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(Citation)

Funkcialaj Ekvacioj, 44(1):171-200

(Issue Date)

2001-04

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100501180>

(URL)

<https://hdl.handle.net/20.500.14094/0100501180>



Large Time Behavior of Solutions to the Landau-Ginzburg Type Equations

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1. Introduction

We consider the Cauchy problem for the following nonlinear dissipative equations

$$(1.1) \quad \begin{cases} u_t - \alpha u_{xx} + \beta |u|^2 u + \gamma |u|^{p-1} u = 0, & x \in \mathbf{R}, \quad t > 0, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$, $\Re \alpha > 0$, $p > 3$. We assume that $\Re \delta \geq 0$, where $\delta = \delta(\alpha, \beta) = \beta / \sqrt{2|\alpha|^2 + \alpha^2}$. Equation (1.1) is known as the complex Landau-Ginzburg equation. Our main purpose is to study the large time asymptotics of solutions to the Cauchy problem (1.1). Existence of solutions to the Cauchy problem for (1.1) was studied by many authors (see, e.g. [6], [7], [14]). Blow-up of positive solutions to the Cauchy problem

$$(1.2) \quad u_t - u_{xx} - u^p = 0, \quad u(0, x) = u_0(x) > 0$$

was investigated in papers [4], [10], [12]. It was shown that solutions of (1.2) blow up in finite time for the case $p > 3$ in [4] and for the case $p = 3$ in [9], [12]. In [1], [5], [13] the asymptotic behavior in time of solutions to the Cauchy problem

$$(1.3) \quad u_t - u_{xx} + u^3 = 0, \quad u(0, x) = u_0(x) > 0$$

was considered and L^∞ norm of solutions was shown to decay in time faster than $t^{-1/2}$. However their methods do not work for the complex Landau-Ginzburg equation (1.1). Asymptotic behavior in time of solutions to the equation (1.3) with nonlinearity u^3 , replaced by $|u|^{p-1}u$, (with $p \neq 3$) was also studied by many authors (see, e.g., [1], [2] for $1 < p < 3$ and [5], [6], [7] for $p > 3$ and references therein.)

In our previous paper [10] we considered the case $\beta = i$ and $\gamma = 0$ and found the asymptotic behavior of solutions to the Cauchy problem. In the present paper we consider more general equations including (1.3). In the case

$$\Re \frac{\beta}{\sqrt{2|\alpha|^2 + \alpha^2}} = 0, \quad \text{and} \quad \Im \alpha \neq 0$$

we need more careful computations than that in the previous case $\beta = i$, $\gamma = 0$ since in this case our analysis depends on sign of the value of

$$\mu = \frac{\theta^4 (\Im \delta(\alpha, \beta))^2}{(4\pi)^2} \Re(2\nu_1 - \nu_2),$$

where $\nu_j = \log(4(1 - \sqrt{1 - \kappa_j})/\{\kappa_j(1 + \sqrt{1 - \kappa_j})\})$, $\kappa_1 = (\alpha + \bar{\alpha})^2/(\alpha + 2\bar{\alpha})^2$, $\kappa_2 = 2(|\alpha|^2 + \bar{\alpha}^2)/|\alpha + 2\bar{\alpha}|^2$, $\theta = |\int u_0(x)dx| = \sqrt{2\pi}|\hat{u}_0(0)|$. Note that when $\beta = i$, $\gamma = 0$, μ is strictly positive.

In order to state our results precisely we introduce *Notation and function spaces*. $\mathcal{F}\phi$ or $\hat{\phi}$ is the Fourier transform of ϕ defined by $\hat{\phi}(\xi) = (1/\sqrt{2\pi}) \int e^{-ix\xi} \phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ or $\check{\phi}(x)$ is the inverse Fourier transform of ϕ , i.e. $\check{\phi}(x) = (1/\sqrt{2\pi}) \int e^{ix\xi} \phi(\xi)d\xi$. Let $\partial = \partial_x = \partial/\partial x$ and $\partial_t = \partial/\partial t$. We denote by L^p the usual Lebesgue space with norm $\|\phi\|_p = (\int_{\mathbf{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess. sup}\{|\phi(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space is defined as follows $\mathbf{H}^{m,s} = \{\phi \in L^2; \|\phi\|_{m,s} = \|\langle x \rangle^s \langle i\partial_x \rangle^m \phi\| < \infty\}$, $m, s \in \mathbf{R}^+$, $\langle x \rangle = \sqrt{1 + x^2}$ are the Japanese brackets. By $\mathbf{C}(\mathbf{I}; \mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. Denote

$$\mathbf{W} = \mathbf{H}^{1,0} \cap \mathbf{H}^{0,1} \quad \text{with} \quad \|\phi\|_{\mathbf{W}} = \|\phi\|_{1,0} + \|\phi\|_{0,1}.$$

Different positive constants might be denoted by the same letter C .

In the present paper we prove the following result.

Theorem 1.1. *We assume that $u_0 \in \mathbf{W}$ with sufficiently small norm $\|u_0\|_{\mathbf{W}} = \varepsilon$ and $\theta = |\int u_0(x)dx| > 0$. Then there exists a unique solution $u(t, x) \in L^\infty((0, \infty) \times \mathbf{R}) \cap \mathbf{C}([0, \infty); \mathbf{W})$ of the Cauchy problem (1.1) satisfying the following time decay estimate*

$$\|u(t)\|_\infty \leq \frac{C\varepsilon}{\sqrt{1+t}\sqrt{1+\eta\log(1+t)}},$$

if $\eta = (\theta^2/2\pi)\Re\delta > 0$, and

$$\|u(t)\|_\infty \leq \frac{C\varepsilon}{\sqrt{1+t}\sqrt[4]{1+\mu\log(1+t)}},$$

if $\eta = 0$, $\mu = \{\theta^4(\Im\delta)^2/(4\pi)^2\}\Re(2\nu_1 - \nu_2) > 0$, finally

$$\|u(t)\|_\infty \leq \frac{C\varepsilon}{\sqrt{1+t}\sqrt[6]{1+\varkappa\log(1+t)}},$$

if $\eta = 0$, $\mu = 0$, $\varkappa > 0$, where $\varkappa$ is a positive constant defined in (2.28) explicitly. Furthermore the following asymptotic formulas of the solution are valid

$$\begin{aligned} u(t, x) &= \frac{\theta}{\sqrt{4\pi\alpha t}} \frac{1}{\sqrt{1+\eta\log t}} \exp\left(-\frac{x^2}{4\alpha t} - \frac{i\omega}{\eta} \log\log t + i \arg \hat{u}_0(0)\right) \\ &+ O\left(\frac{\varepsilon^{5/2}}{\sqrt{t\log t\log t}}\right), \end{aligned}$$

if $\eta > 0$,

$$\begin{aligned} u(t, x) &= \frac{\theta}{\sqrt{4\pi\alpha t}} \frac{1}{\sqrt[4]{1+\mu\log t}} \exp\left(-\frac{x^2}{4\alpha t} - \frac{2i\omega}{\sqrt{\mu}} \sqrt{\log t} + i \arg \hat{u}_0(0)\right) \\ &+ O\left(\frac{\varepsilon^3}{\sqrt{t}\sqrt[4]{\log t}\sqrt{\log t}}\right), \end{aligned}$$

if $\eta = 0$, $\mu > 0$, and finally

$$\begin{aligned} u(t, x) &= \frac{\theta}{\sqrt{4\pi\alpha t}} \frac{1}{\sqrt[6]{1+\varkappa\log t}} \exp\left(-\frac{x^2}{4\alpha t} - \frac{3i\omega}{2\sqrt[3]{\varkappa}} \sqrt[3]{\log^2 t} + i \arg \hat{u}_0(0)\right) \\ &+ O\left(\frac{\varepsilon^3}{\sqrt{t}\sqrt[6]{\log t}\sqrt[3]{\log t}}\right), \end{aligned}$$

if $\eta = 0$, $\mu = 0$, where $\omega = (\theta^2/4\pi)\Im\delta(\alpha, \beta)$.

Remark 1.1. We put $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, where $\alpha_1 > 0$, $\alpha_2, \beta_1, \beta_2 \in \mathbf{R}$. Then our condition $\Re\delta(\alpha, \beta) \geq 0$ is equivalent to the following inequality

$$0 \leq \beta_2\alpha_1\alpha_2 + \beta_1\left(\frac{1}{2}\sqrt{9\alpha_1^4 + \alpha_2^4 + 10\alpha_1^2\alpha_2^2 + 3\alpha_1^2 + \alpha_2^2}\right).$$

This inequality implies that there exist $\beta_2, \alpha_2 > 0$ such that even if $\beta_1 < 0$ the solution does not blow up.

Remark 1.2. In order to check the inequality $\Re(2\nu_1 - \nu_2) > 0$ for the values of α such that $\Re\alpha > 0$, we have substituted $\alpha = e^{i\phi}$ with $\phi \in (-\pi/2, \pi/2)$ into the function

$$\begin{aligned}
f(\phi) &= \Re(2v_1 - v_2) \\
&= 2 \log \left| \frac{4 \left(1 - \sqrt{1 - \left(\frac{1 + e^{-2i\phi}}{2e^{-2i\phi} + 1} \right)^2} \right)}{\left(\frac{1 + e^{-2i\phi}}{2e^{-2i\phi} + 1} \right)^2 \left(1 + \sqrt{1 - \left(\frac{1 + e^{-2i\phi}}{2e^{-2i\phi} + 1} \right)^2} \right)} \right| \\
&\quad - \log \left| \frac{2 \left(1 - \sqrt{1 - 2 \frac{e^{-2i\phi} + 1}{|2e^{-2i\phi} + 1|^2}} \right)}{\frac{e^{-2i\phi} + 1}{|2e^{-2i\phi} + 1|^2} \left(1 + \sqrt{1 - 2 \frac{e^{-2i\phi} + 1}{|2e^{-2i\phi} + 1|^2}} \right)} \right|
\end{aligned}$$

and plotted it numerically applying the program “Maple”. Then it appears that the function $f(\phi)$ is positive for all $\phi \in (-\phi_0, \phi_0)$, and is negative for all $\phi \in (-\pi/2, -\phi_0) \cup (\phi_0, \pi/2)$, where $\phi_0 \approx 1.146035015$. The function $f(\phi)$ has a maximum value approximately 0.27 at the point $\phi = 0$. Also $f(\pi/2) = 0 = f(-\pi/2)$.

Remark 1.3. When $\eta = 0, \mu = 0$, using the program “Maple” we see that $\alpha = \alpha_0 \equiv |\alpha|e^{i\phi_0}$ and $\beta = \beta_1 \equiv |\beta|e^{i\phi_1}$, or $\beta = \beta_2 \equiv |\beta|e^{i\phi_2}$, where $\phi_0 \approx 1.146035015$, $\phi_1 \approx -2.9111$ and $\phi_2 \approx 0.2305$. We have computed numerically by formula (2.28) the value of $\varkappa$ using the program “Maple” to get $\varkappa \approx -0.03i\delta^3\theta^6$. Then we see that if we choose the first value $\beta = \beta_1$, then we obtain $\delta = -i|\delta|$, hence $\varkappa > 0$. Therefore, we consider the values (α_0, β_1) in the case $\eta = 0, \mu = 0$.

Remark 1.4. We do not know the asymptotic behavior in time of solutions to (1.1) in the following three cases: 1) $\eta < 0$, 2) $\eta = 0, \mu < 0$ and 3) $\eta = 0, \mu = 0, \varkappa < 0$.

For the convenience of the reader we state our strategy of the proof. We consider (1.1) with $\gamma = 0$ (for details, see Section 3). As in [10] we make a change of the dependent variable $u = ve^{-\varphi(t) + i\psi(t)}$. Then for the new function v we get the equation

$$v_t - \alpha v_{xx} + e^{-2\varphi} \beta |v|^2 v - (\varphi' - i\psi')v = 0,$$

where we assume that

$$\int (\beta e^{-2\varphi} |v|^2 v - (\varphi' - i\psi')v) dx = 0$$

and $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta/\sqrt{2\pi} > 0$. Namely we consider the Cauchy problem for the new dependent variables (v, φ)

$$\begin{cases} v_t = \alpha v_{xx} - \beta e^{-2\varphi} \left(|v|^2 - \frac{1}{\theta} \int |v|^2 v dx \right) v, \\ \varphi' = \frac{1}{\theta} e^{-2\varphi} \Re \int \beta |v|^2 v dx, \\ v(0, x) = v_0(x), \quad \varphi(0) = 0. \end{cases}$$

We look for the solution v in the neighborhood of the first approximation $\Phi = \mathcal{G}(t)v_0 = \mathcal{F}^{-1}e^{-\alpha\xi^2 t}\hat{v}_0 = \int G(t, x-y)v_0(y)dy$, which is a solution to the linear Cauchy problem $\Phi_t - \alpha\Phi_{xx} = 0$, $\Phi(0, x) = v_0$, where $G(t, x) = (1/\sqrt{4\pi\alpha t})e^{-x^2/4\alpha t}$. Then we put $r = v - \Phi$ and since Φ is nearly equal to $\theta G(t)$ we get the Cauchy problem

$$\begin{cases} r_t = \alpha r_{xx} - \beta e^{-2\varphi(t)} \theta^3 \left(|G|^2 - \int |G|^2 G dx \right) G - Q, \\ \varphi' = \theta^2 e^{-2\varphi(t)} \Re \int \beta |G|^2 G dx + R, \\ r(0, x) = 0, \quad \varphi(0) = 0, \end{cases}$$

where Q and R are considered as the remainder terms. By a simple computation we have $\Re \int \beta |G|^2 G(t, x) dx = (1/4\pi t) \Re \delta(\alpha, \beta)$, where $\delta(\alpha, \beta) = \beta/\sqrt{2|\alpha|^2 + \alpha^2}$. If $\Re \delta(\alpha, \beta) > 0$, then we can apply the contraction mapping principle to the above system (see Section 3) and find the decay estimate $\|r(t)\|_\infty \leq Ct^{-1/2}(\log(1+t))^{-1}$. Also we show in Section 3 that

$$e^{-2\varphi(t)} = \frac{C}{1 + \eta \log(1+t)}.$$

Hence from the representation $u = ve^{-\varphi+i\psi} = (r + \Phi)e^{-\varphi+i\psi}$ we get the decay estimate of the theorem.

When $\eta = 0$, i.e. $\Re \delta(\alpha, \beta) = 0$, the first summand in the right-hand side of equation for φ' vanishes so we need to investigate the second approximation, namely, we find the solution of the Cauchy problem (1.1) in the neighborhood of $\Phi + \Psi$, where the second approximation Ψ is defined by the linear Cauchy problem

$$\Psi_t - \alpha\Psi_{xx} + \beta e^{-2\varphi} \left(|\Phi|^2 - \frac{1}{\theta} \int |\Phi|^2 \Phi dx \right) \Phi = 0, \quad \Psi(0, x) = 0.$$

We again use the fact that $\Phi = \theta G + \text{remainder terms}$, to see that Ψ satisfies approximately to the equation

$$\Psi_t - \alpha \Psi_{xx} + \beta \theta^3 e^{-2\varphi} \left(|G|^2 - \int |G|^2 G dx \right) G = 0.$$

Putting $r = v - \Phi - \Psi$, we obtain the system

$$\begin{cases} r(t) = -\beta \int_0^t e^{-2\varphi(\tau)} \mathcal{G}(t-\tau) \left(|v|^2 v(\tau) - |\Phi(\tau)|^2 \Phi(\tau) \right. \\ \quad \left. - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx + \frac{\Phi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right) d\tau \\ \varphi'(t) = \frac{1}{\theta} e^{-2\varphi(t)} \Re \int \beta |v|^2 v(t, x) dx. \end{cases}$$

Below in Section 2 in Lemma 2.3 by virtue of the fact that $\Re \int \beta |G|^2 G(t, x) dx = 0$ we prove the representation $|v|^2 v - |\Phi|^2 \Phi = \theta^2 (2|G|^2 \Psi + G^2 \bar{\Psi}) + \text{remainder terms}$, and so we can approximate $\varphi(t)$ as the solution of the ordinary differential equation

$$\varphi'(t) = \theta^2 e^{-2\varphi(t)} \Re \int \beta (2|G|^2 \Psi + G^2 \bar{\Psi}) dx.$$

Since Ψ can be considered as a function of G multiplied by $e^{-2\varphi}$, the right-hand side of the above equation has a form $\mu e^{-4\varphi(t)} t^{-1} + \text{remainder term}$, where μ is a positive constant. Hence we obtain $\varphi'(t) = \mu e^{-4\varphi(t)} t^{-1} + \text{remainder term}$, whence it follows that $e^{4\varphi(t)} = 1 + \mu \log(1+t) + \text{remainder term}$. In Lemma 2.3 below in Section 2 we compute with more details the coefficient μ and the remainder term. Similar considerations we carry out in the case $\eta = 0$, $\mu = 0$. Then we need to take into account the third approximation of the perturbation theory. See for details Lemma 2.4 below in Section 2 and Section 5 for the proof of the theorem.

2. Preliminaries

The solution of the linear Cauchy problem

$$\begin{cases} u_t - \alpha u_{xx} = f(t, x), & x \in \mathbf{R}, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbf{R} \end{cases}$$

can be written by the Duhamel formula $u(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)f(\tau)d\tau$, where the Green operator $\mathcal{G}$ is given by

$$\mathcal{G}(t)\phi = \mathcal{F}^{-1} e^{-\alpha \xi^2 t} \hat{\phi}(\xi) = \int G(t, x-y)\phi(y)dy,$$

with the Green function $G(t, x) = (1/\sqrt{4\pi\alpha t})e^{-x^2/4\alpha t}$.

Lemma 2.1. *Let the function $f(t, x)$ have the zero mean value $\hat{f}(t, 0) = 0$ and satisfy the estimate $\|f(t)\|_{\mathbf{W}} \leq \varepsilon(1+t)^{-3/4-\lambda}$, for all $t > 0$, where $\lambda \in [0, 1/4)$. We also assume that the function $g(t)$ satisfies the inequalities*

$$\frac{1}{2}(1 + \eta \log(1+t)) \leq g(t) \leq 2(1 + \eta \log(1+t))$$

for all $t > 0$ with some $\eta > 0$. Then the following inequalities are valid

$$\begin{aligned} \sqrt[4]{1+t} \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{1,0} + \frac{1}{\sqrt[4]{t}} \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{0,1} &\leq C\varepsilon, \\ \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\infty} &\leq C\varepsilon g^{-q}(t)(1+t)^{-1/2-\lambda}, \end{aligned}$$

where $q \in (0, 1]$.

Proof. Since the function $f(t, x)$ has the zero mean value $\hat{f}(t, 0) = 0$, by the Plancherel theorem and the inequality $|\hat{f}(\tau, \xi)| = |\hat{f}(\tau, \xi) - \hat{f}(\tau, 0)| \leq C\sqrt{|\xi|} \|f(\tau)\|_{0,1}$ we get

$$\begin{aligned} &\|\mathcal{G}(t-\tau)f(\tau)\|^2 \\ &= \int e^{-2(\Re\alpha)\xi^2(t-\tau)} |\hat{f}(\tau, \xi)|^2 d\xi \leq C\|f(\tau)\|_{0,1}^2 \int e^{-2(\Re\alpha)\xi^2(t-\tau)} |\xi| d\xi \\ &= C(t-\tau)^{-1} \|f(\tau)\|_{0,1}^2 \int e^{-2(\Re\alpha)y^2} |y| dy \leq C(t-\tau)^{-1} \|f(\tau)\|_{0,1}^2. \end{aligned}$$

Since we also have $\|\mathcal{G}(t-\tau)f(\tau)\|^2 \leq C\|f(\tau)\|^2$, therefore

$$(2.1) \quad \|\mathcal{G}(t-\tau)f(\tau)\|^2 \leq C(1+t-\tau)^{-1} \|f(\tau)\|_{0,1}^2$$

for all $t > \tau > 0$. Similarly, we find that

$$(2.2) \quad \|\partial_x \mathcal{G}(t-\tau)f(\tau)\|^2 \leq C(1+t-\tau)^{-2} \|f(\tau)\|_{\mathbf{W}}^2.$$

In the same way as in the proof of (2.1) we obtain

$$\begin{aligned} &\|x\mathcal{G}(t-\tau)f(\tau)\|^2 \\ &\leq C \int |\partial_\xi(e^{-\alpha\xi^2(t-\tau)} \hat{f}(\tau, \xi))|^2 d\xi \leq C \int e^{-2\Re\alpha\xi^2(t-\tau)} |\partial_\xi \hat{f}(\tau, \xi)|^2 d\xi \\ &\quad + C(t-\tau)^2 \int |\xi|^2 e^{-2\Re\alpha\xi^2(t-\tau)} |\hat{f}(\tau, \xi)|^2 d\xi \\ &\leq C\|f(\tau)\|_{0,1}^2 + C(t-\tau)^2 \|f(\tau)\|_{0,1}^2 \int |\xi|^3 e^{-2\Re\alpha\xi^2(t-\tau)} d\xi \leq C\|f(\tau)\|_{0,1}^2. \end{aligned}$$

Thus in view of the estimate $g^{-q}(t) \leq C$ we get

$$\begin{aligned} & \sqrt[4]{1+t} \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{1,0} + \frac{1}{\sqrt[4]{t}} \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{0,1} \\ & \leq C\varepsilon \int_0^t \left(\frac{\sqrt[4]{1+t}}{\tau^{3/4}\sqrt{t-\tau}} + \frac{1}{\tau^{3/4}\sqrt[4]{t}} \right) d\tau \leq C\varepsilon. \end{aligned}$$

This implies the first estimate of the lemma. Note that for $0 \leq t \leq 4$ the second estimate follows from the first one. So we consider now $t > 4$. Inequalities (2.1) and (2.2) yield

$$\begin{aligned} \|\mathcal{G}(t-\tau)f(\tau)\|_\infty^2 & \leq C\|\mathcal{G}(t-\tau)f(\tau)\| \|\partial_x \mathcal{G}(t-\tau)f(\tau)\| \\ & \leq C(1+t-\tau)^{-3/2} \|f(\tau)\|_{\mathbf{W}}^2 \leq C(1+t-\tau)^{-3/2} (1+\tau)^{-3/2-2\lambda} \end{aligned}$$

for all $t > \tau > 0$. From the condition of the lemma for the function $g(t)$ we have the estimates $(1+t)^{\lambda/2-1/8} \leq Cg^{-q}(t)$ and $\sup_{\tau \in [\sqrt{t}, t]} g^{-q}(\tau) \leq C(1+\eta \log(1+\sqrt{t}))^{-q} \leq C(1+(\eta/2) \log(1+t))^{-q} \leq Cg^{-q}(t)$, whence

$$\begin{aligned} & \left\| \int_0^t g^{-q}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_\infty \\ & \leq C\varepsilon \int_0^t \frac{g^{-q}(\tau) d\tau}{(t-\tau)^{3/4} \tau^{3/4+\lambda}} \\ & \leq Ct^{-3/4} \left(\int_0^{\sqrt{t}} \frac{d\tau}{\tau^{3/4+\lambda}} + g^{-q}(t) \int_{\sqrt{t}}^{t/2} \frac{d\tau}{\tau^{3/4+\lambda}} + g^{-q}(t) \int_{t/2}^t \frac{d\tau}{(t-\tau)^{3/4+\lambda}} \right) \\ & \leq Ct^{-1/2-\lambda} (t^{\lambda/2-1/8} + g^{-q}(t)) \leq Cg^{-q}(t)(1+t)^{-1/2-\lambda} \end{aligned}$$

for all $t > 4$. Thus the second estimate of the lemma is true. Lemma 2.1 is proved. ■

The following lemma is needed to prove the result for the case $\Re\delta(\alpha, \beta) > 0$.

Lemma 2.2. *We assume that $v_0 \in \mathbf{W}$, the norm $\|v_0\|_{\mathbf{W}} = \varepsilon$ is sufficiently small, $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta/\sqrt{2\pi} \geq C\varepsilon > 0$, a function $v(t, x)$ satisfies the estimates $\|v(t)\|_\infty \leq \varepsilon(1+t)^{-1/2}$, $\|v(t)\| \leq \varepsilon(1+t)^{-1/4}$ for all $t > 0$ and has the asymptotic representation $\|v(t) - \mathcal{G}(t)v_0\|_\infty \leq C\varepsilon^{5/2}/(g(t)\sqrt{t+1})$ for $t > 0$. We also assume that the function $g(t)$ is such that*

$$\frac{1}{2}(1+\eta \log(1+t)) \leq g(t) \leq 2(1+\eta \log(1+t))$$

for $t > 0$, where $\eta = (\theta^2/2\pi)\Re\delta(\alpha, \beta) > 0$.

Then the following inequalities are valid

$$(2.3) \quad \frac{1}{\theta} \left| \Re \int \beta |v|^2 v(t, x) dx \right| \leq \frac{C\varepsilon^2}{t+1}$$

and

$$(2.4) \quad \begin{aligned} \frac{9}{10} (1 + \eta \log(t+1)) &\leq 1 + \frac{2}{\theta} \int_0^t \Re \int \beta |v|^2 v(\tau, x) dx d\tau \\ &\leq \frac{11}{10} (1 + \eta \log(1+t)) \end{aligned}$$

for all $t > 0$.

Proof. For $t \in [0, 1]$ the estimates (2.3), (2.4) are trivial. We consider the case $t > 1$. We have $\mathcal{G}(t)v_0 = \theta G(t) + R$ and the remainder term

$$R = (\mathcal{G}(t)v_0 - \theta G(t)) = \mathcal{F}^{-1} e^{-\alpha \xi^2 t} (\hat{v}_0(\xi) - \hat{v}_0(0))$$

has the zero mean value $\hat{R}(t, 0) = 0$, so by the Plancherel theorem (as in Lemma 2.1) we get $\|\partial_x R\|^2 \leq C\varepsilon^2 \int |\xi|^3 e^{-C\xi^2 t} d\xi \leq C\varepsilon^2 (1+t)^{-2}$. In the same manner we have the estimates $\|R\|_{0,1} \leq C\varepsilon$ and $\|R\|_\infty \leq C\varepsilon(1+t)^{-3/4}$. We have by an easy computation with $r = v - \mathcal{G}v_0$

$$\begin{aligned} \| |v|^2 v - \theta^3 |G|^2 G \|_1 &= \|\theta R \bar{G}(\theta G + \mathcal{G}v_0) + \bar{R}(\mathcal{G}v_0)^2 + r \overline{\mathcal{G}v_0}(\mathcal{G}v_0 + v) + \bar{r}v^2\|_1 \\ &\leq C(\|R\|_\infty + \|v - \mathcal{G}v_0\|_\infty)(\theta^2 \|G\|^2 + \|\mathcal{G}v_0\|^2 + \|v\|^2) \\ &\leq \frac{C\varepsilon^{9/2}}{g(t)(1+t)} + \frac{C\varepsilon^3}{(1+t)^{5/4}} \end{aligned}$$

and

$$\begin{aligned} &\theta^3 \Re \int \beta |G|^2 G(t, x) dx \\ &= \frac{\theta^3}{(4\pi t)^{3/2} |\alpha|} \Re \int \exp\left(-\frac{x^2}{4t} \left(\frac{2}{\alpha} + \frac{1}{\bar{\alpha}}\right)\right) \frac{\beta dx}{\sqrt{\alpha}} = \frac{\theta^3}{4\pi t} \Re \delta(\alpha, \beta). \end{aligned}$$

Therefore by virtue of the estimate $g^{-1}(t) \leq C(1 + \eta \log(1+t))^{-1}$ we get

$$\left| \Re \int \beta |v|^2 v(t, x) dx - \frac{\theta^3 \Re \delta(\alpha, \beta)}{4\pi(1+t)} \right| \leq \frac{C\varepsilon^{9/2}}{(1+t)(1 + \eta \log(1+t))} + \frac{C\varepsilon^3}{(1+t)^{5/4}}$$

for all $t > 0$, whence estimate (2.3) follows and

$$\begin{aligned}
& \left| \frac{2}{\theta} \int_0^t d\tau \Re \int \beta |v|^2 v(\tau, x) dx - \eta \log(1+t) \right| \\
& \leq \int_0^t \frac{C\varepsilon^{7/2} d\tau}{(1+\tau)(1+\eta \log(1+\tau))} + \int_0^t \frac{C\varepsilon^2 d\tau}{(1+\tau)^{5/4}} \\
& \leq C\varepsilon^2 + C\varepsilon^{3/2} \log(1+\eta \log(1+t)) \leq \varepsilon(1+\eta \log(1+t)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\eta \log(1+t) - \varepsilon(1+\eta \log(1+t)) & \leq \frac{2}{\theta} \int_0^t d\tau \Re \int \beta |v|^2 v(\tau, x) dx \\
& \leq \eta \log(1+t) + \varepsilon(1+\eta \log(1+t)),
\end{aligned}$$

hence

$$\begin{aligned}
1 + \eta \log(1+t) - \varepsilon(1+\eta \log(1+t)) & \leq 1 + \frac{2}{\theta} \int_0^t d\tau \Re \int \beta |v|^2 v(\tau, x) dx \\
& \leq 1 + \eta \log(1+t) + \varepsilon(1+\eta \log(1+t))
\end{aligned}$$

whence estimate (2.4) follows. Lemma 2.2 is proved. ■

We need the following lemma to prove the theorem in the case $\eta = 0$, $\mu > 0$. We define Ψ as a solution to the Cauchy problem

$$\Psi_t - \alpha \Psi_{xx} + \beta g^{-1/2} \left(|\Phi|^2 - \frac{1}{\theta} \int |\Phi|^2 \Phi dx \right) \Phi = 0, \quad \Psi(0, x) = 0,$$

where $\Phi = \mathcal{G}v_0$. We remind that $\mu = (\theta^4 (\Im \delta)^2 / (4\pi)^2) \Re(2v_1 - v_2)$, where $v_j = \log(4(1 - \sqrt{1 - \kappa_j}) / \{\kappa_j(1 + \sqrt{1 - \kappa_j})\})$, $\kappa_1 = (\alpha + \bar{\alpha})^2 / (\alpha + 2\bar{\alpha})^2$, $\kappa_2 = 2(|\alpha|^2 + \bar{\alpha}^2) / |\alpha + 2\bar{\alpha}|^2$.

Lemma 2.3. *We assume that $v_0 \in \mathbf{W}$, the norm $\|v_0\|_{\mathbf{W}} = \varepsilon$ is sufficiently small, $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta/\sqrt{2\pi} \geq C\varepsilon > 0$, $\eta = 0$, $\mu > 0$, a function $v(t, x)$ satisfies the estimates $\|v(t)\|_{\infty} \leq \varepsilon(1+t)^{-1/2}$, $\|v(t)\| \leq \varepsilon(1+t)^{-1/4}$ for all $t > 0$ and has the asymptotic representation*

$$\|v(t) - \Phi - \Psi\|_{\infty} \leq \frac{C\varepsilon^{9/2}}{g(t)\sqrt{t+1}}$$

for all $t > 0$. We also assume that the function $g(t)$ is such that $\frac{1}{2}(1 + \mu \log(1+t)) \leq g(t) \leq 2(1 + \mu \log(1+t))$ and $|g'(t)| \leq C\varepsilon^4/(1+t)$ for all $t > 0$.

Then the following inequalities are valid

$$(2.5) \quad \frac{1}{\theta} \left| \sqrt{g(t)} \Re \int \beta |v|^2 v(t, x) dx \right| \leq \frac{C\varepsilon^2}{t+1}$$

and

$$(2.6) \quad \begin{aligned} \frac{9}{10} (1 + \mu \log(t+1)) &\leq 1 + \frac{4}{\theta} \int_0^t d\tau \sqrt{g(\tau)} \Re \int \beta |v|^2 v(\tau, x) dx \\ &\leq \frac{11}{10} (1 + \mu \log(1+t)) \end{aligned}$$

for all $t > 0$.

Proof. As in the proof of Lemma 2.2 we have only to consider the case $t \geq 1$. We put $r = v - \Phi - \Psi$ and write the identity

$$(2.7) \quad \begin{aligned} |v|^2 v - |\Phi|^2 \Phi - 2|\Phi|^2 \Psi - \Phi^2 \bar{\Psi} \\ = O(|r|(|v|^2 + |\Phi|^2) + |\Psi|^2(|\Phi| + |v| + |\Psi|)). \end{aligned}$$

First let us prove the representation

$$(2.8) \quad \Psi = \frac{\theta^3 \delta}{4\pi} g^{-1/2}(t) \Gamma(t) + O(\varepsilon^3 (1+t)^{-1/2} g^{-3/2}(t) e^{-Cx^2/t}),$$

where

$$\Gamma(t) = -\frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \left(\frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha t(1-\sigma z))} - e^{-x^2/(4\alpha t)} \right) \frac{dz}{z}$$

with $\sigma = (\bar{\alpha} + \alpha)/(2\bar{\alpha} + \alpha)$. We have

$$\begin{aligned} \Psi &= -\beta \theta^3 \int_0^t g^{-1/2}(\tau) \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) d\tau \\ &\quad - \beta \int_0^t g^{-1/2}(\tau) \mathcal{G}(t-\tau) \left(|\Phi|^2 \Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi|^2 \Phi(\tau, x) dx \right. \\ &\quad \left. - \theta^3 |G|^2 G(\tau) - G(\tau) \theta^3 \int |G|^2 G(\tau, x) dx \right) d\tau. \end{aligned}$$

Dividing the domain of integration into two parts: $0 < \tau < \sqrt{t}$ and $\sqrt{t} < \tau < t$ and integrating by parts with respect to y in the first domain $0 < \tau < \sqrt{t}$ we get

$$\begin{aligned}
& \int_0^t \left| g^{-1/2}(\tau) \mathcal{G}(t-\tau) \left(|\Phi|^2 \Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi|^2 \Phi(\tau, x) dx \right. \right. \\
& \quad \left. \left. - \theta^3 |G|^2 G(\tau) - G(\tau) \theta^3 \int |G|^2 G(\tau, x) dx \right) \right| d\tau \\
& \leq C\varepsilon^3 \int_0^{\sqrt{t}} \frac{d\tau}{\sqrt{t-\tau}} \left(\int_{-\infty}^0 dy e^{-C(x-y)^2/(t-\tau)} \frac{|x-y|}{(t-\tau)\tau^{3/2}} \int_{-\infty}^y e^{-Cz^2/\tau} dz \right. \\
& \quad \left. + \int_0^{\infty} dy e^{-C(x-y)^2/(t-\tau)} \frac{|x-y|}{(t-\tau)\tau^{3/2}} \int_y^{\infty} e^{-Cz^2/\tau} dz \right) \\
& \quad + C\varepsilon^3 \int_{\sqrt{t}}^t \frac{d\tau}{\sqrt{t-\tau}} \int dy e^{-C(x-y)^2/(t-\tau)} \frac{1}{\tau^{7/4}} e^{-Cy^2/\tau} \\
& \leq C\varepsilon^3 e^{-Cx^2/t} \left(\int_0^{\sqrt{t}} \frac{d\tau}{(t-\tau)\sqrt{\tau}} + \int_{\sqrt{t}}^t \frac{d\tau}{\sqrt{t-\tau}\tau^{5/4}} \right) \\
& \leq C\varepsilon^3 e^{-Cx^2/t} (1+t)^{-5/8}.
\end{aligned}$$

Thus

$$\begin{aligned}
\Psi &= -\beta\theta^3 \int_0^t g^{-1/2}(\tau) \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) d\tau \\
& \quad + O(\varepsilon^3 (1+t)^{-5/8} e^{-Cx^2/t}).
\end{aligned}$$

Then as above dividing the domain of integration into two parts: $0 < \tau < \sqrt{t}$ and $\sqrt{t} < \tau < t$ and integrating by parts with respect to time t we obtain

$$\begin{aligned}
\Psi &= -\beta\theta^3 g^{-1/2}(t) \int_0^t \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) d\tau \\
& \quad + C\varepsilon^3 \int_0^t d\tau g^{-3/2}(\tau) g'(\tau) \int_0^{\tau} \mathcal{G}(t-s) \left(|G|^2 G(s) - G(s) \int |G|^2 G(s) dx \right) ds \\
& \quad + O(\varepsilon^3 (1+t)^{-5/8} e^{-Cx^2/t}) \\
&= -\beta\theta^3 g^{-1/2}(t) \int_0^t \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) d\tau \\
& \quad + C\varepsilon^3 \int_0^t d\tau g^{-3/2}(\tau) O((1+\tau)^{-3/4} t^{-3/4} e^{-Cx^2/t}) + O(\varepsilon^3 (1+t)^{-5/8} e^{-Cx^2/t}) \\
&= -\beta\theta^3 g^{-1/2}(t) \int_0^t \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) d\tau \\
& \quad + O(\varepsilon^3 (1+t)^{-1/2} g^{-3/2}(t) e^{-Cx^2/t}).
\end{aligned}$$

By a direct calculation we have

$$\begin{aligned} |G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx &= \frac{1}{(4\pi\tau)^{3/2} \sqrt{\alpha}} \left(\frac{1}{|\alpha|} \exp\left(-\frac{x^2}{4\tau} \left(\frac{2}{\alpha} + \frac{1}{\bar{\alpha}}\right)\right) \right. \\ &\quad \left. - \frac{1}{\sqrt{2|\alpha|^2 + \alpha^2}} \exp\left(-\frac{x^2}{4\alpha\tau}\right) \right). \end{aligned}$$

Hence using the formula $\mathcal{G}(t-\tau)e^{-bx^2} = 1/\sqrt{1+4\alpha b(t-\tau)} \exp(-bx^2/\{1+4\alpha b(t-\tau)\})$ we obtain

$$\begin{aligned} &\beta \mathcal{G}(t-\tau) \left(|G|^2 G(\tau) - G(\tau) \int |G|^2 G(\tau) dx \right) \\ &= \frac{\beta}{(4\pi\tau)^{3/2} |\alpha| \sqrt{\alpha}} \left(\mathcal{G}(t-\tau) \exp\left(-\frac{x^2}{4\tau} \left(\frac{2}{\alpha} + \frac{1}{\bar{\alpha}}\right)\right) \right. \\ &\quad \left. - \frac{|\alpha|}{\sqrt{2|\alpha|^2 + \alpha^2}} \mathcal{G}(t-\tau) \exp\left(-\frac{x^2}{4\alpha\tau}\right) \right) \\ &= \frac{\delta}{(4\pi)^{3/2} \tau \sqrt{\alpha}} \left(\frac{1}{\sqrt{t-\sigma\tau}} e^{-x^2/(4\alpha(t-\sigma\tau))} - \frac{1}{\sqrt{t}} e^{-x^2/(4\alpha t)} \right), \end{aligned}$$

where $\sigma = (\bar{\alpha} + \alpha)/(2\bar{\alpha} + \alpha)$. This implies

$$\begin{aligned} \Psi &= -\frac{\theta^3 \delta}{(4\pi)^{3/2} \sqrt{\alpha}} g^{-1/2}(t) \int_0^t \left(\frac{1}{\sqrt{t-\sigma\tau}} e^{-x^2/(4\alpha(t-\sigma\tau))} - \frac{1}{\sqrt{t}} e^{-x^2/(4\alpha t)} \right) \frac{d\tau}{\tau} \\ &\quad + O(\varepsilon^3 (1+t)^{-1/2} g^{-3/2}(t) e^{-Cx^2/t}), \end{aligned}$$

whence (2.8) follows. By (2.7) and (2.8) we have

$$(2.9) \quad \| |v|^2 v - |\Phi|^2 \Phi - 2|\Phi|^2 \Psi - \Phi^2 \bar{\Psi} \|_1 \leq C\varepsilon^5 (1+t)^{-1} g^{-1}(t).$$

We next prove (2.5) and (2.6). We have by (2.7), (2.9) and the fact that $\eta = 0$, i.e. $\Re \int \beta |\Phi|^2 \Phi(\tau, x) dx = O(\varepsilon^3 (1+t)^{-1-\lambda})$

$$\begin{aligned} (2.10) \quad \Re \int \beta |v|^2 v(\tau, x) dx &= 2\Re \int \beta |\Phi|^2 \Psi(\tau, x) dx \\ &\quad + \Re \int \beta \Phi^2 \bar{\Psi}(\tau, x) dx + O(\varepsilon^2 (1+t)^{-1} g^{-1}(t)). \end{aligned}$$

We consider the right hand side of (2.10). Since $\Phi = \theta G + O(\varepsilon t^{-3/4})$ (see the proof of Lemma 2.2) and by virtue of (2.8) we get

$$\begin{aligned} \int |\Phi|^2 \Psi(t, x) dx &= \theta^2 \int |G|^2 \Psi(t, x) dx + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)) \\ &= -\frac{\theta^5 \delta}{(4\pi)^2 \sqrt{\alpha t} |\alpha| t} g^{-1/2}(t) \int_0^1 \frac{dz}{z} \\ &\quad \times \int \left(\frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha t(1-\sigma z)) - (x^2/4t)(1/\alpha + 1/\bar{\alpha})} - e^{-(x^2/4t)(2/\alpha + 1/\bar{\alpha})} \right) dx \\ &\quad + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)). \end{aligned}$$

We have by a simple calculation

$$\begin{aligned} \frac{1}{\sqrt{t}} \int \frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha t(1-\sigma z)) - (x^2/4t)(1/\alpha + 1/\bar{\alpha})} dx &- \int e^{-(x^2/4t)(2/\alpha + 1/\bar{\alpha})} dx \\ &= \frac{2\sqrt{\pi}|\alpha|}{\sqrt{2\bar{\alpha} + \alpha}} \left(\frac{1}{\sqrt{1-\sigma^2 z}} - 1 \right). \end{aligned}$$

Thus we get

$$(2.11) \quad \int |\Phi|^2 \Psi(t, x) dx = -\frac{\theta^5 \delta^2}{(4\pi)^2 \beta t} g^{-1/2}(t) v_1 + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)),$$

where $v_1 = \int_0^1 ((1/\sqrt{1-\sigma^2 z}) - 1)(dz/z) = \log(4(1-\sqrt{1-\kappa_1})/\{\kappa_1(1+\sqrt{1-\kappa_1})\})$, $\kappa_1 = \sigma^2 = (\bar{\alpha} + \alpha)^2/(2\bar{\alpha} + \alpha)^2$. In the same manner we consider the term $\int \Phi^2 \bar{\Psi}(t, x) dx$. We have

$$\begin{aligned} (2.12) \quad &\int \Phi^2 \bar{\Psi}(t, x) dx \\ &= \frac{\theta^2}{4\alpha\pi t} \int e^{-(2x^2/4t\alpha)} \bar{\Psi}(\tau, x) dx + O(\varepsilon^5(1+t)^{-1-\lambda}) \\ &= -\frac{\theta^5 \bar{\delta}}{(4\pi)^{5/2} t \alpha \sqrt{\bar{\alpha}} t} g^{-1/2}(t) \iint_0^1 \left(\frac{1}{\sqrt{1-\bar{\sigma} z}} \exp\left(-\frac{x^2}{4\bar{\alpha} t(1-\bar{\sigma} z)} - \frac{x^2}{2t\alpha}\right) \right. \\ &\quad \left. - \exp\left(-\frac{x^2}{4t} \left(\frac{1}{\bar{\alpha}} + \frac{2}{\alpha}\right)\right) \right) \frac{dz}{z} dx + O(\varepsilon^5(1+t)^{-1-\lambda}) \\ &= -\frac{\theta^5 |\delta|^2}{(4\pi)^2 \beta t} g^{-1/2}(t) v_2 + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)), \end{aligned}$$

where $v_2 = \int_0^1 ((1/\sqrt{1-\kappa_2 z}) - 1) dz/z = \log(4(1-\sqrt{1-\kappa_2})/\{\kappa_2(1+\sqrt{1-\kappa_2})\})$, $\kappa_2 = 2\bar{\alpha}\bar{\sigma}/(\alpha + 2\bar{\alpha}) = 2(\bar{\alpha}^2 + |\alpha|^2)/|2\alpha + \bar{\alpha}|^2$. Then since $\Re\delta(\alpha, \beta) = 0$ we have by (2.10), (2.11) and (2.12)

$$(2.13) \quad \Re \int \beta |v|^2 v(\tau, x) dx = \frac{(\Im\delta)^2 \theta^5}{(4\pi)^2 t} g^{-1/2}(t) \Re(2v_1 - v_2) + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)) \\ = -\frac{\theta\mu}{4t} g^{-1/2}(t) + O(\varepsilon^3(1+t)^{-1} g^{-1}(t)).$$

Via (2.13) we get (2.5). We again use (2.13) to obtain

$1 + (4/\theta) \int_0^t \sqrt{g(\tau)} \Re \int \beta |v|^2 v(\tau) dx d\tau = 1 + \mu \log(1+t) + O(\varepsilon^2 \sqrt{\log(1+t)})$, whence we have (2.6). Lemma 2.3 is proved. ■

We need the following lemma in order to prove Theorem 1.1 in the case $\eta = 0$, $\mu = 0$. We define Ψ as a solution to the Cauchy problem

$$\Psi_t - \alpha \Psi_{xx} + \beta g^{-1/3} \left(|\Phi|^2 - \frac{1}{\theta} \int |\Phi|^2 \Phi dx \right) \Phi = 0, \quad \Psi(0, x) = 0,$$

where $\Phi = \mathcal{G}v_0$ and let V be a solution to the Cauchy problem

$$V_t - \alpha V_{xx} + \beta g^{-1/3} \left(2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau) \right. \\ \left. - \frac{\Phi(\tau)}{\theta} \int (2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau)) dx d\tau - \frac{\Psi(\tau)}{\theta} \int |\Phi|^2 \Phi(\tau) dx \right) d\tau = 0, \\ V(0, x) = 0,$$

thus we have

$$\Psi(t) = -\beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \left(|\Phi|^2 \Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi|^2 \Phi(\tau) dx \right) d\tau,$$

and

$$V(t) = -\beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \left(2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau) \right. \\ \left. - \frac{\Phi(\tau)}{\theta} \int (2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau)) dx d\tau - \frac{\Psi(\tau)}{\theta} \int |\Phi|^2 \Phi(\tau) dx \right) d\tau.$$

Lemma 2.4. *We assume that $v_0 \in \mathbf{W}$, the norm $\|v_0\|_{\mathbf{W}} = \varepsilon$ is sufficiently small, $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta/\sqrt{2\pi} \geq C\varepsilon > 0$, $\eta = 0$, $\mu = 0$, a function $v(t, x)$ satisfies*

the estimates $\|v(t)\|_\infty \leq \varepsilon(1+t)^{-1/2}$, $\|v(t)\| \leq \varepsilon(1+t)^{-1/4}$ for all $t > 0$ and has the asymptotic representation

$$\|v(t) - \Phi - \Psi - V\|_\infty \leq \frac{C\varepsilon^6}{g(t)\sqrt{t+1}}$$

for all $t > 0$. We also assume that the function $g(t)$ is such that $\frac{1}{2}(1 + \kappa \log(1+t)) \leq g(t) \leq 2(1 + \kappa \log(1+t))$ and $|g'(t)| \leq C\varepsilon^4/(1+t)$ for all $t > 0$, where $\kappa > 0$ is defined below in (2.28).

Then the following inequalities are valid

$$(2.14) \quad \frac{1}{\theta} \left| \sqrt[3]{g^2(t)} \Re \int \beta |v|^2 v(t, x) dx \right| \leq \frac{C\varepsilon^2}{t+1}$$

and

$$(2.15) \quad \begin{aligned} \frac{9}{10} (1 + \kappa \log(t+1)) &\leq 1 + \frac{6}{\theta} \int_0^t d\tau \sqrt[3]{g^2(\tau)} \Re \int \beta |v|^2 v(\tau, x) dx \\ &\leq \frac{11}{10} (1 + \kappa \log(1+t)) \end{aligned}$$

for all $t > 0$.

Proof. As in the proof of Lemma 2.2 we need only to consider the case $t \geq 1$. We put $r = v - \Phi - \Psi - V$ and write the identity

$$(2.16) \quad \begin{aligned} |v|^2 v - |\Phi|^2 \Phi - 2|\Phi|^2 \Psi - \Phi^2 \bar{\Psi} - 2\Phi |\Psi|^2 - \bar{\Phi} \Psi^2 - 2|\Phi|^2 V - \Phi^2 \bar{V} \\ = O(|r|(|v|^2 + |\Phi|^2 + |\Psi|^2 + |V|^2) \\ + |V|(|\Psi| + |V| + |r|)(|v| + |\Phi| + |\Psi| + |V|) + |\Psi|^3). \end{aligned}$$

In the same manner as in the proof of Lemma 2.3 we obtain the representation

$$(2.17) \quad \Psi = \frac{\theta^3 \delta}{4\pi} g^{-1/3}(t) \Gamma(t) + O(\varepsilon^3 g^{-4/3}(t)(1+t)^{-1/2}),$$

where

$$\Gamma(t) = -\frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \left(\frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha t(1-\sigma z))} - e^{-x^2/(4\alpha t)} \right) \frac{dz}{z},$$

as above $\sigma = (\bar{\alpha} + \alpha)/(2\bar{\alpha} + \alpha)$. Now let us prove the asymptotic formula

$$(2.18) \quad V = \frac{\theta^5 \delta^2}{(4\pi)^2} g^{-2/3}(t) A(t) + O(e^5 e^{-Cx^2/t} g^{-4/3}(t) (1+t)^{-1/2}),$$

where

$$\begin{aligned} A(t) = & \frac{2}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z\sqrt{1-\sigma^2 z}} \int_0^1 \frac{d\xi}{\xi} \\ & \times \left(\frac{1}{\sqrt{1-h\xi}} e^{-x^2/(4\alpha t(1-h\xi))} - \frac{1}{\sqrt{1-\sigma\xi}} e^{-x^2/(4\alpha t(1-\sigma\xi))} \right) \\ & + \frac{2}{\sqrt{4\pi\alpha t}} \int_0^1 \left(\frac{1}{\sqrt{1-\sigma^2 z}} - 1 \right) \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{\sqrt{1-\sigma\xi}} e^{-x^2/(4\alpha t(1-\sigma\xi))} - e^{-x^2/(4\alpha t)} \right) \\ & - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z\sqrt{1-\kappa_2 z}} \\ & \times \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{\sqrt{1-a\xi}} e^{-x^2/(4\alpha t(1-a\xi))} - \frac{1}{\sqrt{1-\sigma\xi}} e^{-x^2/(4\alpha t(1-\sigma\xi))} \right) \\ & - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \left(\frac{1}{\sqrt{1-\kappa_2 z}} - 1 \right) \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{\sqrt{1-\sigma\xi}} e^{-x^2/(4\alpha t(1-\sigma\xi))} - e^{-x^2/(4\alpha t)} \right) \\ & - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{\sqrt{1-\sigma z\xi}} e^{-x^2/(4\alpha t(1-\sigma z\xi))} - e^{-x^2/(4\alpha t)} \right), \end{aligned}$$

where $h = (\sigma - bz)/(1 - \sigma^2 z)$, $a = \{\sigma - (\kappa_2/2)z\}/(1 - \kappa_2 z)$, $\sigma = (\alpha + \bar{\alpha})/(\alpha + 2\bar{\alpha})$, $b = \alpha\sigma/(\alpha + 2\bar{\alpha})$. We have

$$\begin{aligned} (2.19) \quad V = & -\frac{\beta\delta\theta^5}{4\pi} \int_0^t g^{-2/3}(\tau) \mathcal{G}(t-\tau) \\ & \times \left(\left(2|G|^2 \Gamma - G^2 \bar{\Gamma} - G \int (2|G|^2 \Gamma - G^2 \bar{\Gamma}) dx - \Gamma \int |G|^2 G dx \right) d\tau \right. \\ & \left. - \beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \tilde{R}(\tau) d\tau, \right. \end{aligned}$$

with

$$\begin{aligned} \tilde{R} = & \left(2|\Phi|^2 \Psi + \Phi^2 \bar{\Psi} - \frac{\Phi}{\theta} \int (2|\Phi|^2 \Psi + \Phi^2 \bar{\Psi}) dx - \frac{\Psi}{\theta} \int |\Phi|^2 \Phi dx \right) \\ & - \frac{\delta\theta^5}{4\pi} g^{-1/3} \left(2|G|^2 \Gamma - G^2 \bar{\Gamma} - G \int (2|G|^2 \Gamma - G^2 \bar{\Gamma}) dx - \Gamma \int |G|^2 G dx \right). \end{aligned}$$

As above we have $\int |G|^2 G(\tau, x) dx = \delta/(4\pi\beta\tau)$ and similarly $\int (2|G|^2 \Gamma - G^2 \bar{\Gamma}) dx = -(\delta/(4\pi\beta\tau))(2\nu_1 - \nu_2)$. Then integrating by parts as in the proof of Lemma 2.3 we get from (2.19)

$$(2.20) \quad V = -\frac{\beta\delta\theta^5}{(4\pi)^2} g^{-2/3}(t) \int_0^t \mathcal{G}(t-\tau) \left(\frac{2}{|\alpha|} \exp\left(-\frac{x^2}{4\tau} \left(\frac{1}{\alpha} + \frac{1}{\bar{\alpha}}\right)\right) \Gamma(\tau) \right. \\ \left. - \frac{1}{\alpha} \exp\left(-\frac{x^2}{2\alpha\tau}\right) \bar{\Gamma}(\tau) + \frac{\delta}{\beta} (2\nu_1 - \nu_2) G(\tau) - \frac{\delta}{\beta} \Gamma(\tau) \right) \frac{d\tau}{\tau} \\ + O(\varepsilon^3 e^{-Cx^2/t} g^{-4/3}(t)(1+t)^{-1/2}).$$

Hence using the formula $\mathcal{G}(t-\tau)e^{-bx^2} = (1/\sqrt{1+4\alpha b(t-\tau)}) \exp(-bx^2/\{1+4\alpha b(t-\tau)\})$ we obtain

$$\begin{aligned} \Lambda(t) &= \frac{2\beta}{\sqrt{4\pi\alpha}|\alpha|\delta} \int_0^t \frac{d\tau}{\tau^{3/2}} \\ &\quad \times \int_0^1 \frac{dz}{z} \mathcal{G}(t-\tau) \left(\frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha\tau(1-\sigma z)) - (x^2(\alpha+\bar{\alpha})/(4\tau\alpha\bar{\alpha}))} - e^{-(x^2(\alpha+2\bar{\alpha})/(4\tau\alpha\bar{\alpha}))} \right) \\ &\quad - \frac{\beta\delta}{\sqrt{4\pi\alpha}|\alpha|} \int_0^t \frac{d\tau}{\tau^{3/2}} \\ &\quad \times \int_0^1 \frac{dz}{z} \mathcal{G}(t-\tau) \left(\frac{1}{\sqrt{1-\bar{\sigma}z}} e^{-x^2/(4\bar{\alpha}\tau(1-\bar{\sigma}z)) - x^2/(2\alpha\tau)} - e^{-(x^2(\alpha+2\bar{\alpha})/(4\tau\alpha\bar{\alpha}))} \right) \\ &\quad + \frac{(2\nu_1 - \nu_2)}{\sqrt{4\pi\alpha}} \int_0^t \frac{d\tau}{\tau^{3/2}} \mathcal{G}(t-\tau) e^{-x^2/(4\alpha\tau)} \\ &\quad - \frac{1}{\sqrt{4\pi\alpha}} \int_0^t \frac{d\tau}{\tau^{3/2}} \int_0^1 \frac{dz}{z} \mathcal{G}(t-\tau) \left(\frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha\tau(1-\sigma z))} - e^{-x^2/(4\alpha\tau)} \right) \\ &= \frac{2}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z\sqrt{1-\sigma^2 z}} \int_0^1 \frac{d\xi}{\xi\sqrt{1-h\xi}} e^{-x^2/(4\alpha t(1-h\xi))} \\ &\quad - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z\sqrt{1-\kappa_2 z}} \int_0^1 \frac{d\xi}{\xi\sqrt{1-a\xi}} e^{-x^2/(4\alpha t(1-a\xi))} \\ &\quad - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi\sqrt{1-\sigma\xi}} e^{-x^2/(4\alpha t(1-\sigma\xi))} \\ &\quad + \frac{(2\nu_1 - \nu_2)}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{d\xi}{\xi} e^{-x^2/(4\alpha t)} \\ &\quad - \frac{1}{\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{\sqrt{1-\sigma z\xi}} e^{-x^2/(4\alpha t(1-\sigma z\xi))} - e^{-x^2/(4\alpha t)} \right), \end{aligned}$$

whence we obtain (2.18). By (2.16) and (2.18) we have

$$\begin{aligned}
 (2.21) \quad & \| |v|^2 v - |\Phi|^2 \Phi - 2|\Phi|^2 \Psi - \Phi^2 \bar{\Psi} \\
 & - 2\Phi |\Psi|^2 - \bar{\Phi} \Psi^2 - 2|\Phi|^2 V - \Phi^2 \bar{V} \|_1 \\
 & \leq C\varepsilon^6 (1+t)^{-1} g^{-1}(t).
 \end{aligned}$$

We next prove (2.14) and (2.15). By a direct calculations we get

$$\begin{aligned}
 (2.22) \quad \int G|\Gamma|^2 dx &= \frac{1}{4\pi|\alpha|t\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \int dx \left(e^{-x^2/(2\alpha t) - x^2/(4\alpha t)} \right. \\
 &+ \frac{1}{\sqrt{1-\sigma z}\sqrt{1-\bar{\sigma}\bar{\xi}}} e^{-x^2/(4\alpha t) - x^2/(4\alpha t(1-\sigma z)) - x^2/(4\alpha t(1-\bar{\sigma}\bar{\xi}))} \\
 &- \frac{1}{\sqrt{1-\sigma z}} e^{-x^2/(4\alpha t) - x^2/(4\alpha t(1-\sigma z)) - x^2/(4\alpha t)} \\
 &\left. - \frac{1}{\sqrt{1-\bar{\sigma}\bar{\xi}}} e^{-x^2/(2\alpha t) - x^2/(4\alpha t(1-\bar{\sigma}\bar{\xi}))} \right) \\
 &= \frac{1}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(1 - \frac{1}{\sqrt{1-\sigma^2 z}} \right. \\
 &\left. + \frac{1}{\sqrt{1-\sigma^2 z - \kappa_2 \xi} + \frac{\sigma}{2}\kappa_2 z \xi}} - \frac{1}{\sqrt{1-\kappa_2 \xi}} \right) \\
 &= \frac{1}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}} \int_0^1 \frac{dz}{z} \left(\frac{v\left(\frac{\sigma\left(1-\frac{\sigma}{2}z\right)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}} - v(\kappa_2) \right)
 \end{aligned}$$

where $v(y) = \int_0^1 (1/\sqrt{1-y\xi} - 1)d\xi/\xi = \log(4(1-\sqrt{1-y})/\{y(1+\sqrt{1-y})\})$, and in the same manner we have

$$(2.23) \quad \int \bar{G}\Gamma^2 dx = \frac{1}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}} \int_0^1 \frac{dz}{z} \left(\frac{v\left(\frac{\sigma(\sigma - bz)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}} - v(\sigma^2) \right),$$

where $b = \alpha\sigma/(\alpha + 2\bar{\alpha})$. Then by an easy calculations we obtain

$$\begin{aligned}
 (2.24) \quad & \int |G|^2 \bar{A} dx \\
 &= \frac{1}{4\pi|\alpha|t\sqrt{4\pi\alpha t}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\bar{\zeta}}{\bar{\zeta}} \int dx e^{-(x^2(\alpha+\bar{\alpha})/(4\alpha\bar{\alpha}t))} \\
 &\quad \times \left(\frac{2}{\sqrt{1-\sigma^2z}} \left(\frac{1}{\sqrt{1-h\bar{\zeta}}} e^{-x^2/(4\alpha t(1-h\bar{\zeta}))} - \frac{1}{\sqrt{1-\sigma\bar{\zeta}}} e^{-x^2/(4\alpha t(1-\sigma\bar{\zeta}))} \right) \right. \\
 &\quad + 2 \left(\frac{1}{\sqrt{1-\sigma^2z}} - 1 \right) \left(\frac{1}{\sqrt{1-\sigma\bar{\zeta}}} e^{-x^2/(4\alpha t(1-\sigma\bar{\zeta}))} - e^{-x^2/(4\alpha t)} \right) \\
 &\quad - \frac{1}{\sqrt{1-\kappa_2z}} \left(\frac{1}{\sqrt{1-a\bar{\zeta}}} e^{-x^2/(4\alpha t(1-a\bar{\zeta}))} - \frac{1}{\sqrt{1-\sigma\bar{\zeta}}} e^{-x^2/(4\alpha t(1-\sigma\bar{\zeta}))} \right) \\
 &\quad - \left(\frac{1}{\sqrt{1-\kappa_2z}} - 1 \right) \left(\frac{1}{\sqrt{1-\sigma\bar{\zeta}}} e^{-x^2/(4\alpha t(1-\sigma\bar{\zeta}))} - e^{-x^2/(4\alpha t)} \right) \\
 &\quad \left. - \left(\frac{1}{\sqrt{1-\sigma z\bar{\zeta}}} e^{-x^2/(4\alpha t(1-\sigma z\bar{\zeta}))} - e^{-x^2/(4\alpha t)} \right) \right) \\
 &= \frac{1}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}} \int_0^1 \frac{dz}{z} \left(\frac{2(v(\sigma h) - v(\sigma^2))}{\sqrt{1-\sigma^2z}} \right. \\
 &\quad \left. - \frac{(v(\sigma a) - v(\sigma^2))}{\sqrt{1-\kappa_2z}} - v(\sigma^2z) \right) + \frac{v(\sigma^2)(2v(\sigma^2) - v(\kappa_2))}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}},
 \end{aligned}$$

and in the same way we get

$$\begin{aligned}
 (2.25) \quad & \int G^2 \bar{A} dx = \frac{1}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}} \\
 &\quad \times \int_0^1 \frac{dz}{z} \left(\frac{2(v(\sigma h) - v(\sigma^2))}{\sqrt{1-\sigma^2z}} - \frac{(v(\sigma a) - v(\sigma^2))}{\sqrt{1-\kappa_2z}} - v(\sigma^2z) \right) \\
 &\quad + \frac{v(\sigma^2)(2v(\sigma^2) - v(\kappa_2))}{4\pi t\sqrt{\alpha^2 + 2|\alpha|^2}}.
 \end{aligned}$$

Since $\eta = 0$ and $\mu = 0$, i.e.

$$\Re \int \beta |\Phi|^2 \Phi(\tau, x) dx = O(\varepsilon^3(1+t)^{-1-\lambda}) \quad \text{and}$$

$\Re \int \beta (\Phi^2 \bar{\Psi}(\tau, x) + |\Phi|^2 \Psi(\tau, x)) dx = O(\varepsilon^5(1+t)^{-1}g^{-1}(t))$, using the asymptotics $\Phi = \theta G + O(\varepsilon t^{-3/4})$, formulas (2.17), (2.18) and (2.22)–(2.25) we obtain

$$\begin{aligned}
(2.26) \quad & \Re \int \beta |v|^2 v dx \\
&= \frac{\theta^7 \delta^2}{(4\pi)^2} g^{-2/3}(t) \Re \int \beta (-2G|\Gamma|^2 + \bar{G}\Gamma^2 + 2|G|^2 A + G^2 \bar{A}) dx \\
&\quad + O(C\varepsilon^3(1+t)^{-1}g^{-1}(t)) \\
&= \frac{\theta^7 i \delta^3}{(4\pi)^3 t} g^{-2/3}(t) \Im \left(\int_0^1 \frac{dz}{z} \left(\frac{2v \left(\frac{\kappa_2 \left(1 - \frac{\sigma}{2} z \right)}{1 - \sigma^2 z} \right)}{\sqrt{1 - \sigma^2 z}} - 2v(\kappa_2) + 3v(\sigma^2 z) \right. \right. \\
&\quad \left. \left. - \frac{v \left(\frac{\sigma(\sigma - bz)}{1 - \sigma^2 z} \right)}{\sqrt{1 - \sigma^2 z}} + v(\sigma^2) - \frac{6(v(\sigma h) - v(\sigma^2))}{\sqrt{1 - \sigma^2 z}} + \frac{3(v(\sigma a) - v(\sigma^2))}{\sqrt{1 - \kappa_2 z}} \right) \right. \\
&\quad \left. \left. - 3v(\sigma^2)(2v(\sigma^2) - v(\kappa_2)) \right) \right) + O(C\varepsilon^3(1+t)^{-1}g^{-1}(t)).
\end{aligned}$$

Via (2.26) we get (2.14) and the following asymptotics

$$\begin{aligned}
(2.27) \quad & 1 + \frac{6}{\theta} \int_0^t \sqrt[3]{g^2(\tau)} \Re \int \beta |v|^2 v(\tau) dx d\tau \\
&= 1 + \kappa \log(1+t) + O\left(\varepsilon^2 \sqrt[3]{\log^2(1+t)}\right),
\end{aligned}$$

where

$$\begin{aligned}
(2.28) \quad & \kappa = \frac{6i\delta^3\theta^6}{(4\pi)^3} \Im \left(\int_0^1 \frac{dz}{z} \left(\frac{2v \left(\frac{\kappa_2 \left(1 - \frac{\sigma}{2} z \right)}{1 - \sigma^2 z} \right)}{\sqrt{1 - \sigma^2 z}} - 2v(\kappa_2) - \frac{v \left(\frac{\sigma(\sigma - bz)}{1 - \sigma^2 z} \right)}{\sqrt{1 - \sigma^2 z}} + v(\sigma^2) \right. \right. \\
&\quad \left. \left. - \frac{6(v(\sigma h) - v(\sigma^2))}{\sqrt{1 - \sigma^2 z}} + \frac{3(v(\sigma a) - v(\sigma^2))}{\sqrt{1 - \kappa_2 z}} + 3v(\sigma^2 z) \right) \right. \\
&\quad \left. \left. - 3v(\sigma^2)(2v(\sigma^2) - v(\kappa_2)) \right) \right).
\end{aligned}$$

Note that from the condition $\mu = 0$ we have $\alpha = e^{i\phi_0}$, where $\phi_0 \approx 1.146035015$ and by the condition $\eta = 0$ we have for β the following two possibilities $\beta = |\beta|e^{i\phi_1}$ or $\beta = |\beta|e^{i\phi_2}$, where $\phi_1 \approx -2.9111$ and $\phi_2 \approx 0.2305$. We have computed numerically via formula (2.28) the value of κ using the program “Maple” to get $\kappa \approx -0.03i\delta^3\theta^6$. Therefore we see that if we choose the first value $\beta = |\beta|e^{i\phi_1}$, then we obtain $\delta = -i|\delta|$, hence $\kappa > 0$.

Now (2.27) yields the estimates (2.15). Lemma 2.4 is proved. ■

3. Proof of Theorem 1.1 in the case $\eta > 0$

First we state the local existence of classical solutions for the Cauchy problem (1.1) (for the proof see [6], [7]).

Theorem 3.1. *Let the initial data $u_0 \in \mathbf{W}$. Then for some $T > 0$ there exists a unique solution $u(t, x) \in L^\infty((0, T) \times \mathbf{R}) \cap C([0, T]; \mathbf{W})$ of the Cauchy problem (1.1). Moreover if the initial data are sufficiently small $\|u_0\|_{\mathbf{W}} \leq \varepsilon$, then there exists a time $T > 1$ such that the solution $u(t, x)$ of the Cauchy problem (1.1) satisfy the estimates $\|u\|_{\mathbf{W}} \leq 2\varepsilon$ for all $t \in [0, T]$.*

We make a change of the dependent variable $u(t, x) = e^{-\varphi(t) + i\psi(t)}v(t, x)$ then we get from (1.1)

$$(3.1) \quad v_t - \alpha v_{xx} + \beta e^{-2\varphi}|v|^2v + \gamma e^{-(p-1)\varphi}|v|^{p-1}v - (\varphi' - i\psi')v = 0.$$

Now we assume that the real-valued functions $\varphi(t)$ and $\psi(t)$ satisfy the following condition

$$\int (\beta e^{-2\varphi}|v|^2v + \gamma e^{-(p-1)\varphi}|v|^{p-1}v - (\varphi' - i\psi')v) dx = 0,$$

whence via equation (3.1) we get $\frac{d}{dt} \int v(t, x) dx = 0$ for all $t > 0$ therefore

$$\int (\beta e^{-2\varphi}|v|^2v + \gamma e^{-(p-1)\varphi}|v|^{p-1}v) dx = \sqrt{2\pi}(\varphi' - i\psi')\hat{v}_0(0).$$

If we choose $\varphi(0) = 0$ and $\psi(0) = \arg \hat{u}_0(0)$, we can write $\hat{v}(t, 0) = \hat{v}_0(0) = e^{\varphi(0) - i\psi(0)}\hat{u}_0(0) \equiv |\hat{u}_0(0)| = \theta > 0$ and we have the equations

$$(3.2) \quad \begin{aligned} \varphi'(t) &= \frac{1}{\theta} \Re \left(\beta e^{-2\varphi} \int |v|^2 v dx + \gamma e^{-(p-1)\varphi} \int |v|^{p-1} v dx \right), \\ \psi'(t) &= -\frac{1}{\theta} \Im \left(\beta e^{-2\varphi} \int |v|^2 v dx + \gamma e^{-(p-1)\varphi} \int |v|^{p-1} v dx \right). \end{aligned}$$

Thus we obtain the Cauchy problem

$$(3.3) \quad \begin{cases} v_t - \alpha v_{xx} + \beta e^{-2\varphi} \left(|v|^2 - \frac{1}{\theta} \int |v|^2 v dx \right) v \\ + \gamma e^{-(p-1)\varphi} \left(|v|^{p-1} - \frac{1}{\theta} \int |v|^{p-1} v dx \right) v = 0, \\ \varphi'(t) = \frac{1}{\theta} \Re \left(\beta e^{-2\varphi} \int |v|^2 v dx + \gamma e^{-(p-1)\varphi} \int |v|^{p-1} v dx \right), \\ v(0, x) = v_0(x), \quad \varphi(0) = 0, \end{cases}$$

where $v_0(x) = u(0, x) \exp(-i \arg \int u(0, x) dx)$. Multiplying the second equation of system (3.3) by the factor $e^{2\varphi(t)}$, then integrating with respect to time $t > 0$ and making a change of the dependent variables $v = \mathcal{G}(t)v_0 + r$, and $e^{2\varphi(t)} = g(t)$, we get the system of integral equations $(r, g) = (\mathcal{A}(r, g), \mathcal{B}(r, g))$ for the first approximation of the perturbation theory

$$(3.4) \quad \begin{cases} \mathcal{A}(r, g)(t) = -\beta \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) \left(|v|^2 v(\tau) - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx \right) d\tau \\ - \gamma \int_0^t g^{-(p-1)/2}(\tau) \mathcal{G}(t-\tau) \left(|v|^{p-1} v(\tau) - \frac{v(\tau)}{\theta} \int |v|^{p-1} v(\tau, x) dx \right) d\tau \\ \mathcal{B}(r, g)(t) = 1 + \frac{2}{\theta} \int_0^t \Re \left(\beta \int |v|^2 v(\tau, x) dx + \gamma g^{-(p-3)/2} \int |v|^{p-1} v(\tau, x) dx \right) d\tau. \end{cases}$$

We prove that $(\mathcal{A}, \mathcal{B})$ is the contraction mapping in the set

$$\begin{aligned} \mathbf{X} = & \left\{ r \in \mathbf{C}([0, \infty); \mathbf{W}), g \in \mathbf{C}(0, \infty): \right. \\ & \sup_{t>0} (g(t)\sqrt{1+t}\|r(t)\|_\infty + \sqrt[4]{1+t}\|r(t)\|_{1,0} + (1+t)^{-1/4}\|r(t)\|_{0,1}) \\ & \left. \leq \varepsilon^{5/2}; \frac{1}{2}(1 + \eta \log(1+t)) \leq g(t) \leq 2(1 + \eta \log(1+t)), \text{ for all } t > 0 \right\}, \end{aligned}$$

where $\eta = (\theta^2/(2\pi)^{3/2})\Re\delta(\alpha, \beta)$. First we prove that the mapping $(\mathcal{A}, \mathcal{B})$ transforms the set $\mathbf{X}$ into itself if ε is sufficiently small. When $(r, g) \in \mathbf{X}$ we get

$$\begin{aligned} & \left\| |v|^2 v(t) - \frac{v(t)}{\theta} \int |v|^2 v(t, x) dx \right\|_{\mathbf{W}} + \left\| |v|^{p-1} v(t) - \frac{v(t)}{\theta} \int |v|^{p-1} v(t, x) dx \right\|_{\mathbf{W}} \\ & \leq C\varepsilon^3(1+t)^{-3/4}. \end{aligned}$$

Therefore applying Lemma 2.1 we get the estimates $\|\mathcal{A}(r)(t)\|_\infty \leq C\varepsilon^3(1+t)^{-1/2}g^{-1}(t)$, $\|\mathcal{A}(r)(t)\|_{1,0} \leq C\varepsilon^3/\sqrt[4]{1+t}$ and $\|\mathcal{A}(r)(t)\|_{0,1} \leq C\varepsilon^3\sqrt[4]{1+t}$ for all

$t > 0$. Furthermore applying Lemma 2.2 we get $\frac{1}{2}(1 + \eta \log(1 + t)) \leq g(t) \leq 2(1 + \eta \log(1 + t))$ for all $t > 0$. Thus $(\mathcal{A}, \mathcal{B})$ transforms the set $\mathbf{X}$ into itself. We have by $\mathbf{L}^2 - \mathbf{L}^2$ estimate of solutions to the heat kernel and the fact that $(r_j, g_j) \in \mathbf{X}$, $j = 1, 2$

$$\begin{aligned} & \sup_{t>0} (1+t)^{-\lambda} \|\mathcal{A}(r_1)(t) - \mathcal{A}(r_2)(t)\| \\ & \leq C \sup_{t>0} (1+t)^{-\lambda} \varepsilon^2 \int_0^t \frac{d\tau}{\tau^{1-\lambda}} \left(\sup_{t>0} (1+t)^{-\lambda} \|r_1(t) - r_2(t)\| \right. \\ & \quad \left. + \sup_{t>0} (1+t)^{-\lambda-1/4} |g_1(t) - g_2(t)| \right) \\ & \leq \varepsilon \left(\sup_{t>0} (1+t)^{-\lambda} \|r_1(t) - r_2(t)\| + \sup_{t>0} (1+t)^{-\lambda-1/4} |g_1(t) - g_2(t)| \right) \end{aligned}$$

with some small $\lambda > 0$ and we get

$$\begin{aligned} & \sup_{t>0} (1+t)^{-\lambda-1/4} |g_1(t) - g_2(t)| \\ & \leq \sup_{t>0} C \varepsilon^2 (1+t)^{-\lambda-1/4} \int_0^t \frac{d\tau}{\tau^{3/4-\lambda}} \left(\sup_{t>0} (1+t)^{-\lambda} \|r_1(t) - r_2(t)\| \right. \\ & \quad \left. + \sup_{t>0} (1+t)^{-\lambda-1/4} |g_1(t) - g_2(t)| \right) \\ & \leq \varepsilon \left(\sup_{t>0} (1+t)^{-\lambda} \|r_1(t) - r_2(t)\| + \sup_{t>0} (1+t)^{-\lambda-1/4} |g_1(t) - g_2(t)| \right). \end{aligned}$$

Thus we see that the transformation $(\mathcal{A}, \mathcal{B})$ is the contraction mapping. Therefore there exists a unique solution (r, g) of the system of integral equations (3.4) in the set $\mathbf{X}$. From the proof of Lemma 2.2 we have $g(t) = \eta \log t + O(\varepsilon \log(\eta \log t))$ for $t \rightarrow \infty$ and by (3.2) we obtain

$$\begin{aligned} \psi'(t) &= -\frac{1}{\theta} g^{-1}(t) \Im \int \beta |v|^2 v dx + O\left(\frac{\varepsilon^2}{(1+t)^{(p-1)/2}}\right) \\ &= -\theta^2 g^{-1}(t) \Im \int \beta |G|^2 G dx + O\left(\frac{\varepsilon^2}{1+t} g^{-2}(t)\right) \\ &= -\frac{\theta^2}{4\pi} \frac{\Im \delta(\alpha, \beta)}{(1 + \eta \log(1 + t))(1 + t)} + O\left(\frac{\varepsilon^2}{1+t} g^{-2}(t)\right) \\ &= -\frac{\omega}{(1 + \eta \log(1 + t))(1 + t)} + O\left(\frac{\varepsilon^2}{1+t} g^{-2}(t)\right), \end{aligned}$$

where $\omega = (\theta^2/4\pi)\Im\delta(\alpha, \beta)$. Hence we get

$$\begin{aligned}\psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{1 + \eta \log(1 + \tau)} + O(\varepsilon^2 g^{-2}(\tau)) \right) \frac{d\tau}{1 + \tau} \\ &= \arg \hat{u}_0(0) - \frac{\omega}{\eta} \log(1 + \eta \log(1 + t)) + O\left(\frac{\varepsilon^2}{\log t}\right).\end{aligned}$$

Therefore via formulas $u(t, x) = e^{-\varphi(t) + i\psi(t)} v(t, x) = e^{-\varphi(t) + i\psi(t)} (\mathcal{G}v_0 + r)$ and the estimate of r we obtain the asymptotics of the solution to the Cauchy problem (1.1). Theorem 1.1 is proved for the case $\eta > 0$.

4. Proof of Theorem 1.1 in the case $\eta = 0$, $\mu > 0$

In this section we consider the case $\eta = 0$, $\mu > 0$. As in the case $\eta > 0$ the higher order term is not difficult to treat. So let us consider (1.1) without the higher order term for the shortness (i.e. we consider the case $\gamma = 0$.) Multiplying the second equation of system (3.3) by the factor $g(t) = e^{4\varphi(t)}$ and then integrating with respect to time $t > 0$, we obtain

$$g(t) = 1 + \frac{4}{\theta} \int_0^t g^{1/2}(\tau) \Re \int \beta |v|^2 v(\tau, x) dx d\tau.$$

Therefore we have the system of integral equations

$$\begin{cases} v(t) = \mathcal{G}(t)v_0 - \beta \int_0^t g^{-1/2}(\tau) \mathcal{G}(t - \tau) \left(|v|^2 v(\tau) - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx \right) d\tau, \\ g(t) = 1 + \frac{4}{\theta} \int_0^t \sqrt{g(\tau)} \Re \int \beta |v|^2 v(\tau, x) dx d\tau. \end{cases}$$

We find a solution v of the above system of equations in the neighborhood of the second approximation of the perturbation theory $\Phi(t) + \Psi(t)$, where $\Phi(t) = \mathcal{G}(t)v_0$ and

$$\Psi(t) = -\beta \int_0^t g^{-1/2}(\tau) \mathcal{G}(t - \tau) \left(|\Phi(\tau)|^2 \Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right) d\tau.$$

We put $r = v - \Phi - \Psi$ then we get

$$\begin{cases} r(t) = -\beta \int_0^t g^{-1}(\tau) \mathcal{G}(t - \tau) f(\tau) d\tau, \\ g(t) = 1 + \frac{4}{\theta} \int_0^t \sqrt{g(\tau)} \Re \int \beta |v|^2 v(\tau, x) dx d\tau, \end{cases}$$

where

$$f(\tau) = \sqrt{g(\tau)} \left(|v|^2 v(\tau) - |\Phi(\tau)|^2 \Phi(\tau) - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx + \frac{\Phi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right).$$

We define the mappings $\mathcal{A}(r, g)$ and $\mathcal{B}(r, g)$ by

$$\begin{cases} \mathcal{A}(r, g) = -\beta \int_0^t g^{-1}(\tau) \mathcal{G}(t - \tau) f(\tau) d\tau, \\ \mathcal{B}(r, g) = 1 + \frac{4}{\theta} \int_0^t \sqrt{g(\tau)} \Re \int \beta |v|^2 v(\tau, x) dx d\tau. \end{cases}$$

We introduce the functional space

$$\mathbf{X} = \left\{ r \in \mathbf{C}([0, \infty); \mathbf{W}); g \in \mathbf{C}(0, \infty): \right.$$

$$\sup_{t>0} (g(t) \sqrt{1+t} \|r(t)\|_\infty + \sqrt[4]{1+t} \|r(t)\|_{1,0} + (1+t)^{-1/4} \|r(t)\|_{0,1})$$

$$\leq \varepsilon^{9/2}; \frac{1}{2} (1 + \mu \log(t+1)) \leq g(t) \leq 2(1 + \mu \log(1+t)),$$

$$\left. |g'(t)| \leq \frac{C\varepsilon^4}{1+t} \text{ for all } t > 0 \right\},$$

where $\mu = \{\theta^2(\Im \delta)^2 / (2\pi)^2\} \Re(2v_1 - v_2) > 0$. By the proof of Lemma 2.3 we have $\|f(\tau)\|_{\mathbf{W}} \leq C\varepsilon^3(1+\tau)^{-3/4}$. Hence in the same way as in the case $\eta > 0$, applying Lemma 2.1 with $q = 1$ and Lemma 2.3 we can show that the mappings $\mathcal{A}(r, g)$ and $\mathcal{B}(r, g)$ are the contraction mappings from the set $\mathbf{X}$ into itself and as above we get

$$\begin{aligned} \psi'(t) &= -\frac{1}{\theta} g^{-1/2}(t) \Im \int \beta |v|^2 v dx \\ &= -\theta^2 g^{-1/2}(t) \Im \int \beta |G|^2 G dx + O\left(\frac{\varepsilon^2}{1+t} g^{-1}(t)\right) \\ &= -\frac{\omega}{\sqrt{1+\mu \log(1+t)}(1+t)} + O\left(\frac{\varepsilon^2}{1+t} g^{-1}(t)\right), \end{aligned}$$

where as above we denote $\omega = \frac{\theta^2}{4\pi} \Im \delta(\alpha, \beta)$. Hence we get

$$\begin{aligned} \psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{\sqrt{1+\mu \log(1+\tau)}} + O(\varepsilon g^{-1}(\tau)) \right) \frac{d\tau}{1+\tau} \\ &= \psi(0) - 2\omega \mu^{-1/2} \sqrt{\log t} + O(\varepsilon \log \log t). \end{aligned}$$

Therefore via formulas $u(t, x) = e^{-\varphi(t)+i\psi(t)}v(t, x) = e^{-\varphi(t)+i\psi(t)}(\mathcal{G}v_0 + \Psi + r)$ and the estimate of Ψ given in Lemma 2.3 we obtain the result of Theorem 1.1 in the case $\eta = 0$, $\mu > 0$.

5. Proof of Theorem 1.1 in the case $\eta = 0$, $\mu = 0$, $\kappa > 0$

In this section we consider the case $\eta = 0$, $\mu = 0$. As in the case $\eta = 0$, $\mu > 0$ the higher order term is not difficult to treat. So let us consider (1.1) without the higher order term for the shortness (i.e. $\gamma = 0$.) Multiplying the second equation of system (3.3) by the factor $g(t) = e^{6\varphi(t)}$ and then integrating with respect to time $t > 0$, we obtain

$$g(t) = 1 + \frac{6}{\theta} \int_0^t g^{2/3}(\tau) \Re \int \beta |v|^2 v(\tau, x) dx d\tau.$$

Therefore we have the system of integral equations

$$\begin{cases} v(t) = \mathcal{G}(t)v_0 - \beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \left(|v|^2 v(\tau) - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx \right) d\tau, \\ g(t) = 1 + \frac{6}{\theta} \int_0^t g^{2/3}(\tau) \Re \int \beta |v|^2 v(\tau, x) dx d\tau. \end{cases}$$

We find a solution v of the above system of equations in the neighborhood of the third approximation of the perturbation theory $\Phi(t) + \Psi(t) + V(t)$, where $\Phi(t)$, is defined as above

$$\Psi(t) = -\beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \left(|\Phi(\tau)|^2 \Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right) d\tau$$

and

$$\begin{aligned} V(t) = & -\beta \int_0^t g^{-1/3}(\tau) \mathcal{G}(t-\tau) \left(2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau) \right. \\ & - \frac{\Phi(\tau)}{\theta} \int (2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau)) dx d\tau \\ & \left. - \frac{\Psi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right) d\tau. \end{aligned}$$

We put $r = v - \Phi - \Psi - V$, then we get

$$\begin{cases} r(t) = -\beta \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau, \\ g(t) = 1 + \frac{6}{\theta} \int_0^t g^{2/3}(\tau) \Re \int \beta |v|^2 v(\tau, x) dx d\tau, \end{cases}$$

where

$$\begin{aligned} f(\tau) = & g^{2/3}(\tau) \left(|v|^2 v(\tau) - |\Phi(\tau)|^2 \Phi(\tau) - (2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau)) \right. \\ & - \frac{v(\tau)}{\theta} \int |v|^2 v(\tau, x) dx + \frac{\Phi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \\ & \left. + \frac{\Phi(\tau)}{\theta} \int (2|\Phi|^2 \Psi(\tau) + \Phi^2 \bar{\Psi}(\tau)) dx + \frac{\Psi(\tau)}{\theta} \int |\Phi(\tau)|^2 \Phi(\tau) dx \right). \end{aligned}$$

We define the mappings $\mathcal{A}(r, g)$ and $\mathcal{B}(r, g)$ by

$$\begin{cases} \mathcal{A}(r, g) = -\beta \int_0^t g^{-1}(\tau) \mathcal{G}(t - \tau) f(\tau) d\tau, \\ \mathcal{B}(r, g) = 1 + \frac{6}{\theta} \int_0^t g^{2/3}(\tau) \Re \int \beta |v|^2 v(\tau, x) dx d\tau. \end{cases}$$

We introduce the functional space

$$\begin{aligned} \mathbf{X} = & \left\{ r \in \mathbf{C}([0, \infty); \mathbf{W}); g \in \mathbf{C}(0, \infty): \right. \\ & \sup_{t>0} (g(t) \sqrt{1+t} \|r(t)\|_\infty + \sqrt[4]{1+t} \|r(t)\|_{1,0} + (1+t)^{-1/4} \|r(t)\|_{0,1}) \\ & \leq \varepsilon^{13/2}, \frac{1}{2} (1 + \varkappa \log(t+1)) \leq g(t) \leq 2(1 + \varkappa \log(1+t)), \\ & \left. |g'(t)| \leq \frac{C\varepsilon^4}{1+t} \text{ for all } t > 0 \right\}, \end{aligned}$$

where $\varkappa > 0$ is taken from (2.28). We easily see that $\hat{f}(\tau, 0) = 0$ and by the proof of Lemma 2.4 we have $\|f(\tau)\|_{\mathbf{W}} \leq C\varepsilon^3(1+\tau)^{-3/4}$. Therefore in the same way as in the case $\eta = 0$, $\mu > 0$, applying Lemma 2.1 with $q = 1$ and Lemma 2.4 we can show that the mappings $\mathcal{A}(r, g)$ and $\mathcal{B}(r, g)$ are the contraction mappings from the set $\mathbf{X}$ into itself and as above we get

$$\begin{aligned} \psi'(t) = & -\frac{1}{\theta} g^{-1/3}(t) \Im \int \beta |v|^2 v dx \\ = & -\theta^2 g^{-1/3}(t) \Im \int \beta |G|^2 G dx + O\left(\frac{\varepsilon^2}{1+t} g^{-2/3}(t)\right) \\ = & -\frac{\omega}{\sqrt[3]{1+\varkappa \log(1+t)}(1+t)} + O\left(\frac{\varepsilon^2}{1+t} g^{-2/3}(t)\right), \end{aligned}$$

where $\omega = (\theta^2/4\pi)\Im\delta(\alpha, \beta)$. Hence we get

$$\begin{aligned}\psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{\sqrt[3]{1 + \kappa \log(1 + \tau)}} + O(\varepsilon g^{-2/3}(\tau)) \right) \frac{d\tau}{1 + \tau} \\ &= \psi(0) - \frac{3\omega}{2\sqrt[3]{\kappa}} \sqrt[3]{\log^2 t} + O(\varepsilon \sqrt[3]{\log t}).\end{aligned}$$

Therefore via formulas $u(t, x) = e^{-\varphi(t) + i\psi(t)} v(t, x) = e^{-\varphi(t) + i\psi(t)} (\mathcal{G}v_0 + \Psi + V + r)$ and the estimates of Ψ , V given in Lemma 2.4 we obtain the result of Theorem 1.1 in the case $\eta = 0$, $\mu = 0$, $\kappa > 0$.

Acknowledgment. One of the authors (N.H.) would like to thank Departamento de Ciencias Básicas, Instituto Tecnológico de Morelia for kind hospitality and financial support during his stay in Mexico in September, 1999. The works of E.I.K. and P.I.N. are partially supported by CONACYT.

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(Ricevita la 10-an de februaro, 2000)