigma_1} M(p_2)^\alpha \quad \text{for}$$

$$p_1, p_2 \in \mathbf{R}^N, \quad |p_1 - p_2| \leq 1,$$

where $c_2 > 0$, $0 < \sigma_1 \leq 1$ and $\alpha \geq 0$.

Assumptions on $u_0(x)$

$$(A4) \quad u_0(x) \in L^1(\mathbf{R}^N).$$

$$(A5) \quad |\hat{u}_0(p)| \leq \varepsilon M(p)^{-s-\alpha} \quad \text{for } p \in \mathbf{R}^N,$$

$$(A6) \quad |\hat{u}_0(p_1) - \hat{u}_0(p_2)| \leq \varepsilon |p_1 - p_2|^{\sigma_2} M(p_2)^{-s} \quad \text{for}$$

$$p_1, p_2 \in \mathbf{R}^N, \quad |p_1 - p_2| \leq 1,$$

where $0 < \sigma_2 \leq 1$, $s > N$ and ε is a small positive constant.

Under these assumptions we shall prove the following theorem.

Theorem. *Let $K(p)$ and $u_0(x)$ satisfy the above (A1) ~ (A6). Then for the solution $u(x, t)$ of the Cauchy problem (1.1) the following asymptotics is valid for $t \rightarrow \infty$ uniformly with respect to $y = x/t^{1/\delta} \in \mathbf{R}^N$:*

$$u(x, t) = A_0 (2\pi)^{-N} t^{-N/\delta} \int_{\mathbf{R}^N} \cos(y \cdot \eta) e^{-\lambda|\eta|^\delta} d\eta + O(t^{-N/\delta-\theta})$$

for some $\theta > 0$, where

$$A_0 = \lim_{t \rightarrow \infty} \int_{\mathbf{R}^N} u(x, t) dx.$$

In this paper, we do not discuss exactly the existence and uniqueness of classical solutions of the Cauchy problem (1.1). If we assume that the operator $\mathbf{K}$ has a finite order, i.e., there exist numbers $\alpha_0 \geq 0$ and $c_0 \geq 0$ such that

$$|K(p)| \leq c_0 (1 + |p|)^{\alpha_0}, \quad p \in \mathbf{R}^N,$$

and if we choose $s > N + \alpha_0$ in (A5) and (A6), then there exists by any means a unique classical solution. It follows at once from Proposition 3.1 below, and

fully justifies all our calculations. Indeed, the above assumptions guarantee the necessary smoothness of classical solution and, after that, the uniqueness follows from standard methods like energetic estimates or contraction mapping principle (see e.g., Naumkin-Shishmarev [7]).

Remark 1. A_0 can be expressed explicitly in terms of the symbol $K(p)$ and the initial data $u_0(x)$. See Naumkin-Shishmarev [6].

Remark 2. In the previous paper [8] by Shishmarev was studied the asymptotic behavior as $t \rightarrow \infty$ of small solutions to the Cauchy problem

$$(2.1) \quad \begin{cases} u_t + \sum_{k=4}^n a_k u^k + \mathbf{K}(u) = 0, & x \in \mathbf{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where $n \geq 4$ is an integer and a_k are arbitrary real numbers. In this case δ in (A1) is required to satisfy $\delta \in (0, 3N)$. We combine this result and Theorem 1. Then the same asymptotic representation as in Theorem 1 can be obtained for the Cauchy problem

$$(2.2) \quad \begin{cases} u_t + \sum_{k=3}^n a_k u^k + \mathbf{K}(u) = 0, & x \in \mathbf{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N \end{cases}$$

if δ is more restricted to satisfy $\delta \in (0, 2N)$.

Remark 3. A simple example of our problem (1.1) is the Cauchy problem

$$(2.3) \quad \begin{cases} u_t + a|u|^{\ell-1}u - \lambda \Delta u = 0, & x \in \mathbf{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N. \end{cases}$$

with $\ell = 3$ and $N \geq 2$. In this case $k(p) = \lambda|p|^2$, and the above theorem shows that for small L^1 -initial data $u_0(x)$, the solution behaves like

$$u(x, t) = A_0(4\pi\lambda t)^{-N/2} e^{-|x|^2/4\lambda t} + O(t^{-N/2-\theta})$$

as $t \rightarrow \infty$. The restriction $N \geq 2$ comes from the condition $\delta = 2 < 2N$. A similar conclusion has been obtained by Kawanago [2] for (2.3) with more general exponent $\ell > 1 + 2/N$ (as for the positive solutions, see also Kamin-Peletier [1] and Mochizuki [5]). However, his methods depends strongly on the differential equation (2.3), and it is difficult to apply them directly to our problem (1.1).

Remark 4. If we consider the Cauchy problem for the equation

$$u_t + u^3 + \Delta^2 u = 0,$$

the asymptotics of small solutions is given for $N \geq 3$ by Theorem with $\delta = 4$. On the other hand, if we consider the equation

$$u_t + u^3 - \Delta u + \Delta^2 u = 0,$$

the asymptotics is given for $N \geq 2$ by Theorem with $\delta = 2$. A more interesting example is the Cauchy problem for the nonlocal reaction-diffusion equation in open systems

$$u_t + u^3 - \Delta u + \beta \int_{\mathbf{R}^N} \frac{\Delta u(y, t) dy}{(1 + |x - y|^2)^{\alpha/2}} = 0$$

with $\alpha > N$ and $\beta \in \mathbf{R}$, see e.g., Klimontovich [3]. We have $(\Delta u)^\wedge = -|p|^2 \hat{u}(p, t)$ and

$$[(1 + |x|^2)^{-\alpha/2}]^\wedge = \frac{(2\pi)^{N/2}}{2^{\alpha/2-1} \Gamma(\alpha/2)} \frac{K_{(N-\alpha)/2}(|p|)}{|p|^{(N-\alpha)/2}},$$

where $K_\nu(|p|)$ is the Macdonald function of order ν . So, if $N \geq 2$, assumptions (A.1) $\sim$ (A.3) are fulfilled with $\delta = 2$, and the asymptotics of small solutions is the same as in the case of (2.3) with $\ell = 3$.

The proof of our theorem will be based on the integral equation

$$(2.4) \quad \hat{u}(p, t) = \hat{u}_0(p) e^{-K(p)t} - a \int_0^t e^{-K(p)(t-\tau)} d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \hat{u}(p-q, \tau) \hat{u}(q-r, \tau) \hat{u}(r, \tau) dq dr,$$

which comes from the Fourier transformation of (1.1). We shall apply the perturbation methods to this equation, that is, we denote

$$(2.5) \quad \hat{u}_0(p) = \varepsilon v(p),$$

and represent the solution $\hat{u}(p, t)$ in the form of the power series in parameter ε :

$$(2.6) \quad \hat{u}(p, t) = \sum_{n=0}^{\infty} \varepsilon^{2n+1} v_n(p, t).$$

Substitute (2.6) into (2.4). Then comparing the coefficients at the same powers of ε , we obtain

$$v_0(p, t) = v(p) e^{-K(p)t},$$

$$(2.7) \quad v_n(p, t) = -a \int_0^t e^{-K(p)(t-\tau)} d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{k=1}^n \sum_{j=1}^k v_{j-1}(p-q, \tau) \\ \times v_{k-j}(q-r, \tau) v_{n-k}(r, \tau) dq dr \quad (n \geq 1).$$

We shall show that $v_n(p, t)$ can be determined successively in a suitable L^1 -class in p of functions (Proposition 3.1). Then for sufficiently small ε , (2.6) converges uniformly in $(p, t) \in \mathbf{R}^N \times [0, \infty)$, and as the Fourier inverse transform of $\hat{u}(p, t)$, the solution $u(x, t)$ of (1.1) is obtained in the form

$$(2.8) \quad u(x, t) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{ix \cdot p} \hat{u}(p, t) dp = \sum_{n=0}^{\infty} \varepsilon^{2n+1} \psi_n(x, t),$$

where

$$(2.9) \quad \psi_n(x, t) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{ix \cdot p} v_n(p, t) dp.$$

Thus, our problem reduces to the study of the asymptotics of each function $\psi_n(x, t)$ as $t \rightarrow \infty$ (Proposition 4.1).

3. Estimates for functions $v_n(p, t)$

In this section we shall prepare several propositions which estimate the behavior of function $v_n(p, t)$.

Proposition 3.1. *For given any $\gamma > 1 + \delta$, there exists a constant $A_1 = A_1(\gamma) \geq 1$ such that*

$$|v_n(p, t)| \leq A_1^n (n+1)^{-\gamma} M(p)^{-s-\alpha} e^{-\kappa_n(p)t}, \quad (p, t) \in \mathbf{R}^N \times [0, \infty)$$

for any $n \geq 0$, where $\kappa_n(p) = \kappa(p/(2n+1))$.

Proof. It follows from (A2) and (A5) that

$$(3.1) \quad |v_0(p, t)| = |\varepsilon^{-1} \hat{u}_0(p) e^{-K(p)t}| \leq M(p)^{-s-\alpha} e^{-2\kappa(p)t}.$$

This proves the assertion for $n = 0$.

For general $n \geq 1$, the proof will be done by induction. Assume that the assertion is valid for all $v_\ell(p, t)$, $\ell = 1, 2, \dots, n-1$. Then we have for $v_n(p, t)$,

$$(3.2) \quad |v_n(p, t)| \leq |a| A_1^{n-1} \sum_{k=1}^n \sum_{j=1}^k j^{-\gamma} (k-j+1)^{-\gamma} (n-k+1)^{-\gamma} \\ \times \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(p-q)^{-s_1} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \\ \times \int_0^t e^{-2\kappa(p)t + L_n(p, q, r)\tau} d\tau,$$

where $s_1 = s + \alpha$ and

$$L_n(p, q, r) = 2\kappa(p) - \{\kappa_{j-1}(p - q) + \kappa_{k-j}(q - r) + \kappa_{n-k}(r)\}.$$

Lemma 3.1. *Let $\gamma, s_1 > 0$. Then*

$$(3.3) \quad \begin{aligned} j^{-\gamma}(k - j + 1)^{-\gamma}(n - k + 1)^{-\gamma} \\ \leq 3^\gamma(n + 2)^{-\gamma}\{j^{-\gamma}(k - j + 1)^{-\gamma} \\ + (k - j + 1)^{-\gamma}(n - k + 1)^{-\gamma} + j^{-\gamma}(n - k + 1)^{-\gamma}\}, \end{aligned}$$

$$(3.4) \quad \begin{aligned} M(p - q)^{-s_1} M(q - r)^{-s_1} M(r)^{-s_1} \\ \leq 3^{s_1} M(p)^{-s_1} \{M(p - q)^{-s_1} M(q - r)^{-s_1} \\ + M(q - r)^{-s_1} M(r)^{-s_1} + M(p - q)^{-s_1} M(r)^{-s_1}\}. \end{aligned}$$

Proof. For $h > 0$ put $F(r) = r^h$, $r > 0$. Then for any $b_i \geq 0$ ($i = 1, 2, 3$) such that $b_1 + b_2 + b_3 = 1$, we have

$$(3.5) \quad F(b_1 r + b_2 s + b_3 t) \leq F(r) + F(s) + F(t).$$

This inequality is rewritten as

$$\frac{F(b_1 r + b_2 s + b_3 t)}{F(r)F(s)F(t)} \leq \frac{1}{F(r)F(s)} + \frac{1}{F(s)F(t)} + \frac{1}{F(r)F(t)},$$

and the assertions of the lemma easily follow from this inequality.

Lemma 3.2. *We have*

$$\int_0^t e^{-2\kappa(p)t + L_n(p, q, r)\tau} d\tau \leq \frac{e^{-\kappa_n(p)t}}{\max\{\kappa(p), |L_n(p, q, r)|\}}.$$

Proof. Since

$$\kappa_{j-1}(p - q) + \kappa_{k-j}(q - r) + \kappa_{n-k}(r) \geq \kappa_n(p)$$

by (3.5), integration of the left side gives

$$\int_0^t e^{-2\kappa(p)t + L_n(p, q, r)\tau} d\tau = \frac{e^{-2\kappa(p)t} \{e^{L_n(p, q, r)t} - 1\}}{L_n(p, q, r)} \leq \frac{e^{-\kappa_n(p)t}}{|L_n(p, q, r)|}.$$

Moreover, if $|L_n(p, q, r)| \leq \kappa(p)$, then

$$\int_0^t e^{-2\kappa(p)t + L_n(p, q, r)\tau} d\tau \leq \frac{e^{-2\kappa(p)t} \{e^{\kappa(p)t} - 1\}}{\kappa(p)} \leq \frac{e^{-\kappa_n(p)t}}{\kappa(p)}.$$

These two inequalities conclude the lemma. □

Lemma 3.3. *There exists $c_3 > 0$ independent of j, k, n such that*

$$\max\{\kappa(p), |L_n(p, q, r)|\} \geq c_3 \max\{j^{-\delta}(k-j+1)^{-\delta}, (k-j+1)^{-\delta}(n-k+1)^{-\delta}, j^{-\delta}(n-k+1)^{-\delta}\} \kappa(p),$$

where $\rho = \sqrt{|q|^2 + |r|^2}$ and $\kappa(\rho) = c_1(\min\{1, \rho\})^\delta$.

Proof. Note that the inequality

$$(3.6) \quad \max\{\kappa(p), |L_n(p, q, r)|\} \geq \max\left\{\kappa(p), \frac{1}{3}[\kappa_{j-1}(p-q) + \kappa_{k-j}(q-r) + \kappa_{n-k}(r)]\right\}$$

holds. In fact, if $|L_n(p, q, r)| \leq \kappa(p)$, we have

$$\frac{1}{3}[\dots] \leq \kappa(p) \leq [\dots],$$

where $[\dots] = \kappa_{j-1}(p-q) + \kappa_{k-j}(q-r) + \kappa_{n-k}(r)$, and the first inequality implies (3.6). On the other hand, if $|L_n(p, q, r)| \geq \kappa(p)$, we have

$$\kappa(p) \geq [\dots] \quad \text{or} \quad \kappa(p) \leq \frac{1}{3}[\dots].$$

It then follows that

$$\begin{aligned} L_n(p, q, r) &\geq \kappa(p) = \max\left\{\kappa(p), \frac{1}{3}[\dots]\right\} \quad \text{or} \\ L_n(p, q, r) &\leq -\frac{1}{3}[\dots] \leq -\max\left\{\kappa(p), \frac{1}{3}[\dots]\right\}, \end{aligned}$$

and (3.6) is also obtained.

By use of (3.6) we shall show

$$(3.7) \quad \max\{\kappa(p), |L_n(p, q, r)|\} \geq c j^{-\delta}(k-j+1)^{-\delta} \kappa(p).$$

Here and in the following the letter c denotes various constants independent of n . If $|q-r| \geq 2^{-1}|r|$, then it follows that

$$\begin{aligned} \max\{\kappa(p), |L_n(p, q, r)|\} &\geq \frac{1}{2} \left\{ \kappa(p) + \frac{1}{3}[\kappa_{j-1}(p-q) + \kappa_{k-j}(q-r) + \kappa_{n-k}(r)] \right\} \\ &\geq \frac{1}{6} \{ \kappa(p) + \kappa_{j-1}(p-q) + \kappa_{k-j}(q-r) \} \\ &\geq \frac{1}{6} \{ (2j-1)^{-\delta} \{ \kappa(p) + \kappa(p-q) \} + \kappa_{k-j}(q-r) \} \\ &\geq \frac{1}{6} \{ (2j-1)^{-\delta} \kappa(2^{-1}q) + (2k-2j+1)^{-\delta} \kappa(2^{-1}r) \} \\ &\geq c j^{-\delta}(k-j+1)^{-\delta} \kappa(p). \end{aligned}$$

If $|q - r| \leq 2^{-1}|r|$, then since $|q| \geq |r| - |q - r| \geq 2^{-1}|r|$, it follows that

$$\begin{aligned} \max\{\kappa(p), |L_n(p, q, r)|\} &\geq \frac{1}{6} \{\kappa(p) + \kappa_{j-1}(p - q)\} \\ &\geq \frac{1}{6} (2j - 1)^{-\delta} \{\kappa(p) + \kappa(p - q)\} \geq \frac{1}{6} (2j - 1)^{-\delta} \kappa(2^{-1}q). \\ &\geq \frac{1}{12} (2j - 1)^{-\delta} \{\kappa(2^{-1}q) + \kappa(4^{-1}r)\} \geq cj^{-\delta} \kappa(p). \end{aligned}$$

These prove (3.7).

Inequalities

$$\max\{\kappa(p), |L_n(p, q, r)|\} \geq c(k - j + 1)^{-\delta} (n - k + 1)^{-\delta} \kappa(p),$$

$$\max\{\kappa(p), |L_n(p, q, r)|\} \geq cj^{-\delta} (n - k + 1)^{-\delta} \kappa(p)$$

are similarly proved, and the lemma is concluded. $\square$

Proof of Proposition 3.1 (continue). We write (3.2) as

$$\begin{aligned} (3.8) \quad |v_n(p, t)| &\leq |a| A_1^{n-1} V_{s_1}(p, t); \\ V_{s_1}(p, t) &\equiv \sum_{k=1}^n \sum_{j=1}^k j^{-\gamma} (k - j + 1)^{-\gamma} (n - k + 1)^{-\gamma} \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(p - q)^{-s_1} \\ &\quad \times M(q - r)^{-s_1} M(r)^{-s_1} dq dr \int_0^t e^{-2\kappa(p)t + L_n(p, q, r)\tau} d\tau. \end{aligned}$$

By means of Lemmas 3.1 and 3.2,

$$\begin{aligned} (3.9) \quad V_{s_1}(p, t) &\leq 3^{\gamma+s_1} (n + 2)^{-\gamma} M(p)^{-s_1} e^{-\kappa_n(p)t} \sum_{k=1}^n \sum_{j=1}^k \{j^{-\gamma} (k - j + 1)^{-\gamma} \\ &\quad + (k - j + 1)^{-\gamma} (n - k + 1)^{-\gamma} + j^{-\gamma} (n - k + 1)^{-\gamma}\} \\ &\quad \times \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \{M(p - q)^{-s_1} M(q - r)^{-s_1} + M(q - r)^{-s_1} M(r)^{-s_1} \\ &\quad + M(p - q)^{-s_1} M(r)^{-s_1}\} \frac{dq dr}{\max\{\kappa(p), |L_n(p, q, r)|\}}. \end{aligned}$$

We shall show that the term

$$\begin{aligned} I_1 &\equiv \sum_{k=1}^n \sum_{j=1}^k j^{-\gamma} (k - j + 1)^{-\gamma} \\ &\quad \times \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(p - q)^{-s_1} M(q - r)^{-s_1} \frac{dq dr}{\max\{\kappa(p), |L_n(p, q, r)|\}} \end{aligned}$$

is bounded by a constant independent of n . It follows from Lemma 3.3 that

$$I_1 \leq c_3^{-1} \sum_{k=1}^n \sum_{j=1}^k j^{-\gamma+\delta} (k-j+1)^{-\gamma+\delta} \\ \times \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(p-q)^{-s_1} M(q-r)^{-s_1} \kappa(\rho)^{-1} dq dr.$$

Since $\gamma > 1 + \delta$, the sum $\sum_{k=1}^n \sum_{j=1}^k j^{-\gamma+\delta} (k-j+1)^{-\gamma+\delta}$ is estimated from above by a constant independent of n . Thus,

$$I_1 \leq c \left\{ \iint_{\rho < 1} \kappa(\rho)^{-1} dq dr + \iint_{\rho > 1} M(p-q)^{-s_1} M(q-r)^{-s_1} dq dr \right\} \\ \leq c \int_{S^{2N-1}} dS \int_0^1 \rho^{2N-1-\delta} d\rho + c \left\{ \int_{\mathbf{R}^N} M(r)^{-s_1} dr \right\}^2 \leq c,$$

where in the last inequality we have used the fact that $\delta < 2N$ and $s_1 > N$.

The other terms of the right side of (3.9) are similarly estimated, and we conclude

$$(3.10) \quad V_{s_1}(p, t) \leq c_4(n+1)^{-\gamma} M(p)^{-s_1} e^{-\kappa_n(p)t}$$

for some $c_4 = c_4(\gamma) > 0$ independent of n .

Substitute this into (3.8). Then choosing $A_1 = \max\{1, |a|c_4\}$, we conclude the inequality of the proposition for $v_n(p, t)$. $\square$

Proposition 3.2. *Let $p_1, p_2 \in \mathbf{R}^N$ satisfy $|p_2| \leq |p_1|$ and $|p_1 - p_2| \leq 1$. Then we have for any n ,*

$$|v_n(p_1, t) - v_n(p_2, t)| \leq c_5 A_2^n (n+1)^{-\gamma} (1+t^\nu) |p_1 - p_2|^\sigma M(p_2)^{-s} e^{-\kappa_n(p_2)t},$$

where $0 < \nu \leq 1$, $\sigma = \min\{\sigma_2, \sigma_1 \nu\}$, $c_5 = 1 + 2c_2^\nu$ and $A_2 = \max\{A_1, 3|a|c_4\}$.

Lemma 3.4. *Let $p_1, p_2 \in \mathbf{R}^N$ be as in the above proposition. Then for any $0 < \nu \leq 1$,*

$$|e^{-K(p_1)t} - e^{-K(p_2)t}| \leq 2(c_2 |p_1 - p_2|^{\sigma_1} t)^\nu M(p_2)^\alpha e^{-2\kappa(p_2)t}.$$

Proof. For definiteness we assume $\operatorname{Re} K(p_1) \geq \operatorname{Re} K(p_2)$. It then follows from (A3) that

$$|1 - e^{-K(p_1)t + K(p_2)t}| \leq 2 \min\{1, |K(p_1) - K(p_2)|t\} \\ \leq 2(\min\{1, |K(p_1) - K(p_2)|t\})^\nu \leq 2(c_2 M(p_2)^\alpha |p_1 - p_2|^{\sigma_1} t)^\nu.$$

This proves the lemma since $M(p_2) \geq 1$ and $e^{-\operatorname{Re} K(p_2)t} \leq e^{-2\kappa(p_2)t}$ by (A2). $\square$

Proof of Proposition 3.2. It follows from (A2), (A5), (A6) and Lemma 3.4 that

$$\begin{aligned}
|v_0(p_1, t) - v_0(p_2, t)| &= \varepsilon^{-1} |\hat{u}_0(p_1) e^{-K(p_1)t} - \hat{u}_0(p_2) e^{-K(p_2)t}| \\
&\leq \varepsilon^{-1} |\hat{u}_0(p_1) - \hat{u}_0(p_2)| e^{-2\kappa(p_1)t} + \varepsilon^{-1} |\hat{u}_0(p_2)| |e^{-K(p_1)t} - e^{-K(p_2)t}| \\
&\leq |p_1 - p_2|^{\sigma_2} M(p_2)^{-s} e^{-2\kappa(p_1)t} \\
&\quad + M(p_2)^{-s-\alpha} 2c_2^v |p_1 - p_2|^{\sigma_1 v} t^v M(p_2)^\alpha e^{-2\kappa(p_2)t} \\
&\leq (1 + 2c_2^v) |p_1 - p_2|^\sigma (1 + t^v) M(p_2)^{-s} e^{-2\kappa(p_2)t}.
\end{aligned}$$

which proves the proposition for $n = 0$.

For $n \geq 1$, we put

$$\begin{aligned}
(3.11) \quad v_n(p_1, t) - v_n(p_2, t) &= -a\{I_2 + I_3\}; \\
I_2 &\equiv \int_0^t \{e^{-K(p_1)(t-\tau)} - e^{-K(p_2)(t-\tau)}\} d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{k=1}^n \sum_{j=1}^k v_{j-1}(p_2 - q, \tau) \\
&\quad \times v_{k-j}(q - r, \tau) v_{n-k}(r, \tau) dq dr, \\
I_3 &\equiv \int_0^t e^{-K(p_1)(t-\tau)} d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \sum_{k=1}^n \sum_{j=1}^k \{v_{j-1}(p_1 - q, \tau) - v_{j-1}(p_2 - q, \tau)\} \\
&\quad \times v_{k-j}(q - r, \tau) v_{n-k}(r, \tau) dq dr.
\end{aligned}$$

By use of Lemma 3.4 and Proposition 3.1,

$$|I_2| \leq 2(c_2 |p_1 - p_2|^{\sigma_1} t^v M(p_2)^\alpha A_1^{n-1} V_{s_1}(p_2, t),$$

where $V_{s_1}(p, t)$ satisfies (3.10). Thus, it follows that

$$(3.12) \quad |I_2| \leq 2c_2^v A_1^{n-1} t^v |p_1 - p_2|^\sigma c_4(n+1)^{-\gamma} M(p_2)^{-s_1} e^{-\kappa_n(p_2)t}.$$

To estimate I_3 we assume that Proposition 3.2 is valid for all $v_\ell(p_1, t) - v_\ell(p_2, t)$, $\ell = 1, 2, \dots, n-1$. Then, since $s > N$, as is easily seen

$$\begin{aligned}
|I_3| &\leq \int_0^t e^{-2\kappa(p_1)(t-\tau)} d\tau \left\{ \iint_{|p_1-q| < |p_2-q|} + \iint_{|p_1-q| \geq |p_2-q|} \right\} \\
&\leq (1 + 2c_2^v) A_2^{n-1} (1 + t^v) |p_1 - p_2|^\sigma \{V_s(p_1, t) + V_s(p_2, t)\},
\end{aligned}$$

where $V_s(p, t)$ satisfies (3.8) with s_1 replaced by $s > N$. Thus, by (3.10)

$$\begin{aligned}
|I_3| &\leq (1 + 2c_2^v) A_2^{n-1} (1 + t^v) |p_1 - p_2|^\sigma c_4(n+1)^{-\gamma} \\
&\quad \times \{M(p_1)^{-s} e^{-\kappa_n(p_1)t} + M(p_2)^{-s} e^{-\kappa_n(p_2)t}\}.
\end{aligned}$$

Since $|p_2| \leq |p_1|$, it then follows that

$$(3.13) \quad |I_3| \leq 2(1 + 2c_2^v)A_2^{n-1}(1 + t^v)|p_1 - p_2|^\sigma c_4(n+1)^{-\gamma} M(p_2)^{-s} e^{-\kappa_n(p_2)t}.$$

(3.11)~(3.13) prove the inequality of the proposition for $v_n(p_1, t) - v_n(p_2, t)$. $\square$

Proposition 3.3. *Assume that $|p| \leq 1$. Then we have for any n ,*

$$|v_n(p, t)e^{K(p)t} - v_n(0, t)| \leq c_6 A_2^n (n+1)^{2N+\delta-\gamma} t^{-N/\delta-\theta_1} |p|^{-N+\theta_2} e^{\operatorname{Re} K(p)t},$$

where $c_6 > 0$ and small constants $\theta_1, \theta_2 > 0$ are independent of n .

Proof. Let $n = 0$. Then we have

$$|v_0(p, t)e^{K(p)t} - v_0(0, t)| = |v(p) - v(0)| \leq |p|^{\sigma_2} M(p)^{-s}.$$

Note that

$$1 = e^{\operatorname{Re} K(p)t} e^{-\operatorname{Re} K(p)t} \leq e^{\operatorname{Re} K(p)t} (\mu^{-1} \operatorname{Re} K(p)t)^{-\mu} \quad \text{for any } \mu > 0.$$

Then since $\operatorname{Re} K(p) \geq 2\kappa(p) \geq 2c_1|p|^\delta$ by assumption $|p| \leq 1$, choosing $0 < \theta_1 < \sigma_2/\delta$, $\theta_2 = \sigma_2 - \theta_1\delta$, $\mu = N/\delta + \theta_1$ and $c_6 \geq (2c_1\mu^{-1})^{-\mu}$, we obtain the assertion.

Next let $n \geq 1$. Then we have

$$\begin{aligned} v_n(p, t)e^{K(p)t} - v_n(0, t) = & \\ & -a \sum_{k=1}^n \sum_{j=1}^k \left\{ \int_0^t (e^{K(p)\tau} - 1) d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} v_{j-1}(-q, \tau) v_{k-j}(q-r, \tau) v_{n-k}(r, \tau) dq dr \right. \\ & \left. + \int_0^t e^{K(p)\tau} d\tau \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} [v_{j-1}(p-q, \tau) - v_{j-1}(-q, \tau)] v_{k-j}(q-r, \tau) v_{n-k}(r, \tau) dq dr \right\}. \end{aligned}$$

Since $|p| \leq 1$, it follows from Lemma 3.4 with $p_2 = 0$ that

$$|e^{K(p)t} - 1| = e^{\operatorname{Re} K(p)t} |1 - e^{-K(p)t}| \leq 2e^{\operatorname{Re} K(p)t} (c_2|p|^{\sigma_1} t)^v$$

for any $0 < v \leq 1$. We apply this and Propositions 3.1 and 3.2 to the right side of above equality. Then putting

$$C(j, k, n) = j^{-\gamma} (k-j+1)^{-\gamma} (n-k+1)^{-\gamma},$$

$$P_n(p, q, r) = \kappa_{j-1}(p-q) + \kappa_{k-j}(q-r) + \kappa_{n-k}(r),$$

we obtain

$$(3.14) \quad |v_n(p, t)e^{K(p)t} - v_n(0, t)| \leq |a|A_2^{n-1}|p|^\sigma \sum_{j=1}^k \sum_{k=1}^n C(j, k, n) \{2c_2^v I_4 + I_5 + I_6\};$$

$$I_4 \equiv \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(-q)^{-s_1} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - P_n(0, q, r)\tau} d\tau,$$

$$I_5 \equiv \iint_{|q| < |p-q|} M(-q)^{-s} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - P_n(0, q, r)\tau} d\tau,$$

$$I_6 \equiv \iint_{|q| \geq |p-q|} M(p-q)^{-s} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - P_n(p, q, r)\tau} d\tau.$$

Since $M(-q)$, $M(p-q) \geq 1$ and $\kappa_{j-1}(-q)$, $\kappa_{j-1}(p-q) \geq 0$, we see that each I_ℓ ($\ell = 4, 5, 6$) satisfies the inequality

$$I_\ell \leq \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - \{\kappa_{k-j}(q-r) + \kappa_{n-k}(r)\}\tau} d\tau.$$

Lemma 3.5. *Let $\rho = \sqrt{|q|^2 + |r|^2}$. Then we have*

$$\kappa_{k-j}(q-r) + \kappa_{n-k}(r) \geq \kappa_n(\rho/2) \geq 2^{-\delta} \kappa_n(\rho).$$

Proof. If $|q-r| \geq |q|/2$,

$$\kappa_{k-j}(q-r) + \kappa_{n-k}(r) \geq \kappa_{k-j}(q/2) + \kappa_{n-k}(r/2) \geq \kappa_{n-j+1}(\rho/2).$$

On the other hand, if $|q-r| \leq |q|/2$, we have $|r| \geq |q| - |q-r| \geq |q|/2$, and it follows that

$$\kappa_{k-j}(q-r) + \kappa_{n-k}(r) \geq \kappa_{n-k}(r) \geq \kappa_{n-k}(\rho/2).$$

These prove the assertion. □

Lemma 3.6. *For any $\mu > 0$ there exists $c_7 = c_7(\mu) > 0$ such that*

$$\begin{aligned} & \int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - 2^{-\delta} \kappa_n(\rho)\tau} d\tau \\ & \leq c_7(n+1)^{\delta(1+v+\mu)} t^{-\mu} \\ & \quad \times \kappa(p)^{-v} (\max\{\kappa(p), \kappa(\rho)\})^{-1} (\min\{\kappa(p), \kappa(\rho)\})^{-\mu} e^{\operatorname{Re} K(p)t}. \end{aligned}$$

Proof. Note that $(v^{-1}\omega'\kappa_n(\rho)\tau)^v \leq e^{\omega'\kappa_n(\rho)\tau}$ for any $\omega' > 0$. Then choosing $\omega = 2^{-\delta} - \omega' > 0$, we obtain

$$\int_0^t (1 + \tau^v) e^{\operatorname{Re} K(p)\tau - 2^{-\delta} \kappa_n(\rho)\tau} d\tau \leq 2(v^{-1}\omega'\kappa_n(\rho))^{-v} \int_0^t e^{\operatorname{Re} K(p)\tau - \omega\kappa_n(\rho)\tau} d\tau.$$

Let us estimate the right side. First assume that $\rho \leq |p| \leq 1$. Then since $\operatorname{Re} K(p) \geq 2\kappa(p) \geq 2\kappa_n(p)$, we have for any $\mu > 0$,

$$\int_0^t e^{\operatorname{Re} K(p)\tau - \omega\kappa_n(p)\tau} d\tau \leq \frac{e^{\operatorname{Re} K(p)t - \omega\kappa_n(p)t}}{\kappa_n(p)} \leq \frac{(\mu^{-1}\omega\kappa_n(p)t)^{-\mu} e^{\operatorname{Re} K(p)t}}{\kappa_n(p)}.$$

Next assume that $\rho > |p|$. Then since $\kappa_n(p) \leq \kappa_n(\rho)$, choosing $0 < \tilde{\omega} < \omega$, we obtain

$$\begin{aligned} \int_0^t e^{\operatorname{Re} K(p)\tau - \omega\kappa_n(p)\tau} d\tau &\leq e^{\operatorname{Re} K(p)t - \tilde{\omega}\kappa_n(p)t} \int_0^t e^{\tilde{\omega}\kappa_n(p)\tau - \omega\kappa_n(p)\tau} d\tau \leq \frac{e^{\operatorname{Re} K(p)t - \tilde{\omega}\kappa_n(p)t}}{\omega\kappa_n(p) - \tilde{\omega}\kappa_n(p)} \\ &\leq ((\omega - \tilde{\omega})\kappa_n(p))^{-1} (\mu^{-1}\tilde{\omega}\kappa_n(p)t)^{-\mu} e^{\operatorname{Re} K(p)t}. \end{aligned}$$

Note here that $\kappa_n(p) \geq (2n+1)^{-\delta}\kappa(p)$. Then the above inequalities lead us to the conclusion. $\square$

Proof of Proposition 3.3 (continue). By use of Lemmas 3.5 and 3.6, we have

$$\begin{aligned} I_\ell &\leq c_7(n+1)^{\delta(1+v+\mu)} t^{-\mu} e^{\operatorname{Re} K(p)t} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M(q-r)^{-s_1} M(r)^{-s_1} \\ &\quad \times \kappa(p)^{-v} (\max\{\kappa(p), \kappa(\rho)\})^{-1} (\min\{\kappa(p), \kappa(\rho)\})^{-\mu} dq dr \\ &\leq c(n+1)^{\delta(1+v+\mu)} t^{-\mu} e^{\operatorname{Re} K(p)t} \left\{ |p|^{-\delta} \int_0^{|p|} \rho^{2N-1-\delta(v+\mu)} d\rho \right. \\ &\quad \left. + |p|^{-\delta\mu} \int_{|p|}^1 \rho^{2N-1-\delta(1+v)} d\rho + \iint_{\rho>1} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \right\}. \end{aligned}$$

We put $\mu = N/\delta + \theta_1 > 0$, and choose $v, \theta_1 \in (0, 1)$ to satisfy

$$(3.15) \quad 2N - \delta(v + \mu) = N - \delta(v + \theta_1) > 0, \quad 2N - \delta(v + 1) > 0.$$

Then the right side becomes finite and we have

$$I_\ell \leq c(n+1)^{\delta(1+v+\mu)} t^{-N/\delta - \theta_1} \{ |p|^{N-\delta(1+v+\theta_1)} + |p|^{-N-\delta\theta_1} + c \} e^{\operatorname{Re} K(p)t}.$$

If we further restrict v, θ_1 to satisfy

$$(3.16) \quad N - \delta(1 + v + \theta_1) \geq -N - \delta\theta_1 > -N - \sigma,$$

then the above inequality implies

$$(3.17) \quad I_\ell \leq c_8(n+1)^{\delta(1+v+\mu)} t^{-N/\delta - \theta_1} |p|^{-N-\delta\theta_1} e^{\operatorname{Re} K(p)t}.$$

Here we note that conditions (3.15) and (3.16) are equivalent to

$$(3.18) \quad v + \theta_1 < \frac{N}{\delta}, \quad v \leq \frac{2N}{\delta} - 1 \text{ and } \theta_1 < \frac{\sigma}{\delta} = \frac{\min\{\sigma_2, v\sigma_1\}}{\delta},$$

and the existence of such a positive pair is guaranteed since we have assumed $\delta \in (0, 2N)$.

By Lemma 3.1,

$$\begin{aligned} \sum_{k=1}^n \sum_{j=1}^k C(j, k, n) &\leq 3^\gamma (n+2)^{-\gamma} \sum_{k=1}^n \sum_{j=1}^k \{j^{-\gamma} (k-j+1)^{-\gamma} \\ &\quad + (k-j+1)^{-\gamma} (n-k+1)^{-\gamma} + j^{-\gamma} (n-k+1)^{-\gamma}\} \leq c_9 (n+1)^{-\gamma}. \end{aligned}$$

We apply this and (3.17) in (3.14). Then putting $\theta_2 = \sigma - \delta\theta_1 > 0$ and $c_6 = \max\{2c_4^{-1}c_5c_8c_9, (2c_1\mu^{-1})^{-\mu}\}$, we conclude the assertion of the proposition for $n \geq 1$.

Proposition 3.4. *There exists the limit $\varphi_n = \lim_{t \rightarrow \infty} v_n(0, t)$ and we have*

$$|v_n(0, t) - \varphi_n| \leq c_{10} A_2^n (n+1)^{2N-\gamma} t^{-\theta_3},$$

where $c_{10} > 0$ and $\theta_3 > 0$ are independent of n .

Proof. Since $v_0(0, t)$ is independent of t , we have only to consider the case $n \geq 1$. By Proposition 3.1, we have for any $t > t_1 > 0$,

$$\begin{aligned} &|v_n(0, t) - v_n(0, t_1)| \\ &\leq |a| A_1^{n-1} \sum_{k=1}^n \sum_{j=1}^k C(j, k, n) \\ &\quad \times \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M(-q)^{-s_1} M(q-r)^{-s_1} M(r)^{-s_1} dq dr \int_{t_1}^t e^{-P_n(0, q, r)\tau} d\tau \\ &\leq |a| A_1^{n-1} c(n+1)^{-\gamma} \int_{t_1}^t d\tau \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-2^{-\delta}\kappa_n(\rho)\tau} M(q-r)^{-\gamma} M(r)^{-\gamma} dq dr. \end{aligned}$$

Since $e^{-2^{-\delta}\kappa_n(\rho)\tau} \leq (\mu^{-1}2^{-\delta}\kappa_n(\rho)\tau)^{-\mu}$ for any $\mu > 0$, as in the above proof of Proposition 3.3,

$$\begin{aligned} &\int_{t_1}^t d\tau \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \dots \\ &\leq c(2n+1)^{\delta\mu} \int_{t_1}^t \tau^{-\mu} \left\{ \int_0^1 \rho^{2N-1-\delta\mu} d\rho + \iint_{\rho>1} M(q-r)^{-\gamma} M(r)^{-\gamma} dq dr \right\} d\tau. \end{aligned}$$

By the condition $0 < \delta < 2N$, we can choose μ to satisfy $1 < \mu < 2N/\delta$. Then the above inequalities show

$$|v_n(0, t) - v_n(0, t_1)| \leq |a| A_1^{n-1} c_{11} (n+1)^{\delta\mu-\gamma} \frac{1}{-\mu+1} (t^{-\mu+1} - t_1^{-\mu+1}).$$

Thus, $v_n(0, t)$ converges as $t \rightarrow \infty$, and the inequality of the proposition holds with $c_{10} = c_{11}/3c_4(\mu - 1)$ and $\theta_3 = \mu - 1$.

4. Large time asymptotics of functions $\psi_n(x, t)$

It is proved in Proposition 3.1 that each function $v_n(p, t)$ belongs to the class

$$BC(\mathbf{R}^N \times [0, \infty)) \cap BC([0, \infty); L^1(\mathbf{R}^N))$$

(BC means bounded continuous functions). So, we see that its Fourier inverse transform $\psi_n(x, t)$ is also in $BC(\mathbf{R}^N \times [0, \infty))$.

In this section we shall show

Proposition 4.1. *For each n , the following asymptotics is valid for $t \rightarrow \infty$ uniformly with respect to $y = x/t^{1/\delta} \in \mathbf{R}^N$:*

$$(2\pi)^N \psi_n(x, t) = \varphi_n t^{-N/\delta} \int_{\mathbf{R}^N} \cos(y \cdot \eta) e^{-\lambda|\eta|^\delta} d\eta + A_3^n O(t^{-N/\delta-\theta}),$$

where φ_n is as given in Proposition 3.4 and $A_3 \geq 1$ and $\theta > 0$ are independent of n .

Proof. We represent $\psi_n(x, t)$ in the following form:

$$\begin{aligned} (4.1) \quad \psi_n(x, t) &= \int_{|p| \geq 1} e^{ip \cdot x} v_n(p, t) dp + \int_{|p| \leq 1} e^{ip \cdot x} v_n(p, t) \{1 - e^{i \operatorname{Im} K(p)t}\} dp \\ &\quad + \int_{|p| \leq 1} e^{ip \cdot x - \operatorname{Re} K(p)t} \{v_n(p, t) e^{K(p)t} - v_n(0, t)\} dp \\ &\quad + \int_{|p| \leq 1} e^{ip \cdot x - \operatorname{Re} K(p)t} \{v_n(0, t) - \varphi_n\} dp + \varphi_n \int_{|p| \leq 1} e^{ip \cdot x - \operatorname{Re} K(p)t} dp \\ &= J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

By use of Proposition 3.1, J_1 is estimated as follows:

$$\begin{aligned} (4.2) \quad |J_1| &\leq A_1^n (n+1)^{-\gamma} e^{-2c_1 t/(2n+1)^\delta} \int_{|p| \geq 1} M(p)^{-s-\alpha} dp \\ &\leq c A_1^n (n+1)^{-\gamma} (\mu^{-1} 2c_1 t/(2n+1)^\delta)^{-\mu} \\ &\leq c_{12} A_1^n (n+1)^{-\gamma+\delta\mu} t^{-\mu} = c_{12} A_1^n (n+1)^{N+\delta\theta_1-\gamma} t^{-N/\delta-\theta_1}. \end{aligned}$$

Here, since $\mu > 0$ is any constant, we have chosen $\mu = N/\delta + \theta_1$. By use of Proposition 3.3, J_3 is estimated as follows:

$$\begin{aligned} (4.3) \quad |J_3| &\leq c A_2^n (n+1)^{2N+\delta-\gamma} t^{-N/\delta-\theta_1} \int_{|p| < 1} |p|^{-N+\theta_2} dp \\ &\leq c_{13} A_2^n (n+1)^{2N+\delta-\gamma} t^{-N/\delta-\theta_1}, \end{aligned}$$

To estimate J_2 , J_4 and J_5 we need the following

Lemma 4.1. *Let $b > 0$ and $h > -N$. Then we have*

$$\int_{\mathbf{R}^N} e^{-b|p|^\delta t} |p|^h dp \leq c(bt)^{-N/\delta-h/\delta}.$$

Proof. We put $q = (bt)^{1/\delta} p$. Then as is easily seen

$$\int_{\mathbf{R}^N} e^{-b|p|^\delta t} |p|^h dp = (bt)^{-N/\delta-h/\delta} \int_{\mathbf{R}^N} e^{-|q|^\delta} |q|^h dq.$$

Since $h > -N$, this proves the lemma. $\square$

Proof of Proposition 4.1 (continue). It follows from Proposition 3.1 that

$$|J_2| \leq A_1^n (n+1)^{-\gamma} \int_{|p|<1} e^{-\kappa_n(p)t} |e^{i\operatorname{Im} K(p)t} - 1| dp.$$

By (A1), we have $|\operatorname{Im} K(p)t| \leq c|p|^\beta t$ for $|p| \leq 1$. Then

$$|e^{i\operatorname{Im} K(p)t} - 1| \leq c|\operatorname{Im} K(p)t| \leq c|p|^\beta t,$$

and hence by Lemma 4.1 with $b = c_1/(2n+1)^\delta$ and $h = \beta$,

$$\begin{aligned} (4.4) \quad |J_2| &\leq cA_1^n (n+1)^{-\gamma} \int_{|p|<1} e^{-c_1|p|^\delta t/(2n+1)^\delta} |p|^\beta dp \\ &\leq c_{14} A_1^n (n+1)^{N+\beta-\gamma} t^{-N/\delta-\beta/\delta+1}. \end{aligned}$$

On the other hand, by Proposition 3.4 and Lemma 4.1 with $b = 2c_1$ and $h = 0$,

$$\begin{aligned} (4.5) \quad |J_4| &\leq A_1^n c(n+1)^{2N-\gamma} t^{-\theta_3} \int_{|p|<1} e^{-2\kappa(p)t} dp \\ &\leq c_{15} A_1^n (n+1)^{2N-\gamma} t^{-N/\delta-\theta_3}. \end{aligned}$$

Finally, to estimate J_5 , we devide it as follows:

$$\begin{aligned} J_5 &= \varphi_n \int_{|p|<1} e^{ip \cdot x - \operatorname{Re} K(p)t} dp = \varphi_n \left\{ \int_{|p|^\beta < t^{-1}} e^{ip \cdot x - \lambda|p|^\delta t} dp \right. \\ &\quad \left. + \int_{|p|^\beta < t^{-1}} e^{ip \cdot x - \lambda|p|^\delta t} (e^{O(|p|^\beta)t} - 1) dp + \int_{t^{-1} \leq |p|^\beta < 1} e^{ip \cdot x - \operatorname{Re} K(p)t} dp \right\} \\ &= \varphi_n \{J_{5,1} + J_{5,2} + J_{5,3}\}. \end{aligned}$$

By use of (A2) we have for any $\mu > 0$,

$$\begin{aligned} |J_{5,3}| &\leq \int_{t^{-1} \leq |p|^\beta < 1} e^{-2c_1|p|^\delta t} dp \leq ce^{-2c_1 t^{1-\delta/\beta}} \\ &\leq (\mu^{-1} 2c_1 t^{1-\delta/\beta})^{-\mu} \leq ct^{-(1-\delta/\beta)\mu}. \end{aligned}$$

By Lemma 4.1 with $b = \lambda/2$ and $h = 0$ we have for any $\mu' > 0$,

$$\begin{aligned} \left| J_{5,1} - \int_{\mathbf{R}^N} e^{ip \cdot x - \lambda|p|^\delta t} dp \right| &\leq \int_{|p|^\beta \geq t^{-1}} e^{-\lambda|p|^\delta t} dp \\ &\leq e^{-2^{-1}\lambda t^{1-\delta/\beta}} \int_{|p|^\beta > t^{-1}} e^{-2^{-1}\lambda|p|^\delta t} dp \\ &\leq (\mu'^{-1} 2^{-1} \lambda t^{1-\delta/\beta})^{-\mu'} c t^{-N/\delta} \leq c t^{-N/\delta - (1-\delta/\beta)\mu'}. \end{aligned}$$

On the other hand, if $|p| < t^{-1/\beta}$, then $0 < (|p|^\beta)t \leq c$, and we have

$$|e^{O(|p|^\beta)t} - 1| \leq c e^c |p|^\beta t.$$

Hence by Lemma 4.1 with $b = \lambda$ and $h = \beta$,

$$|J_{5,2}| \leq c t \int_{|p| < t^{-1/\beta}} e^{-\lambda|p|^\delta t} |p|^\beta dp \leq c t^{-N/\delta - \beta/\delta + 1}.$$

If we choose μ, μ' to satisfy

$$(1 - \delta/\beta)\mu = N/\delta + (1 - \delta/\beta)\mu' = N/\delta + \beta/\delta - 1,$$

and if we note

$$\int_{\mathbf{R}^N} e^{ip \cdot x - \lambda|p|^\delta t} dp = \frac{1}{2} \int_{\mathbf{R}^N} (e^{ip \cdot x} + e^{-ip \cdot x}) e^{-\lambda|p|^\delta t} dp = \int_{\mathbf{R}^N} \cos(p \cdot x) e^{-\lambda|p|^\delta t} dp,$$

then summarizing the above inequalities, we obtain

$$(4.6) \quad \left| J_5 - \varphi_n t^{-N/\delta} \int_{\mathbf{R}^N} \cos(y \cdot \eta) e^{-\lambda|\eta|^\delta} d\eta \right| \leq c_{16} t^{-N/\delta - \beta/\delta + 1},$$

where $y = x/t^{1/\delta}$.

Now we put

$$\theta = \min\{\theta_1, \theta_3, \beta/\delta - 1\} > 0, \quad A_3 = bA_2,$$

where b is chosen to satisfy

$$\begin{aligned} b^n &\geq \max\{1, c_{12}(n+1)^{N+\delta\theta_1-\gamma}, c_{14}(n+1)^{N+\beta-\gamma}, \\ &\quad c_{13}(n+1)^{2N+\delta-\gamma}, c_{15}(n+1)^{2N-\gamma}, A_2^{-n} c_{16}\}, \end{aligned}$$

for any n . Then (4.1)~(4.6) show the assertion of the proposition (the uniformity of the convergence in $y = x/t^{1/\delta}$ follows from (4.6)). $\square$

5. Proof of Theorem

Proposition 3.1 shows that for sufficiently small ε , say $\varepsilon < A^{-1/2} \leq A_1^{-1/2}$, the right side of (2.6) converges uniformly in $(p, t) \in \mathbf{R}^N \times [0, \infty)$ and satisfies

the integral equation (2.4). Let $u(x, t)$ be the Fourier inverse transform of $\hat{u}(p, t)$. Then it is in $BC(\mathbf{R}^N \times [0, \infty))$ since

$$|\hat{u}(p, t)| \leq cM(p)^{-s_1} \in L^1(\mathbf{R}^N),$$

and gives the unique weak solution of the Cauchy problem (1.1).

To obtain the large time asymptotics of $u(x, t)$, we use its expression (2.8) and Proposition 4.1.

Note that

$$|\varphi_n| = \lim_{t \rightarrow \infty} |v_n(0, t)| \leq A_1^n.$$

Then it follows from (2.6) that

$$(2\pi)^N \sum_{n=0}^{\infty} \varepsilon^{2n+1} \varphi_n = \lim_{t \rightarrow \infty} \hat{u}(0, t) = \lim_{t \rightarrow \infty} \int_{\mathbf{R}^N} u(x, t) dx = A_0.$$

Moreover, since

$$(2\pi)^N \sum_{n=0}^{\infty} \varepsilon^{2n+1} A_3^n \equiv A_4 < \infty,$$

(2.8) and Proposition 4.1 give the expression

$$(2\pi)^N u(x, t) = A_0 t^{-N/\delta} \int_{\mathbf{R}^N} \cos(y \cdot \eta) e^{-\lambda|\eta|^\delta} d\eta + A_3 O(t^{-N/\delta-\theta})$$

which holds as $t \rightarrow \infty$ uniformly in $y = xt^{-1/\delta} \in \mathbf{R}^N$.

Thus, the proof of Theorem is completed.

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