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Determinant Formulas for the Toda and Discrete Toda Equations

By

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1. Introduction

In the works [1, 2, 3, 4, 5, 6, 7, 13, 14, 15], the determinant formulas for the τ functions of the Painlevé equations are obtained. These determinant formulas arise as a consequence of the Toda equation which describes Bäcklund (or Schlesinger) transformations of the Painlevé equations [3, 4, 5, 6]. They provide a proof of the miraculous polynomiality of the special polynomials arising as the special solutions of the Painlevé equations [16, 17].

Recently, it is clarified that these determinant formulas can be also applied not only for the special (classical) solutions but also for generic (transcendental) ones [17, 8]. It is natural to expect the existence of such determinant formulas for the general solutions of the Toda equation independent of the Painlevé equations and their (special) solutions.

In this paper, we will give a formula of Hankel type determinant for the solution τ_n of the Toda equation (viewed as a recurrence relation)

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n-1}\tau_{n+1}, \quad n \in \mathbf{Z},$$

for general initial conditions τ_0 and τ_1 . In the case of $n \geq 0$ with $\tau_0 = 1$, such a formula is known as Darboux's formula (cf. [3]). As an application of the formula, we will consider the τ functions of the Painlevé equations and give a new direct proof of their polynomiality for generic cases. Of course, such formulas can be regarded as generalizations of known determinant formulas for classical solutions of the Painlevé equations. We also present a similar determinant formula for the discrete Toda equation

$$\rho_n^{l+1}\rho_n^{l-1} - (\rho_n^l)^2 = \varepsilon^2\rho_{n+1}^{l+1}\rho_{n-1}^{l-1}.$$

2. Determinant formulas

In this section, we present the determinant formulas for general solutions of Toda and discrete Toda equations.

We consider the following recursion relation of a sequence $\{\tau_n\}_{n \in \mathbf{Z}}$,

$$(1) \quad \tau_n''\tau_n - \tau_n'^2 = \tau_{n+1}\tau_{n-1} - \psi\phi\tau_n^2,$$

with

$$(2) \quad \tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi,$$

where ψ and φ are arbitrary functions, and $'$ denotes a derivation. Equation (1) is called the Toda equation in bilinear form. For given ψ and φ , τ_n are uniquely determined as rational functions in them and their derivatives. However, as we shall show below, τ_n are actually polynomials in ψ , φ and their derivatives, and moreover, they are expressed in determinantal forms.

Theorem 2.1. *Let $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ be two sequences defined recursively as*

$$(3) \quad a_n = a'_{n-1} + \psi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} a_i a_j, \quad a_0 = \varphi,$$

$$(4) \quad b_n = b'_{n-1} + \varphi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} b_i b_j, \quad b_0 = \psi.$$

For any integer n , we define $|n| \times |n|$ Hankel determinant τ_n by

$$(5) \quad \tau_n = \begin{cases} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix}, & n > 0, \\ 1, & n = 0, \\ \begin{vmatrix} b_0 & b_1 & \cdots & b_{-n-1} \\ b_1 & b_2 & \cdots & b_{-n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{-n-1} & b_{-n} & \cdots & b_{-2n-2} \end{vmatrix}, & n < 0. \end{cases}$$

Then, τ_n satisfies equation (1) with initial condition (2).

Remark 2.2.

1. Equation (1) is transformed to the bilinear form of Toda equation in "usual" form,

$$(6) \quad \sigma_n'' \sigma_n - \sigma_n'^2 = \sigma_{n+1} \sigma_{n-1},$$

by applying suitable gauge transformation on τ_n , where σ_n is given by

$$(7) \quad (\log \sigma_n)'' = (\log \tau_n)'' + \psi \varphi.$$

2. Since σ_n involves two arbitrary functions, Theorem 2.1 gives a determinant formula for the general solution of the Toda equation.
3. In the case of $\varphi = 0$ or $\psi = 0$, Theorem 2.1 recovers the well-known Darboux's formula, namely, the determinant expression for solutions of the so-called Toda molecule equation [9, 10].

It is also possible to construct a similar formula for the discrete Toda equation. Let Φ^l and Ψ^l be arbitrary functions in l , and $\{\kappa_n^l\}_{n \in \mathbb{Z}}$ a sequence defined by

$$(8) \quad \kappa_n^{l+1} \kappa_n^{l-1} - (1 - \varepsilon^2 \Phi^{l+1} \Psi^{l-1}) (\kappa_n^l)^2 = \varepsilon^2 \kappa_{n+1}^{l+1} \kappa_{n-1}^{l-1},$$

$$(9) \quad \kappa_{-1}^l = \Phi^l, \quad \kappa_0^l = 1, \quad \kappa_1^l = \Psi^l,$$

where ε is a parameter corresponding to the lattice interval of l . Equation (8) is called the discrete Toda equation in bilinear form [11]. Then we have:

Theorem 2.3. Let $\{c_k^l\}_{k \in \mathbb{N}}$, $\{d_k^l\}_{k \in \mathbb{N}}$ be two sequences defined recursively as

$$(10) \quad c_k^l = \frac{c_{k-1}^l - c_{k-1}^{l-1}}{\varepsilon} + \Psi^{l-2} \sum_{\substack{i+j=k-2 \\ i,j \geq 0}} (c_i^l - \varepsilon c_{i+1}^l) c_j^{l-1}, \quad c_0^l = \Phi^l,$$

$$(11) \quad d_k^l = \frac{d_{k-1}^{l+1} - d_{k-1}^l}{\varepsilon} + \Phi^{l+2} \sum_{\substack{i+j=k-2 \\ i,j \geq 0}} (d_i^l + \varepsilon c_{i+1}^l) d_j^{l+1}, \quad d_0^l = \Psi^l.$$

For any integer n , we define $|n| \times |n|$ Hankel determinant κ_n^l by

$$(12) \quad \kappa_n^l = \begin{cases} \begin{vmatrix} c_0^l & c_1^l & \cdots & c_{n-1}^l \\ c_1^l & c_2^l & \cdots & c_n^l \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1}^l & c_n^l & \cdots & c_{2n-2}^l \end{vmatrix}, & n > 0, \\ 1, & n = 0, \\ \begin{vmatrix} d_0^l & d_1^l & \cdots & d_{-n-1}^l \\ d_1^l & d_2^l & \cdots & d_{-n}^l \\ \vdots & \vdots & \ddots & \vdots \\ d_{-n-1}^l & d_{-n}^l & \cdots & d_{-2n-2}^l \end{vmatrix}, & n < 0. \end{cases}$$

Then, κ_n^l satisfies equation (8) with initial condition (9).

Remark 2.4.

1. Equation (8) is transformed to the bilinear form of discrete Toda equation in “usual” form,

$$(13) \quad \rho_n^{l+1} \rho_n^{l-1} - (\rho_n^l)^2 = \varepsilon^2 \rho_{n+1}^{l+1} \rho_{n-1}^{l-1},$$

by introducing ρ_n^l by

$$(14) \quad \rho_n^l = \frac{1}{\prod_{i=l_0}^l \prod_{j=i_0}^i (1 - \varepsilon^2 \Phi^j \Psi^{j-2})} \kappa_n^l.$$

2. Since ρ_n^l involves two arbitrary functions, Theorem 2.3 gives a determinant expression for general solution of the discrete Toda equation.

3. In the case of $\varphi = 0$ or $\psi = 0$, Theorem 2.3 recovers the determinant expression for solution of the so-called discrete Toda molecule equation [12].
4. Discrete Toda equation (8) and its solution (10)–(12) reduce to the Toda equation (1) and its solution (3)–(5) in the limit of $\varepsilon \rightarrow 0$, respectively.

Let us first prove Theorem 2.1. We consider the case of $n > 0$. Let D be the determinant of an $(n+1) \times (n+1)$ matrix X , and $D \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ the determinant of the matrix obtained from X by removing the rows with indices $i_1, \dots, i_k$ and the columns with indices $j_1, \dots, j_k$. Then we have well-known Jacobi's formula (Lewis Carroll's formula)

$$(15) \quad D \begin{pmatrix} n \\ n \end{pmatrix} D \begin{pmatrix} n+1 \\ n+1 \end{pmatrix} - D \begin{pmatrix} n \\ n+1 \end{pmatrix} D \begin{pmatrix} n+1 \\ n \end{pmatrix} = D \cdot D \begin{pmatrix} n & n+1 \\ n & n+1 \end{pmatrix}.$$

We have the following differential formula for τ_n :

Lemma 2.5. *Putting $D \equiv \tau_{n+1}$, we have,*

$$(16) \quad D \begin{pmatrix} n+1 \\ n+1 \end{pmatrix} = \tau_n, \quad D \begin{pmatrix} n & n+1 \\ n & n+1 \end{pmatrix} = \tau_{n-1},$$

$$(17) \quad D \begin{pmatrix} n \\ n+1 \end{pmatrix} = D \begin{pmatrix} n+1 \\ n \end{pmatrix} = \tau'_n,$$

$$(18) \quad D \begin{pmatrix} n \\ n \end{pmatrix} = \tau''_n + \varphi\psi\tau_n.$$

Then, Theorem 2.1 follows immediately from equation (15) and Lemma 2.5. Therefore, it suffices to prove Lemma 2.5.

Proof of Lemma 2.5. Equation (16) is obvious by definition. To show equation (17), we consider the following equality,

$$(19) \quad D \begin{pmatrix} n \\ n+1 \end{pmatrix} = D \begin{pmatrix} n+1 \\ n \end{pmatrix} \\ = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{pmatrix}.$$

Here, for $n \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$, $A \cdot B$ denotes

$$(20) \quad A \cdot B = \sum_{i,j=1}^n A_{ij} B_{ij} = \text{Tr } A^t B,$$

which is the standard scalar product of matrices, and Δ_{ij} is an (i, j) -cofactor of τ_n . The first matrix of equation (19) is rewritten by using the recursion relation (3) as

$$\begin{aligned}
 (21) \quad & \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix} \\
 &= \begin{pmatrix} a'_0 & a'_1 & \cdots & a'_{n-1} \\ a'_1 & a'_2 & \cdots & a'_n \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n-1} & a'_n & \cdots & a'_{2n-2} \end{pmatrix} \\
 &+ \psi \left[\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{pmatrix} \begin{pmatrix} 0 & a_0 & a_1 & \cdots & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ & & \ddots & \ddots & \vdots \\ & 0 & & 0 & a_0 \\ & & & & 0 \end{pmatrix} \right. \\
 &\quad \left. + \begin{pmatrix} 0 & & & & \\ a_0 & 0 & & & \\ a_1 & a_0 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & \cdots & a_1 & a_0 & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{pmatrix} \right].
 \end{aligned}$$

Applying scalar product of equation (21) with (Δ_{ij}) , we see that the first term of the right hand side of equation (21) gives τ'_n and second and third terms give no contribution. Thus we have shown that equation (17) holds.

Equation (18) is shown in a similar manner by considering the equality,

$$(22) \quad D \binom{n}{n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix} \cdot \begin{pmatrix} \Delta'_{11} & \Delta'_{12} & \cdots & \Delta'_{1n} \\ \Delta'_{21} & \Delta'_{22} & \cdots & \Delta'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta'_{n1} & \Delta'_{n2} & \cdots & \Delta'_{nn} \end{pmatrix},$$

where Δ'_{ij} denotes the (i, j) -cofactor of $\tau'_n = D \binom{n}{n+1}$. ■

The case $n < 0$ of Theorem 2.1 is proved in a manner similar to the case of $n > 0$, and the case $n = 0$ is checked directly. Thus, proof of Theorem 2.1 is completed.

Let us next prove Theorem 2.3. Similarly to the proof of Theorem 2.1, we concentrate on the case of $n > 0$. We have the following lemma:

Lemma 2.6. *For $n \geq 1$, we have*

$$(23) \quad \varepsilon^{n-1} \kappa_n^{l+1} = \begin{vmatrix} c_0^l & c_1^l & \cdots & c_{n-2}^l & C_0^{l+1} \\ c_1^l & c_2^l & \cdots & c_{n-1}^l & C_1^{l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1}^l & c_n^l & \cdots & c_{2n-3}^l & C_{n-1}^{l+1} \end{vmatrix},$$

$$(24) \quad \frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \kappa_n^{l+2} = \begin{vmatrix} c_0^l & c_1^l & \cdots & c_{n-2}^l & C_0^{l+1} \\ c_1^l & c_2^l & \cdots & c_{n-1}^l & C_1^{l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2}^l & c_{n-1}^l & \cdots & c_{2n-4}^l & C_{n-2}^{l+1} \\ C_0^{l+1} & C_1^{l+1} & \cdots & C_{n-2}^{l+1} & \frac{C_0^{l+2}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \end{vmatrix},$$

where C_k^l is defined by

$$(25) \quad C_k^l = c_k^l + \varepsilon \Psi^{l-2} \sum_{i=1}^k c_{k-i}^l c_{i-1}^{l-1}.$$

Theorem 2.3 for the case of $n > 0$ is derived as follows. We put

$$(26) \quad D \equiv \frac{\varepsilon^{2n}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \kappa_{n+1}^{l+2} = \begin{vmatrix} c_0^l & \cdots & c_{n-2}^l & c_{n-1}^l & C_0^{l+1} \\ c_1^l & \cdots & c_{n-1}^l & c_n^l & C_1^{l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2}^l & \cdots & c_{2n-4}^l & c_{2n-3}^l & C_{n-2}^{l+1} \\ c_{n-1}^l & \cdots & c_{2n-3}^l & c_{2n-2}^l & C_{n-1}^{l+1} \\ C_0^{l+1} & \cdots & C_{n-2}^{l+1} & C_{n-1}^{l+1} & \frac{C_0^{l+2}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \end{vmatrix}.$$

Then we have from Lemma 2.6,

$$(27) \quad D \binom{n}{n} = \frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \kappa_n^{l+2}, \quad D \binom{n+1}{n+1} = \kappa_n^l,$$

$$(28) \quad D \binom{n}{n+1} = D \binom{n+1}{n} = \varepsilon^{n-1} \kappa_n^{l+1}, \quad D \binom{n}{n} \binom{n+1}{n+1} = \kappa_{n-1}^l.$$

Therefore, Jacobi's identity (15) yields

$$(29) \quad \frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \kappa_n^{l+2} \kappa_n^l - (\varepsilon^{n-1} \kappa_n^{l+1})^2 = \frac{\varepsilon^{2n}}{1 - \varepsilon^2 \Phi^{l+2} \Psi^l} \kappa_{n+1}^{l+2} \kappa_{n-1}^l,$$

which is equivalent to the discrete Toda equation (8).

Proof of Lemma 2.6. We rewrite

$$\kappa_n^{l+1} = \begin{vmatrix} c_0^{l+1} & c_1^{l+1} & \cdots & c_{n-2}^{l+1} & c_{n-1}^{l+1} \\ c_1^{l+1} & c_2^{l+1} & \cdots & c_{n-1}^{l+1} & c_n^{l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1}^{l+1} & c_n^{l+1} & \cdots & c_{2n-3}^{l+1} & c_{2n-2}^{l+1} \end{vmatrix}$$

by using the recursion relation (10) to obtain eq. (23). We first add j -th column multiplied by $\varepsilon \Psi^{l-1} c_{n-1-j}^l$ to n -th column for $j = 2, \dots, n-1$. Next, adding j -th column multiplied by $\Psi^{l-1} c_{n-2-j}^l$ to n -th column for $j = 1, \dots, n-2$, we have from eq. (10),

$$\kappa_n^{l+1} = \begin{vmatrix} c_0^{l+1} & \cdots & c_{n-2}^{l+1} & -\frac{c_{n-2}^l}{\varepsilon} \\ c_1^{l+1} & \cdots & c_{n-1}^{l+1} & -\frac{c_{n-1}^l}{\varepsilon} + \Psi^{l-1}(c_0^{l+1} - \varepsilon c_1^{l+1})c_{n-2}^l \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2}^{l+1} & \cdots & c_{2n-4}^{l+1} & -\frac{c_{2n-4}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-3} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{2n-5-j}^l \\ c_{n-1}^{l+1} & \cdots & c_{2n-3}^{l+1} & -\frac{c_{2n-3}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-2} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{2n-4-j}^l \end{vmatrix},$$

Applying the similar procedure to $(n-1)$ -th, $\dots$, 2nd columns, we obtain,

$$\kappa_n^{l+1} = \begin{vmatrix} c_0^{l+1} & -\frac{c_0^l}{\varepsilon} & \cdots & -\frac{c_{n-2}^l}{\varepsilon} \\ c_1^{l+1} & -\frac{c_1^l}{\varepsilon} + \Psi^{l-1}(c_0^{l+1} - \varepsilon c_1^{l+1})c_0^l & \cdots & -\frac{c_{n-1}^l}{\varepsilon} + \Psi^{l-1}(c_0^{l+1} - \varepsilon c_1^{l+1})c_{n-2}^l \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2}^{l+1} & -\frac{c_{n-2}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-3} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{n-3-j}^l & \cdots & -\frac{c_{2n-4}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-3} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{2n-5-j}^l \\ c_{n-1}^{l+1} & -\frac{c_{n-1}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-2} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{n-2-j}^l & \cdots & -\frac{c_{2n-3}^l}{\varepsilon} + \Psi^{l-1} \sum_{j=0}^{n-2} (c_j^{l+1} - \varepsilon c_{j+1}^{l+1})c_{2n-4-j}^l \end{vmatrix}$$

Next, we apply the similar procedure in vertical direction. For $k = 2, \dots, n$, we add j -th row multiplied by $\varepsilon \Psi^{l-1}(c_{j-1}^{l+1} - c_j^{l+1})$ to k -th row for $j = 1, \dots, k-1$. Then we get

$$\begin{aligned} \kappa_n^{l+1} &= \begin{vmatrix} C_0^{l+1} & -\frac{c_0^l}{\varepsilon} & \cdots & -\frac{c_{n-2}^l}{\varepsilon} \\ C_1^{l+1} & -\frac{c_1^l}{\varepsilon} & \cdots & -\frac{c_{n-1}^l}{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n-1}^{l+1} & -\frac{c_{n-1}^l}{\varepsilon} & \cdots & -\frac{c_{2n-3}^l}{\varepsilon} \end{vmatrix} \\ &= \varepsilon^{-(n-1)} \begin{vmatrix} c_0^l & \cdots & c_{n-2}^l & C_0^{l+1} \\ c_1^l & \cdots & c_{n-1}^l & C_1^{l+1} \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1}^l & \cdots & c_{2n-3}^l & C_{n-1}^{l+1} \end{vmatrix}, \end{aligned}$$

where C_k^{l+1} is defined recursively by

$$(30) \quad C_k^{l+1} = c_k^{l+1} + \sum_{j=0}^{k-1} \varepsilon \Psi^{l-1}(c_{k-1-j}^{l+1} - \varepsilon c_{k-j}^{l+1}) C_j^{l+1}, \quad C_0^{l+1} = c_0^{l+1}.$$

We can verify that C_k^l is actually given as eq. (25) by induction. Thus we have proved eq. (23). Since eq. (24) is proved by a similar calculation, we omit the detail. This completes the proof of Lemma 2.6. $\blacksquare$

The above discussion proves Theorem 2.3 for the case of $n > 0$. The case of $n < 0$ is proved similarly, and the case of $n = 0$ is checked immediately. Thus we have proved Theorem 2.3.

3. Applications to Painlevé equations

It is established by K. Okamoto that the τ functions of the Painlevé equations $P_{II}, \dots, P_{VI}$ satisfy the Toda equation. Recall that each of the Painlevé equations P_J ($J = II, \dots, VI$) can be written as a Hamiltonian system

$$(31) \quad \delta q = \frac{\partial H}{\partial p}, \quad \delta p = -\frac{\partial H}{\partial q},$$

where H is a certain polynomial in p, q , and the derivation δ is given by $\delta = \partial_t$ for P_{II}, P_{IV} , $\delta = t\partial_t$ for P_{III}, P_V , and $\delta = t(t-1)\partial_t$ for P_{VI} , respectively. (The

explicit formula for H will be given in the next section.) The τ function, which we denote by σ , is defined up to constant factor as

$$(32) \quad \delta(\log \sigma) = H.$$

By using a Bäcklund transformation (Schlesinger transformation) T which acts as a translation on the parameter space, one can define a sequence of functions σ_n by choosing an appropriate normalization factor for $T^n(\sigma)$ ($n \in \mathbb{Z}$).

Theorem 3.1 (Okamoto [3]). *The sequence of τ functions σ_n for the Painlevé equations P_J satisfies the Toda equation*

$$(33) \quad (\delta^2 \sigma_n) \sigma_n - (\delta \sigma_n)^2 = \sigma_{n+1} \sigma_{n-1}.$$

We will review the derivation of the Toda equation in the next section for completeness.

The main result (The T_n function and its polynomiality).

Let us put

$$(34) \quad \sigma_n = \sigma_0 \left(\frac{\sigma_1}{\sigma_0} \right)^n T_n.$$

Then the function T_n satisfy the recurrence relation

$$(35) \quad T_{n+1} T_{n-1} = (\delta^2 T_n) T_n - (\delta T_n)^2 + (n\delta v + u) T_n^2,$$

with initial condition $T_0 = T_1 = 1$. Here $u = \delta^2 \log \sigma_0$, $v = \delta \log \frac{\sigma_1}{\sigma_0}$.

The functions T_n determined by the Toda equation (35) are rational functions in p and q , since we must divide the right hand side of (35) by T_{n-1} to obtain T_{n+1} . However, as observed by H. Umenura (see [17] for example), a remarkable factorization occurs at each recurrence step and the functions T_n are polynomials in p and q . Namely we have the following.

Theorem 3.2. *Let (p, q) be a solution for P_J and define functions $T_n = T_n(p, q, t)$ through the recurrence relation (35). Then the functions T_n are polynomials in p, q .*

Proof. The basic ingredient of this proof is the determinant formula in section 2. We will show for $n \geq 0$. The case $n \leq 0$ is similar. Note that the functions $\tau_n = \sigma_n / \sigma_0$ ($n \in \mathbb{Z}$) satisfy the equation

$$(36) \quad (\delta^2 \tau_n) \tau_n - (\delta \tau_n)^2 = \tau_{n+1} \tau_{n-1} - \psi \varphi \tau_n^2,$$

with

$$(37) \quad \psi = \frac{\sigma_{-1}}{\sigma_0}, \quad \varphi = \frac{\sigma_1}{\sigma_0}.$$

From the determinant formula (Theorem 2.1), the solution of the Toda equation (33) is given by

$$(38) \quad \sigma_n = \sigma_0 \tau_n = \sigma_0 \det(a_{i+j})_{0 \leq i, j \leq n-1}.$$

We introduce g_i ($i \in \mathbb{N}$) by setting $a_i = \sigma_1 g_i / \sigma_0 = \varphi g_i$, so that

$$(39) \quad \sigma_n = \sigma_0 \left(\frac{\sigma_1}{\sigma_0} \right)^n \det(g_{i+j})_{0 \leq i, j \leq n-1}.$$

Putting $u = \delta^2 \log \sigma_0 = \psi \varphi$ and $v = \delta \log \varphi = \delta \log(\sigma_1 / \sigma_0)$, we can rewrite the recurrence formula for a_n to that of g_n :

$$(40) \quad g_0 = 1, \quad \text{and} \quad g_n = \delta g_{n-1} + v g_{n-1} + u \sum_{\substack{i+j=n-2 \\ i, j \geq 0}} g_i g_j, \quad (n \geq 1).$$

As we will see in the next section, the functions u , v and their δ -derivatives are all polynomials in p, q . Hence the matrix elements g_k and their determinants T_n are also polynomials in p, q . ■

4. Derivation of the Toda equations

In order to ensure the polynomiality of the functions u, v and their δ -derivatives, we will review the derivation of the Toda equation following the work by Okamoto [3, 4, 5, 6]. In each case P_J ($J = II, \dots, VI$), we will check the following two facts:

- The Hamiltonian H and its Bäcklund transformation (translation) $T(H)$ are polynomials in p, q .
- The tau functions σ_n satisfying the standard Toda equation (33) are given by

$$(41) \quad \sigma_n = K_n T^n(\sigma).$$

Here the normalization factors K_n depend only on t and parameters α_i . The polynomiality of u, v follows from these facts since we have

$$(42) \quad \begin{aligned} u &= \delta^2 \log \sigma_0 = \delta H + \delta^2 \log K_0, \\ v &= \delta \log \frac{\sigma_1}{\sigma_0} = T(H) - H + \delta \log \frac{K_1}{K_0}. \end{aligned}$$

Note that the polynomiality in p, q is preserved by δ -derivation because of the polynomiality of the Hamiltonian H .

4.1. The second Painlevé equation: P_{II}

The Hamiltonian is

$$(43) \quad H_{II} = \frac{1}{2}p^2 - \left(q^2 + \frac{1}{2}t\right)p - \alpha_1 q.$$

The equation for $y = q$ is the Painlevé equation P_{II} given as follows:

$$(44) \quad y'' = 2y^3 + ty + a,$$

with $a = \alpha_1 - \frac{1}{2}$.

The Bäcklund transformations are given by

$$(45) \quad \begin{array}{c|c|c|c|c} x & \alpha_0 & \alpha_1 & p & q \\ \hline s_0(x) & -\alpha_0 & \alpha_1 + 2\alpha_0 & p + \frac{4\alpha_0 q}{f} + \frac{2\alpha_0^2}{f^2} & q + \frac{\alpha_0}{f} \\ s_1(x) & \alpha_0 + 2\alpha_1 & -\alpha_1 & p & q + \frac{\alpha_1}{p} \\ \pi(x) & \alpha_1 & \alpha_0 & -f & -q \end{array}$$

where $\alpha_0 = 1 - \alpha_1$ and $f = p - 2q^2 - t$.

Let $T = \pi s_1$ be the translation which acts on the parameter as $T(\alpha_0, \alpha_1) = (\alpha_0 + 1, \alpha_1 - 1)$. Then we have

$$(46) \quad T(H) - H = Y = q, \quad \partial_t(H) = -\frac{p}{2},$$

and

$$(47) \quad \partial_t(\log \partial_t H) = T(H) - 2H + T^{-1}(H).$$

Hence we have the Toda equation:

$$(48) \quad \partial_t^2 \log T^n(\sigma) = c(n) \frac{T^{n-1}(\sigma) T^{n+1}(\sigma)}{T^n(\sigma)^2},$$

where $c(n)$ is a non-zero constant. This can be transformed to the standard Toda equation (33) by changing the normalization as

$$(49) \quad \sigma_n = C_n T^n(\sigma).$$

Here and in the followings, the relation between the constants C_n and $c(n)$ are given by $C_{n-1} C_{n+1} C_n^{-2} = c(n)$ (i.e. $C_n = \prod_{k=1}^{n-1} c(k)^{n-k} C_1^n C_0^{1-n}$).

4.2. The third Painlevé equation: P_{III}

The Hamiltonian is

$$(50) \quad H_{III} = q^2 p^2 - (q^2 + v_1 q - t)p + \frac{1}{2}(v_1 + v_2)q.$$

The equation for $y = q/s$ ($t = s^2$) is given by the third Painlevé equation P_{III}

$$(51) \quad \frac{d^2 y}{ds^2} = \frac{1}{y} \left(\frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} (ay^2 + b) + cy^3 + \frac{d}{y},$$

with

$$(52) \quad a = -4v_2, \quad b = 4(v_1 + 1), \quad c = 4, \quad d = -4.$$

The Bäcklund transformations are

(53)

x	v_1	v_2	p	q	t
$s_0(x)$	$-1 - v_2$	$-1 - v_1$	$\frac{q}{t}(q(p-1) - \frac{1}{2}(v_1 - v_2)) + 1$	$-\frac{t}{q}$	t
$s_1(x)$	v_2	v_1	p	$q + \frac{\frac{1}{2}(v_2 - v_1)}{p-1}$	t
$s_2(x)$	v_1	$-v_2$	$1 - p$	$-q$	$-t$

Let $T = s_0 s_2 s_1 s_2$ be the translation which acts on the parameter as $T(v_1, v_2) = (v_1 + 1, v_2 + 1)$. Then

$$(54) \quad T(H) - H = Y = q(1 - p), \quad \partial_t(H) = p.$$

and

$$(55) \quad \delta \log(\partial_t H) = T(H) - 2H + T^{-1}(H).$$

Hence we have the Toda equation

$$(56) \quad \partial_t \delta \log T^n(\sigma) = c(n) \frac{T^{n-1}(\sigma) T^{n+1}(\sigma)}{T^n(\sigma)^2},$$

where $c(n)$ is a non-zero constant. This equation can be transformed to the standard Toda equation by the change of the normalization as

$$(57) \quad \sigma_n = C_n t^{n^2/2} T^n(\sigma).$$

4.3. The fourth Painlevé equation: P_{IV}

The Hamiltonian is

$$(58) \quad H_{IV} = (t - p - q)pq - \alpha_1 q + \alpha_2 p.$$

Equation for $y = q$ is given by

$$(59) \quad y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} - 2ty^2 + 2\left(\frac{t^2}{4} - a\right)y + \frac{b}{y},$$

with

$$(60) \quad a = \frac{1}{2}(\alpha_1 - \alpha_0), \quad b = -\frac{\alpha_2^2}{2}.$$

The Bäcklund transformations are

$$(61) \quad \begin{array}{c|ccccc} x & \alpha_0 & \alpha_1 & \alpha_2 & p & q \\ \hline s_0(x) & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 & p + \frac{\alpha_0}{f} & q - \frac{\alpha_0}{f} \\ s_1(x) & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & p & q + \frac{\alpha_1}{p} \\ s_2(x) & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & p - \frac{\alpha_2}{q} & q \\ \pi(x) & \alpha_1 & \alpha_2 & \alpha_0 & q & f \end{array}$$

where $\alpha_0 = 1 - \alpha_1 - \alpha_2$ and $f = t - p - q$.

Let $T = \pi s_2 s_1$ be the translation which acts on the parameter as $T(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2)$. Then

$$(62) \quad T(H) - H = Y = q, \quad \partial_t(H) = pq.$$

and

$$(63) \quad \delta \log(\partial_t H + \alpha_1) = T(H) - 2H + T^{-1}(H).$$

Hence we have the Toda equation

$$(64) \quad \partial_t \delta \log T^n(\sigma) + (\alpha_1 - n) = c(n) \frac{T^{n-1}(\sigma) T^{n+1}(\sigma)}{T^n(\sigma)^2},$$

where $c(n)$ is a non-zero constant. The change of normalization is given by

$$(65) \quad \sigma_n = C_n e^{1/2(\alpha_1 - n)t^2} T^n(\sigma).$$

4.4. The fifth Painlevé equation: P_V

The Hamiltonian is

$$(66) \quad H_V = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1).$$

Equation for $y = 1 - 1/q$ is given by

$$(67) \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2} \left(ay + \frac{b}{y} \right) + c \frac{y}{t} + d \frac{y(y+1)}{y-1},$$

where

$$(68) \quad a = \frac{\alpha_1^2}{2}, \quad b = -\frac{\alpha_3^2}{2}, \quad c = \alpha_0 - \alpha_2, \quad d = -\frac{1}{2}.$$

The Bäcklund transformations are

$$(69) \quad \begin{array}{c|c|c|c|c|c|c} x & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & p & q \\ \hline s_0(x) & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 & \alpha_3 + \alpha_0 & p & q + \frac{\alpha_0}{p+t} \\ s_1(x) & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & \alpha_3 & p - \frac{\alpha_1}{q} & q \\ s_2(x) & \alpha_0 & \alpha_1 + \alpha_2 & -\alpha_2 & \alpha_3 + \alpha_2 & p & q + \frac{\alpha_2}{p} \\ s_3(x) & \alpha_0 + \alpha_3 & \alpha_1 & \alpha_2 + \alpha_3 & -\alpha_3 & p - \frac{\alpha_3}{q-1} & q \\ \pi(x) & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 & t(q-1) & -\frac{p}{t} \end{array}$$

where $\alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3$.

Let $T = \pi s_3 s_2 s_1$ be the translation which acts on the parameters as $T(\alpha_0, \dots, \alpha_3) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2, \alpha_3)$. Then

$$(70) \quad T(H) - H = Y = p(q-1), \quad \partial_t(H) = pq(q-1) + \alpha_2 q,$$

and

$$(71) \quad \delta \log(\partial H + \alpha_1) = T(H) - 2H + T^{-1}(H).$$

Hence, we obtain the Toda equation

$$(72) \quad \partial_t \delta \log T^n(\sigma) + (\alpha_1 - n) = c(n) \frac{T^{n-1}(\sigma) T^{n+1}(\sigma)}{T^n(\sigma)^2},$$

where $c(n)$ is a non-zero constant. The change of the normalization is given by

$$(73) \quad \sigma_n = C_n t^{n^2/2} e^{(\alpha_1 - n)t} T^n(\sigma).$$

4.5. The sixth Painlevé equation: P_{VI}

The Hamiltonian is

$$(74) \quad H = q(q-1)(q-t)p^2 - [\alpha_4(q-1)(q-t) + \alpha_3q(q-t) + (\alpha_0-1)q(q-1)]p \\ + \alpha_2(\alpha_1 + \alpha_2)(q-t).$$

The equation for $y = q$ is given by

$$y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\ + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[a + b \frac{t}{y^2} + c \frac{t-1}{(y-1)^2} + d \frac{t(t-1)}{(y-t)^2} \right],$$

with

$$(75) \quad a = \alpha_1^2/2, \quad b = -\alpha_4^2/2, \quad c = \alpha_3^2/2, \quad d = -(\alpha_0^2 - 1)/2.$$

($\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$). The Bäcklund transformations are as follows

(76)

x	α_0	α_1	α_2	α_3	α_4	p	q
$s_0(x)$	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	$p - \frac{\alpha_0}{q-t}$	q
$s_1(x)$	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	p	q
$s_2(x)$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	p	$q + \frac{\alpha_2}{p}$
$s_3(x)$	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	$p - \frac{\alpha_3}{q-1}$	q
$s_4(x)$	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	$p - \frac{\alpha_4}{q}$	q
$r(x)$	α_1	α_0	α_2	α_4	α_3	$-\frac{p(q-t)^2 + \alpha_2(q-t)}{t(t-1)}$	$\frac{(q-1)t}{(q-t)}$

Let $T = s_0 s_2 s_3 s_4 s_2 s_0 r$ be the translation which acts on the parameters as $T(\alpha_0, \dots, \alpha_4) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2, \alpha_3, \alpha_4)$. Then we have

$$(77) \quad \begin{aligned} T(H) - H &= Y = pq(1 - q) + \alpha_2(t - q), \\ \partial_t H &= p^2 q(1 - q) - \alpha_2(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)pq - \alpha_4 p, \end{aligned}$$

and

$$(78) \quad \delta \log(\partial_t H + (\alpha_1 + \alpha_2)(1 - \alpha_0)) = T(H) - 2H + T^{-1}(H).$$

The Toda equation is given as

$$(79) \quad \partial_t \delta \log T^n(\sigma) + (\alpha_1 + \alpha_2 - n)(1 - \alpha_0 - n) = c(n) \frac{T^{n-1}(\sigma) T^{n+1}(\sigma)}{T^n(\sigma)^2},$$

where $c(n)$ is a non-zero constant. The change of the normalization is given by

$$(80) \quad \sigma_n = C_n(t(t-1))^{1/2(\alpha_1 + \alpha_2 - n)(1 - \alpha_0 - n)} T^n(\sigma).$$

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