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## Global Solvability and Asymptotic Behaviour of a Hyperbolic Problem with Acoustic Boundary Conditions

By

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### 1. Introduction

This article presents a study of global solvability and asymptotic behaviour of solutions to the mixed problem for the nonlinear hyperbolic equation with nonlinear damping and acoustic boundary conditions:

$$(1.1) \quad u_{tt} - a(u)u_{xx} + g(u_t) = 0, \quad \text{in } Q = (0, 1) \times (0, T),$$

$$(1.2) \quad u(0, t) = 0, \quad t \in (0, T),$$

$$(1.3) \quad -\rho u_t(1, t) = \delta_{tt}(t) + c_1 \delta_t(t) + c_0 \delta(t), \quad t \in (0, T),$$

$$(1.4) \quad u_x(1, t) = \delta_t(t), \quad t \in (0, T),$$

$$(1.5) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \quad \delta(0) = \delta_0;$$

where  $\rho, T, c_0, c_1$  are positive constants,  $a: \mathbf{R} \rightarrow \mathbf{R}$  and  $g: \mathbf{R} \rightarrow \mathbf{R}$  are differentiable functions.

In [3], J. T. Beale and S. I. Rosencrans considered acoustic boundary conditions for a gas, undergoing small irrotational perturbations from the rest, in a domain  $\Omega \subset \mathbf{R}^3$  with a smooth compact boundary  $\partial\Omega = \Gamma$ . They assumed that the boundary  $\Gamma$  is nonporous and locally reacting in the sense that wave motion along the boundary is negligible. Therefore, if  $u(x, t)$  is the velocity potential and  $\delta(x, t)$  denotes the normal displacement of a point  $x \in \Gamma$  at time  $t$ , they must satisfy the equations:

$$(1.6) \quad u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times (0, T);$$

$$(1.7) \quad \delta_{tt} - c_1 \delta_t + c_0 \delta = -\rho u_t, \quad \text{on } \Gamma \times (0, T);$$

$$(1.8) \quad \delta_t = \frac{\partial u}{\partial \nu}, \quad \text{on } \Gamma \times (0, T);$$

here  $\nu$  is the unit outer normal,  $c_1 = d/m$ ,  $c_0 = k/m$ ,  $\rho = \rho_0/m$ ; where  $m$  is the mass per unit area on the boundary,  $d$  is the resistivity of the boundary,  $k$  is the spring constant and  $\rho_0$  is the unperturbed density of the gas.

The mixed problem for (1.6)–(1.8) was considered by J. T. Beale which, using semigroup techniques, carried out a detailed analysis in bounded domain [1] and exterior domains [2]. Beale's idea was to work with an equivalent initial value problem in the form  $u_t = Au$  where  $A$  is an operator on a suitable Hilbert space  $H$ . For bounded domains, in [1], Beale gave a description of the spectrum of the operator  $A$  and proved that there is no uniform rate of decay for solutions to the mixed problem for (1.6)–(1.8).

Recently, C. L. Frota and J. A. Goldstein [4] studied acoustic boundary conditions (1.7), (1.8) for the nonlinear Carrier equation with a nonlinear damping

$$(1.9) \quad u_{tt} - M \left( \int_{\Omega} u^2 dx \right) \Delta u + |u_t|^{\alpha} u_t = 0.$$

The authors proved results on global solvability, uniqueness, regularity and continuous dependence of solutions on the parameters when the boundary  $\Gamma$  is made up of two disjoint pieces  $\Gamma_0, \Gamma_1$ , each having nonempty interior. Acoustic boundary conditions (1.7), (1.8) were imposed for  $(x, t) \in \Gamma_1 \times (0, T)$  and the Dirichlet boundary condition  $u(x, t) = 0$  was imposed for  $(x, t) \in \Gamma_0 \times [0, T]$ . Moreover, for dimensions  $n = 2$  or  $n = 3$ , global solvability and uniqueness for the mixed problem for (1.9) with acoustic boundary conditions on the whole boundary  $\Gamma$  were established. This was obtained as a limit case when the positive measure of  $\Gamma_0$  shrink to zero.

Global solvability and stability of the energy of the mixed problem for (1.9) with the Dirichlet boundary conditions was proved in [5].

In the present paper we consider the nonlinear wave equation (1.1) with local nonlinearities. The function  $a(u)$  depends on a solution while the function  $M(\int_{\Omega} u^2 dx)$  in (1.9) depends on the  $L^2$ -norm of it. This difference makes the study of (1.1)–(1.5) more complicated and we restrict it only to the one-dimensional case. Exponential and algebraic energy decay are proved under some appropriate assumptions on the damping  $g(u_t)$  and when  $c_0$  is sufficiently large.

## 2. Existence and uniqueness

In this section we prove the existence and uniqueness of global solutions to (1.1)–(1.5). We denote the inner product and norm in  $L^2(0, 1)$  by

$$(u, v) = \int_0^1 u(x)v(x)dx, \quad \|u\| = \left( \int_0^1 |u(x)|^2 dx \right)^{1/2},$$

and we define  $V = \{z \in H^1(0, 1); z(0) = 0\}$  and the norm in  $V$  by

$$\|z\|_V = \|z_x\|, \quad z \in V.$$

The functional spaces are standard, see in Lions and Magenes [5].

**Theorem 2.1.** Suppose that the functions  $a$  and  $g$  satisfy

$$(2.1) \quad 0 < a_0 \leq a(\lambda),$$

$$(2.2) \quad |a'(\lambda)| \leq k_0 a(\lambda),$$

$$(2.3) \quad |g(\xi)| \leq k_1(|\xi| + |\xi|^{\alpha+1}),$$

$$(2.4) \quad k_2|\xi|^{\alpha+2} \leq g(\xi)\xi,$$

$$(2.5) \quad k_3|\xi|^\alpha \leq g'(\xi),$$

for all  $\lambda, \xi \in \mathbf{R}$ , where  $k_i$ ,  $i = 0, 1, 2, 3$ , are positive constants and  $\alpha > 1$ . Let  $c_0$ ,  $\rho$  be positive constants and  $c_1 \in \mathbf{R}$ . Given  $\delta_0 \in \mathbf{R}$ ,  $u_0 \in V \cap H^2(0, 1)$  and  $u_1 \in V \cap L^{2\alpha+2}(0, 1)$ , there exists a unique pair of functions  $(u(x, t), \delta(t))$  which is a solution to (1.1)–(1.5) from the class:

$$\begin{aligned} u, u' &\in L^\infty(0, T; C([0, 1])), & u'' &\in L^\infty(0, T; L^2(0, 1)); \\ \delta &\in C^1([0, T]), & \delta'' &\in L^\infty(0, T). \end{aligned}$$

*Proof.* Let  $\{w_j\}$  be a basis of  $V$ , orthonormal in  $L^2(0, 1)$ . For each  $\varepsilon \in (0, 1]$  and  $m \in \mathbf{N}$  we consider  $\delta_{\varepsilon m} : [0, T] \rightarrow \mathbf{R}$  and  $u_{\varepsilon m}(x, t) = \sum_{j=1}^m \varphi_{\varepsilon m j}(t) w_j(x)$ ,  $x \in (0, 1)$ ,  $t \in [0, T_{\varepsilon m}]$ , which are solutions to the following approximate problem

$$\left\{ \begin{aligned} &\left( \left( \varepsilon + \frac{1}{a(u_{\varepsilon m}(t))} \right) u_{\varepsilon m}''(t), w_j \right) + \left( \frac{\partial u_{\varepsilon m}}{\partial x}(t), \frac{\partial w_j}{\partial x} \right) - \delta_{\varepsilon m}'(t) w_j(1) \\ &\quad + \left( \frac{g(u_{\varepsilon m}'(t))}{a(u_{\varepsilon m}(t))}, w_j \right) = 0, \quad 1 \leq j \leq m, \\ &\rho u_{\varepsilon m}'(1, t) + \delta_{\varepsilon m}''(t) + c_1 \delta_{\varepsilon m}'(t) + c_0 \delta_{\varepsilon m}(t) = 0, \\ &u_{\varepsilon m}(0) = u_{0m}, \quad u_{\varepsilon m}'(0) = u_{1m}, \quad \delta_{\varepsilon m}(0) = \delta_0, \quad \delta_{\varepsilon m}'(0) = \frac{\partial u_{0m}}{\partial x}(1); \end{aligned} \right.$$

where  $\delta_0 \in \mathbf{R}$ ,  $u_{0m} = \sum_{j=1}^m (u_0, w_j) w_j$  and  $u_{1m} = \sum_{j=1}^m (u_1, w_j) w_j$ .

Since  $\varepsilon > 0$ , this approximate problem can be transformed into a normal system of ODE for the unknown functions  $\delta_{\varepsilon m}$  and  $\varphi_{\varepsilon m j}$ ,  $1 \leq j \leq m$ . Therefore, it has local solutions, defined at some interval  $[0, T_{\varepsilon m}]$ , which satisfy the following approximate equations:

$$(2.6) \quad \left\{ \begin{aligned} &\left( \left( \varepsilon + \frac{1}{a(u_{\varepsilon m}(t))} \right) u_{\varepsilon m}''(t), v \right) + \left( \frac{\partial u_{\varepsilon m}}{\partial x}(t), \frac{\partial v}{\partial x} \right) - \delta_{\varepsilon m}'(t) v(1) \\ &\quad + \left( \frac{g(u_{\varepsilon m}'(t))}{a(u_{\varepsilon m}(t))}, v \right) = 0, \quad \text{for all } v \in V_m = [w_1, \dots, w_m], \end{aligned} \right.$$

$$(2.7) \quad -\rho u_{\varepsilon m}'(1, t) = \delta_{\varepsilon m}''(t) + c_1 \delta_{\varepsilon m}'(t) + c_0 \delta_{\varepsilon m}(t).$$

*Estimate 1.* Taking  $v = 2u'_{em}(t)$  in (2.6) and multiplying (2.7) by  $2\delta'_{em}(t)$ , we have

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 \right] - 2\delta'_{em}(t)u'_{em}(1, t) \\ & + 2 \int_0^1 \frac{g(u'_{em}(x, t))u'_{em}(x, t)}{a(u_{em}(x, t))} dx = - \int_0^1 \frac{a'(u_{em}(x, t))(u'_{em}(x, t))^3}{(a(u_{em}(x, t)))^2} dx; \\ & - 2\delta'_{em}(t)u'_{em}(1, t) = \frac{d}{dt} \left[ \frac{1}{\rho} (\delta'_{em}(t))^2 + \frac{c_0}{\rho} (\delta_{em}(t))^2 \right] + \frac{2c_1}{\rho} (\delta'_{em}(t))^2. \end{aligned}$$

From here we get

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta'_{em}(t))^2 + \frac{c_0}{\rho} (\delta_{em}(t))^2 \right] \\ & + 2 \int_0^1 \frac{g(u'_{em}(x, t))u'_{em}(x, t)}{a(u_{em}(x, t))} dx \\ & = - \int_0^1 \frac{a'(u_{em}(x, t))(u'_{em}(x, t))^3}{(a(u_{em}(x, t)))^2} dx - \frac{2c_1}{\rho} (\delta'_{em}(t))^2. \end{aligned}$$

This equality, (2.2) and (2.4) give

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta'_{em}(t))^2 + \frac{c_0}{\rho} (\delta_{em}(t))^2 \right] \\ & + 2k_2 \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx \leq k_0 \int_0^1 \frac{|u'_{em}(x, t)|^3}{a(u_{em}(x, t))} dx + \frac{2|c_1|}{\rho} (\delta'_{em}(t))^2. \end{aligned}$$

Using Young's inequality for all  $\eta > 0$ , we obtain

$$\int_0^1 \frac{|u'_{em}(x, t)|^3}{a(u_{em}(x, t))} dx \leq \eta \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx + c(\eta) \int_0^1 \frac{|u'_{em}(x, t)|^2}{a(u_{em}(x, t))} dx.$$

Choosing  $\eta$  sufficiently small, we find

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta'_{em}(t))^2 + \frac{c_0}{\rho} (\delta_{em}(t))^2 \right] \\ & + A_1 \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx \leq A_2 \int_0^1 \frac{|u'_{em}(x, t)|^2}{a(u_{em}(x, t))} dx + A_3 (\delta'_{em}(t))^2; \end{aligned}$$

here and in expressions to follow,  $A_i$ ,  $i \in \mathbb{N}$ , denotes a positive constant which does not depend on  $\varepsilon, m$  and  $t$ . Integrating this inequality from 0 to  $t \leq T_{em}$ , we obtain

$$\begin{aligned} & \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta'_{em}(t))^2 \\ & + \frac{c_0}{\rho} (\delta_{em}(t))^2 + A_1 \int_0^t \int_0^1 \frac{|u'_{em}(x, \tau)|^{\alpha+2}}{a(u_{em}(x, \tau))} dx d\tau \\ & \leq A_4 + A_2 \int_0^t \int_0^1 \frac{|u'_{em}(x, \tau)|^2}{a(u_{em}(x, \tau))} dx d\tau + A_3 \int_0^t (\delta'_{em}(\tau))^2 d\tau. \end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned} (2.8) \quad & \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) (u'_{em}(x, t))^2 dx + \|u_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta'_{em}(t))^2 \\ & + \frac{c_0}{\rho} (\delta_{em}(t))^2 + A_1 \int_0^t \int_0^1 \frac{|u'_{em}(x, \tau)|^{\alpha+2}}{a(u_{em}(x, \tau))} dx d\tau \leq A_5. \end{aligned}$$

Taking into account that  $V \hookrightarrow H^1(0, 1) \hookrightarrow C([0, 1])$  and  $a$  is a continuous function, we can see that there exists a constant  $M$ , independent of  $\varepsilon, m$  and  $t$ , such that

$$(2.9) \quad 0 < a_0 \leq a(u_{em}(x, t)) \leq M; \quad \text{for all } x \in (0, 1), \quad t \in [0, T_{em}].$$

From (2.8) and (2.9), we have

$$(2.10) \quad \|u'_{em}(t)\|^2 + \|u_{em}(t)\|^2 + (\delta'_{em}(t))^2 + (\delta_{em}(t))^2 + \int_0^t \|u'_{em}(\tau)\|_{\alpha+2}^{\alpha+2} d\tau \leq A_6.$$

This is the first estimate.

*Estimate 2.* First of all we put  $t = 0$  in (2.6) and (2.7). It follows

$$\begin{aligned} & \left( \left( \varepsilon + \frac{1}{a(u_{0m})} \right) u''_{em}(0), v \right) - \left( \frac{\partial^2 u_{0m}}{\partial x^2}, v \right) + \left( \frac{g(u_{1m})}{a(u_{0m})}, v \right) = 0, \\ & \delta''_{em}(0) = -\rho u_{1m}(1) - c_1 \frac{\partial u_{0m}}{\partial x}(1) - c_0 \delta_0. \end{aligned}$$

Taking  $v = u''_{em}(0)$  in this equality, we find constants  $A_7$  and  $A_8$ , such that

$$(2.11) \quad \|u''_{em}(0)\| \leq A_7 \quad \text{and} \quad |\delta''_{em}(0)| \leq A_8.$$

Differentiating (2.6) and (2.7) with respect to  $t$ , we have

$$\begin{aligned}
 (2.12) \quad & \left( \left( \varepsilon + \frac{1}{a(u_{em}(t))} \right) u_{em}'''(t), v \right) + \left( \frac{\partial u_{em}'}{\partial x}(t), \frac{\partial v}{\partial x} \right) - \delta_{em}''(t)v(1) \\
 & \left( \frac{g'(u_{em}'(t))u_{em}''(t)}{a(u_{em}(t))}, v \right) - \left( \frac{g(u_{em}'(t))u_{em}'(t)a'(u_{em}(t))}{[a(u_{em}(t))]^2}, v \right) \\
 & - \left( \frac{a'(u_{em}(t))u_{em}'(t)u_{em}''(t)}{[a(u_{em}(t))]^2}, v \right) = 0,
 \end{aligned}$$

$$(2.13) \quad \delta_{em}'''(t) + c_1 \delta_{em}''(t) + c_0 \delta_{em}'(t) = -\rho u_{em}''(1, t).$$

Taking  $v = 2u_{em}''(t)$  in (2.12) and multiplying (2.13) by  $2\delta_{em}''(t)$ , we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) |u_{em}''(x, t)|^2 dx + \|u_{em}'(t)\|_V^2 + \frac{1}{\rho} (\delta_{em}''(t))^2 + \frac{c_0}{\rho} (\delta_{em}'(t))^2 \right] \\
 & + 2 \int_0^1 \frac{g'(u_{em}'(x, t))|u_{em}''(x, t)|^2}{a(u_{em}(x, t))} dx \\
 & = -\frac{2c_1}{\rho} (\delta_{em}''(t))^2 + \int_0^1 \frac{a'(u_{em}(x, t))u_{em}'(x, t)|u_{em}''(x, t)|^2}{[a(u_{em}(x, t))]^2} dx \\
 & + 2 \int_0^1 \frac{a'(u_{em}(x, t))g(u_{em}'(x, t))u_{em}'(x, t)u_{em}''(x, t)}{[a(u_{em}(x, t))]^2} dx.
 \end{aligned}$$

For all  $\eta > 0$ , Young's inequality gives

$$\begin{aligned}
 I_1 &= \int_0^1 \frac{a'(u_{em}(x, t))u_{em}'(x, t)|u_{em}''(x, t)|^2}{[a(u_{em}(x, t))]^2} dx \\
 &\leq \eta \int_0^1 \frac{|u_{em}'(x, t)|^\alpha |u_{em}''(x, t)|^2}{a(u_{em}(x, t))} dx + C(\eta) \int_0^1 \frac{|u_{em}''(x, t)|^2}{a(u_{em}(x, t))} dx; \\
 I_2 &= 2 \int_0^1 \frac{a'(u_{em}(x, t))g(u_{em}'(x, t))u_{em}'(x, t)u_{em}''(x, t)}{[a(u_{em}(x, t))]^2} dx \\
 &\leq 2k_0 k_1 \left( \int_0^1 \frac{|u_{em}'(x, t)|^2 |u_{em}''(x, t)|}{a(u_{em}(x, t))} dx + \int_0^1 \frac{|u_{em}'(x, t)|^{\alpha+2} |u_{em}''(x, t)|}{a(u_{em}(x, t))} dx \right) \\
 &\leq \eta \int_0^1 \frac{|u_{em}'(x, t)|^\alpha |u_{em}''(x, t)|^2}{a(u_{em}(x, t))} dx + C(\eta) \|u_{em}'(t)\|_V^2 \int_0^1 \frac{|u_{em}'(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx \\
 &\quad + 2k_0 k_1 \int_0^1 \frac{|u_{em}''(x, t)|^2}{a(u_{em}(x, t))} dx + A_{11};
 \end{aligned}$$

$$I_3 = 2 \int_0^1 \frac{g'(u'_{em}(x, t)) |u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx \geq 2k_3 \int_0^1 \frac{|u'_{em}(x, t)|^\alpha |u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx;$$

here we have used (2.3) and (2.4).

Therefore,

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) |u''_{em}(x, t)|^2 dx + \|u'_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta''_{em}(t))^2 + \frac{c_0}{\rho} (\delta'_{em}(t))^2 \right] \\ & + 2k_3 \int_0^1 \frac{|u'_{em}(x, t)|^\alpha |u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx \\ & \leq \frac{2|c_1|}{\rho} (\delta''_{em}(t))^2 + \eta \int_0^1 \frac{|u'_{em}(x, t)|^\alpha |u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx \\ & + C(\eta) \|u'_{em}(t)\|_V^2 \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx + \int_0^1 \frac{|u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx + A_{11}. \end{aligned}$$

Choosing  $\eta > 0$  sufficiently small we have

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) |u''_{em}(x, t)|^2 dx + \|u'_{em}(t)\|_V^2 + \frac{1}{\rho} (\delta''_{em}(t))^2 + \frac{c_0}{\rho} (\delta'_{em}(t))^2 \right] \\ & + A_{12} \int_0^1 \frac{|u'_{em}(x, t)|^\alpha |u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx \\ & \leq A_{13} \left( (\delta''_{em}(t))^2 + \int_0^1 \frac{|u''_{em}(x, t)|^2}{a(u_{em}(x, t))} dx + \|u'_{em}(t)\|_V^2 \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx + 1 \right). \end{aligned}$$

Integrating this, we get

$$\begin{aligned} & \Psi_{em}(t) + \int_0^t \int_0^1 \frac{|u'_{em}(x, \tau)|^\alpha |u''_{em}(x, \tau)|^2}{a(u_{em}(x, \tau))} dx d\tau \\ & \leq A_{14} \left( 1 + \int_0^t h_{em}(\tau) \Psi_{em}(\tau) d\tau \right), \end{aligned}$$

where

$$\Psi_{em}(t) = \int_0^1 \left( \varepsilon + \frac{1}{a(u_{em}(x, t))} \right) |u''_{em}(x, t)|^2 dx + \|u'_{em}(t)\|_V^2 + (\delta''_{em}(t))^2 + (\delta'_{em}(t))^2,$$

and

$$h_{em}(t) = 1 + \int_0^1 \frac{|u'_{em}(x, t)|^{\alpha+2}}{a(u_{em}(x, t))} dx.$$

From (2.10) we can see that there exists a constant  $A_{15}$  independent on  $\varepsilon, m, t$  such that

$$\int_0^t |h_{em}(\tau)| d\tau \leq A_{15},$$

then  $h_{em} \in L^1(0, T)$ . This and Gronwall's inequality give

$$(2.14) \quad \Psi_{em}(t) \leq A_{16} e^{A_{14} \int_0^T h_{em}(\tau) d\tau} = A_{17}, \quad \text{for all } t \in [0, T].$$

Hence

$$(2.15) \quad \|u''_{em}(t)\|^2 + \|u'_{em}(t)\|_V^2 + (\delta''_{em}(t))^2 + (\delta'_{em}(t))^2 \leq A_{18}, \quad \text{for all } t \in [0, T],$$

and the second estimate is proved.

Using estimates (2.10) and (2.15) we can pass to the limit as  $m \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  in the approximate problem to get

$$(2.16) \quad \left( \frac{u''(t)}{a(u(t))}, v \right) + \left( \frac{\partial u}{\partial x}(t), \frac{\partial v}{\partial x} \right) - \delta'(t)v(1) + \left( \frac{g(u'(t))}{a(u(t))}, v \right) = 0,$$

for all  $v \in V$ , a.e. on  $[0, T]$ .

$$(2.17) \quad \rho u'(1, t) + \delta''(t) + c_1 \delta'(t) + c_0 \delta(t) = 0, \quad \text{a.e. on } [0, T].$$

$$(2.18) \quad u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \quad \text{a.e. on } [0, 1].$$

$$(2.19) \quad \delta(0) = \delta_0, \quad \delta'(0) = \frac{\partial u_0}{\partial x}(1).$$

Taking in (2.16)  $v = \varphi \in \mathcal{D}(0, 1)$ , we have

$$\left( \frac{u''(t)}{a(u(t))} - \frac{\partial^2 u}{\partial x^2}(t) + \frac{g(u'(t))}{a(u(t))}, \varphi \right) = 0$$

which implies

$$u'' - a(u)u_{xx} + g(u') = 0 \quad \text{a.e. in } (0, 1) \times (0, T).$$

Uniqueness may be proved in the usual way. Theorem 2.1 is proved.

*Remark 2.1.* The result of Theorem 2.1 is valid even for  $\alpha = 1$  if we assume that  $k_2$  and  $k_3$  are sufficiently large.

### 3. Asymptotic behaviour

In this section, we shall prove the exponential and algebraic decay of the energy of (1.1)–(1.5). Therefore, instead of (2.3) and (2.4) we make the following stronger assumptions

$$(3.1) \quad |g(\xi)| \leq \overline{k_1}(|\xi|^{\beta+1} + |\xi|^{\alpha+1}),$$

$$(3.2) \quad \overline{k_2}(|\xi|^{\beta+2} + |\xi|^{\alpha+2}) \leq g(\xi)\xi;$$

for all  $\xi \in \mathbf{R}$ , with  $0 \leq \beta \leq 1$ , where

$$(3.3) \quad \overline{k_2} > \left[ \frac{k_0(\alpha-1)}{2(\alpha-\beta)} \right]^{(\alpha-1)/(\alpha-\beta)} \left[ \frac{k_0(1-\beta)}{(\alpha-\beta)} \right]^{(1-\beta)/(\alpha-\beta)}.$$

**Theorem 3.1.** Let  $(u, \delta)$  be a solution to the problem (1.1)–(1.5). Assume that  $c_1 > 0$  and the coefficient  $c_0$  in (1.3) satisfies the inequality

$$(3.4) \quad \frac{\rho}{c_0} < \frac{1}{\sqrt{8\bar{c}}},$$

where  $\bar{c} > 0$  is the constant of the embedding  $\|\cdot\|_{C([0,1])} \leq \bar{c}\|\cdot\|_V$ . Then there are positive constants  $K$  and  $C$  such that

$$(3.5) \quad \begin{cases} h(t) \leq Ke^{-Ct}, & \text{for all } t \geq 0, \text{ if } \beta = 0 \\ \text{or} \\ h(t) \leq C(1+t)^{-2/\beta}, & \text{for all } t \geq 0, \text{ if } 0 < \beta \leq 1 \end{cases}$$

here  $h(t) = \|u'(t)\|^2 + \|u(t)\|_V^2 + |\delta'(t)|^2 + |\delta(t)|^2$ .

*Proof.* Let us define a function  $\Phi : [0, \infty) \rightarrow \mathbf{R}$  by

$$(3.6) \quad \Phi(t) = \int_0^1 \frac{(u'(x, t))^2}{a(u(x, t))} dx + \|u(t)\|_V^2 + \frac{1}{\rho}(\delta'(t))^2 + \frac{c_0}{\rho}(\delta(t))^2.$$

Multiplying (1.1) by  $2u'(x, t)$  and integrating over  $(0, 1)$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \left[ \int_0^1 \frac{(u'(x, t))^2}{a(u(x, t))} dx + \|u(t)\|_V^2 \right] - 2\delta'(t)u'(1, t) + 2 \int_0^1 \frac{g(u'(x, t))u'(x, t)}{a(u(x, t))} dx \\ &= - \int_0^1 \frac{a'(u(x, t))(u'(x, t))^3}{[a(u(x, t))]^2} dx. \end{aligned}$$

This and (1.3) give

$$(3.7) \quad \begin{aligned} \Phi'(t) + 2 \int_0^1 \frac{g(u'(x, t))u'(x, t)}{a(u(x, t))} dx + \frac{2c_1}{\rho}(\delta'(t))^2 \\ = - \int_0^1 \frac{a'(u(x, t))(u'(x, t))^3}{[a(u(x, t))]^2} dx. \end{aligned}$$

From (2.2) and (3.2), we find

$$\begin{aligned} \Phi'(t) + 2\bar{k}_2 \int_0^1 \frac{|u'(x, t)|^{\beta+2}}{a(u(x, t))} dx + 2\bar{k}_2 \int_0^1 \frac{|u'(x, t)|^{\alpha+2}}{a(u(x, t))} dx + \frac{2c_1}{\rho} (\delta'(t))^2 \\ \leq k_0 \int_0^1 \frac{|u'(x, t)|^3}{a(u(x, t))} dx. \end{aligned}$$

By Young's inequality

$$\begin{aligned} \Phi'(t) + \frac{2c_1}{\rho} (\delta'(t))^2 + (2\bar{k}_2 - \varepsilon) \int_0^1 \frac{|u'(x, t)|^{\alpha+2}}{a(u(x, t))} dx \\ + \left[ 2\bar{k}_2 - \frac{k_0^{(\alpha-\beta)/(\alpha-1)} (\alpha-1)}{(\alpha-\beta)} \left( \frac{(\alpha-\beta)}{(1-\beta)} \varepsilon \right)^{-(1-\beta)/(\alpha-1)} \right] \int_0^1 \frac{|u'(x, t)|^{\beta+2}}{a(u(x, t))} dx \leq 0, \end{aligned}$$

for all  $\varepsilon > 0$ .

Choosing  $\varepsilon = \bar{k}_2$  and taking into account (3.3), we obtain

$$(3.8) \quad \Phi'(t) + \frac{2c_1}{\rho} (\delta'(t))^2 + A_1 \int_0^1 \frac{|u'(x, t)|^{\beta+2}}{a(u(x, t))} dx + \bar{k}_2 \int_0^1 \frac{|u'(x, t)|^{\alpha+2}}{a(u(x, t))} dx \leq 0,$$

where

$$A_1 = 2\bar{k}_2 - \frac{k_0^{(\alpha-\beta)/(\alpha-1)} (\alpha-1)}{(\alpha-\beta)} \left( \frac{(\alpha-\beta)}{(1-\beta)} \bar{k}_2 \right)^{-(1-\beta)/(\alpha-1)} > 0.$$

Inequality (3.8) shows that  $\Phi$  is a monotonically decreasing and bounded function for all  $t \geq 0$ . We define

$$\Theta(t) = [\Phi(t) - \Phi(t+1)]^{1/(\beta+2)}, \quad t \geq 0.$$

Integrating (3.8), we get

$$\begin{aligned} (3.9) \quad \Phi(t) + \frac{2c_1}{\rho} \int_0^t |\delta'(\tau)|^2 d\tau + A_1 \int_0^t \int_0^1 \frac{|u'(x, \tau)|^{\beta+2}}{a(u(x, \tau))} dx d\tau \\ + \bar{k}_2 \int_0^t \int_0^1 \frac{|u'(x, \tau)|^{\alpha+2}}{a(u(x, \tau))} dx d\tau \leq \Phi(0), \end{aligned}$$

and

$$\begin{aligned} (3.10) \quad \int_t^{t+1} |\delta'(\tau)|^2 d\tau + \int_t^{t+1} \|u'(\tau)\|_{L^{\beta+2}(0,1)}^{\beta+2} d\tau \\ + \int_t^{t+1} \|u'(\tau)\|_{L^{\alpha+2}(0,1)}^{\alpha+2} d\tau \leq A_2 \Theta^{\beta+2}(t), \quad \text{for all } t \geq 0. \end{aligned}$$

By the mean value theorem for integrals, there exist  $t_1 \in [t, t+1/4]$  and  $t_2 \in [t+3/4, t+1]$  such that

$$(3.11) \quad \|u'(t_i)\| \leq 2A_2^{1/(\beta+2)}\Theta(t), \quad |\delta'(t_i)| \leq 2A_2^{1/2}\Theta^{(\beta+2)/2}(t), \quad \text{for } i = 1, 2.$$

Moreover, from (3.6), we obtain

$$(3.12) \quad \|u(t)\|_V \leq \Phi^{1/2}(t) \quad \text{and} \quad |\delta(t)| \leq \sqrt{\frac{\rho}{c_0}}\Phi^{1/2}(t).$$

Another simple consequence of the mean value theorem is that there exists a point  $t^* \in [t_1, t_2]$  such that

$$(3.13) \quad \Phi(t^*) \leq 2 \int_{t_1}^{t_2} \Phi(\tau) d\tau.$$

Integrating (3.7) from  $t^*$  to  $t$ , we find

$$(3.14) \quad \begin{aligned} \Phi(t) = \Phi(t^*) + 2 \int_{t^*}^t \int_0^1 \frac{g(u'(x, \tau))u'(x, \tau)}{a(u(x, \tau))} dx d\tau \\ + \frac{2c_1}{\rho} \int_{t^*}^t |\delta'(\tau)|^2 d\tau + \int_{t^*}^t \int_0^1 \frac{a'(u(x, \tau))|u'(x, \tau)|^3}{[a(u(x, \tau))]^2} dx d\tau. \end{aligned}$$

To estimate the right hand side of (3.14), we observe that (2.1), (3.1) and (3.10) give

$$(3.15) \quad \begin{aligned} 2 \int_{t^*}^t \int_0^1 \frac{g(u'(x, \tau))u'(x, \tau)}{a(u(x, \tau))} dx d\tau \\ \leq \frac{2\bar{k}_1}{a_0} \left( \int_{t^*}^t \|u'(\tau)\|_{L^{\beta+2}(0,1)}^{\beta+2} d\tau + \int_{t^*}^t \|u'(\tau)\|_{L^{\alpha+2}(0,1)}^{\alpha+2} d\tau \right) \\ \leq \frac{4\bar{k}_1 A_2}{a_0} \Theta^{\beta+2}(t); \end{aligned}$$

$$(3.16) \quad \frac{2c_1}{\rho} \int_{t^*}^t |\delta'(\tau)|^2 d\tau \leq \frac{2c_1 A_2}{a_0} \Theta^{\beta+2}(t);$$

$$(3.17) \quad \begin{aligned} \int_{t^*}^t \int_0^1 \frac{a'(u(x, \tau))|u'(x, \tau)|^3}{[a(u(x, \tau))]^2} dx d\tau \leq \frac{k_0}{a_0} \int_{t^*}^t \int_0^1 |u'(x, \tau)|^3 dx d\tau \\ \leq A_3 \left( \int_{t^*}^t \|u'(\tau)\|_{L^{\beta+2}(0,1)}^{\beta+2} d\tau + \int_{t^*}^t \|u'(\tau)\|_{L^{\alpha+2}(0,1)}^{\alpha+2} d\tau \right) \leq A_4 \Theta^{\beta+2}(t). \end{aligned}$$

Taking into account (3.13)–(3.17), we find

$$(3.18) \quad \Phi(t) \leq 2 \int_{t_1}^{t_2} \Phi(\tau) d\tau + A_5 \Theta^{\beta+2}(t).$$

We need an estimate for the integral  $\int_{t_1}^{t_2} \Phi(\tau) d\tau$ . Using (3.6) and (3.10), we have

$$(3.19) \quad \int_{t_1}^{t_2} \Phi(\tau) d\tau \leq A_6(\Theta^2(t) + \Theta^{\beta+2}(t)) + \int_{t_1}^{t_2} \|u(\tau)\|_V^2 d\tau + \frac{c_0}{\rho} \int_{t_1}^{t_2} |\delta(\tau)|^2 d\tau.$$

Multiplying (1.1) by  $u(x, t)$  and integrating over  $(0, 1)$  we get

$$\begin{aligned} \|u(t)\|_V^2 &= -\frac{d}{dt} \int_0^1 \frac{u'(x, t)u(x, t)}{a(u(x, t))} dx + \int_0^1 \frac{|u'(x, t)|^2}{a(u(x, t))} dx \\ &\quad - \int_0^1 \frac{a'(u(x, t))|u'(x, t)|^2 u(x, t)}{[a(u(x, t))]^2} dx + \delta'(t)u(1, t) \\ &\quad - \int_0^1 \frac{g(u'(x, t))u(x, t)}{a(u(x, t))} dx. \end{aligned}$$

Integrating this from  $t_1$  to  $t_2$ , we obtain

$$\begin{aligned} (3.20) \quad &\int_{t_1}^{t_2} \|u(\tau)\|_V^2 d\tau \\ &= \int_0^1 \frac{u'(x, t_1)u(x, t_1)}{a(u(x, t_1))} dx - \int_0^1 \frac{u'(x, t_2)u(x, t_2)}{a(u(x, t_2))} dx \\ &\quad + \int_{t_1}^{t_2} \int_0^1 \frac{|u'(x, \tau)|^2}{a(u(x, \tau))} dx d\tau - \int_{t_1}^{t_2} \int_0^1 \frac{a'(u(x, \tau))|u'(x, \tau)|^2 u(x, \tau)}{[a(u(x, \tau))]^2} dx d\tau \\ &\quad + \int_{t_1}^{t_2} \delta'(\tau)u(1, \tau) d\tau - \int_{t_1}^{t_2} \int_0^1 \frac{g(u'(x, \tau))u(x, \tau)}{a(u(x, \tau))} dx d\tau. \end{aligned}$$

Since  $V \hookrightarrow C([0, 1])$ , making use of (3.10)–(3.12), we can estimate each term of the right hand side of (3.20) for all  $\eta > 0$  as follows:

$$\begin{aligned} (3.21) \quad &\int_0^1 \frac{u'(x, t_i)u(x, t_i)}{a(u(x, t_i))} dx \leq \frac{2A_2^{1/(\beta+2)}}{a_0} \Theta(t) \Phi^{1/2}(t) \\ &\leq \eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + C(\eta) \Theta^2(t); \quad \text{for } i = 1, 2. \end{aligned}$$

$$(3.22) \quad \int_{t_1}^{t_2} \int_0^1 \frac{|u'(x, \tau)|^2}{a(u(x, \tau))} dx d\tau \leq \frac{A_2^{2/(\beta+2)}}{a_0} \Theta^2(t);$$

$$\begin{aligned}
 (3.23) \quad & \int_{t_1}^{t_2} \int_0^1 \frac{a'(u(x, \tau)) |u'(x, \tau)|^2 u(x, \tau)}{[a(u(x, \tau))]^2} dx d\tau \\
 & \leq \frac{k_0}{a_0} \int_{t_1}^{t_2} \left( \max_{0 \leq x \leq 1} |u(x, \tau)| \right) \int_0^1 |u'(x, \tau)|^2 dx d\tau \\
 & \leq A_7 \int_{t_1}^{t_2} \|u(\tau)\|_V \|u'(\tau)\|^2 d\tau \leq A_8 \Theta^2(t);
 \end{aligned}$$

$$\begin{aligned}
 (3.24) \quad & \int_{t_1}^{t_2} \delta'(\tau) u(1, \tau) d\tau \leq \int_{t_1}^{t_2} \left( \max_{0 \leq x \leq 1} |u(x, \tau)| \right) |\delta'(\tau)| d\tau \\
 & \leq A_9 \int_{t_1}^{t_2} \|u(\tau)\|_V |\delta'(\tau)| d\tau \leq \eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + C(\eta) \Theta^{\beta+2}(t);
 \end{aligned}$$

$$\begin{aligned}
 (3.25) \quad & \int_{t_1}^{t_2} \int_0^1 \frac{g(u'(x, \tau)) u(x, \tau)}{a(u(x, \tau))} dx d\tau \\
 & \leq \frac{\bar{k}_1}{a_0} \int_{t_1}^{t_2} \int_0^1 (|u'(x, \tau)|^{\beta+1} + |u'(x, \tau)|^{\alpha+1}) |u(x, \tau)| dx d\tau \\
 & \leq \eta \int_{t_1}^{t_2} \left[ \max_{0 \leq x \leq 1} |u(x, \tau)|^\beta + \max_{0 \leq x \leq 1} |u(x, \tau)|^\alpha \int_0^1 |u(x, \tau)|^2 dx \right] d\tau \\
 & \quad + C(\eta) \int_{t_1}^{t_2} (\|u'(\tau)\|_{L^{\beta+2}(0,1)}^{\beta+2} + \|u'(\tau)\|_{L^{\alpha+2}(0,1)}^{\alpha+2}) d\tau \\
 & \leq C\eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + C(\eta) \Theta^{\beta+2}(t).
 \end{aligned}$$

From (3.20)–(3.25), we have

$$(3.26) \quad \int_{t_1}^{t_2} \|u(\tau)\|_V^2 d\tau \leq 4\eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + A(\eta) (\Theta^2(t) + \Theta^{\beta+2}(t)).$$

It remains to estimate the last term of (3.19). We multiply (1.3) by  $\delta(t)$  and integrate from  $t_1$  to  $t_2$ , to obtain

$$\begin{aligned}
 (3.27) \quad & \frac{c_0}{\rho} \int_{t_1}^{t_2} |\delta(\tau)|^2 d\tau \\
 & = u(1, t_1) \delta(t_1) - u(1, t_2) \delta(t_2) + \int_{t_1}^{t_2} u(1, \tau) \delta'(\tau) d\tau + \frac{1}{\rho} \delta'(t_1) \delta(t_1) \\
 & \quad - \frac{1}{\rho} \delta'(t_2) \delta(t_2) + \frac{1}{\rho} \int_{t_1}^{t_2} |\delta'(\tau)|^2 d\tau - \frac{c_1}{\rho} \int_{t_1}^{t_2} \delta(\tau) \delta'(\tau) d\tau.
 \end{aligned}$$

Let us estimate the right hand side of (3.27). First of all, using (3.12), we can see that

$$(3.28) \quad |u(1, t_i)\delta(t_i)| \leq \bar{c}\sqrt{\frac{\rho}{c_0}}\Phi(t), \quad \text{for } i = 1, 2.$$

Here  $\bar{c}$  is given by (3.4), and we have used that  $V \hookrightarrow C([0, 1])$  and the fact that  $\Phi$  is a monotonically decreasing function.

On the other hand,

$$(3.29) \quad \frac{1}{\rho}\delta'(t_i)\delta(t_i) \leq \eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + C(\eta)\Theta^{\beta+2}(t), \quad \text{for } \eta > 0, \quad i = 1, 2;$$

$$(3.30) \quad \frac{1}{\rho} \int_{t_1}^{t_2} |\delta'(\tau)|^2 d\tau \leq \frac{A_2}{\rho} \Theta^{\beta+2}(t);$$

$$(3.31) \quad \frac{c_1}{\rho} \int_{t_1}^{t_2} \delta(\tau)\delta'(\tau) d\tau \leq \frac{\varepsilon c_0}{\rho} \int_{t_1}^{t_2} |\delta(\tau)|^2 d\tau + C(\varepsilon)\Theta^{\beta+2}(t), \quad \text{for all } \varepsilon \geq 0.$$

Taking into account (3.24), (3.27) and (3.28)–(3.31), we get

$$\begin{aligned} \frac{c_0}{\rho} (1 - \varepsilon) \int_{t_1}^{t_2} |\delta(\tau)|^2 d\tau &\leq 2\bar{c}\sqrt{\frac{\rho}{c_0}}\Phi(t) + 3\eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) \\ &\quad + \left( 3C(\eta) + \frac{A_2}{\rho} + C(\varepsilon) \right) \Theta^{\beta+2}(t), \end{aligned}$$

choosing  $\varepsilon = 1/2$ , we find

$$(3.32) \quad \begin{aligned} \frac{c_0}{\rho} \int_{t_1}^{t_2} |\delta(\tau)|^2 d\tau &\leq 4\bar{c}\sqrt{\frac{\rho}{c_0}}\Phi(t) + 6\eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) \\ &\quad + (6C(\eta) + A_{10})\Theta^{\beta+2}(t). \end{aligned}$$

From (3.18), (3.19), (3.26) and (3.32), it follows

$$\left( 1 - 8\bar{c}\sqrt{\frac{\rho}{c_0}} \right) \Phi(t) \leq A_{11}\eta \left( \sup_{t \leq \tau \leq t+1} \Phi(\tau) \right) + \bar{A}(\eta)(\Theta^2(t) + \Theta^{\beta+2}(t)).$$

This inequality and assumption (3.4), for  $\eta > 0$  sufficiently small, imply

$$\sup_{t \leq \tau \leq t+1} \Phi(\tau) \leq A_{12}(\Phi(0))\Theta^2(t), \quad \text{for all } t \geq 0,$$

which can be rewritten as

$$(3.33) \quad \sup_{t \leq \tau \leq t+1} \Phi^{1+\beta/2}(\tau) \leq C(\Phi(0))[\Phi(t) - \Phi(t+1)], \quad \text{for all } t \geq 0.$$

By lemma of M. Nakao (see [6] and [7]), (3.5) holds, there with theorem 3.1 is proved.

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