



On L^p Regularity for Weak Derivatives of Spherically Symmetric Solutions of the Porous Media Equation

Hayasida, Kazuya
Kobayashi, Masatomo

(Citation)

Funkcialaj Ekvacioj, 45(1):23-51

(Issue Date)

2002-04

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100501196>

(URL)

<https://hdl.handle.net/20.500.14094/0100501196>



On L^p Regularity for Weak Derivatives of Spherically Symmetric Solutions of the Porous Media Equation

By

Kazuya HAYASIDA and Masatomo KOBAYASHI

(Kanazawa University and NEC Communication Systems LTD., Japan)

1. Introduction

We consider the Cauchy problem for the porous media equation

$$(1.1) \quad \begin{cases} u_t = \Delta u^m & \text{in } \mathbf{R}^N \times (0, T) \\ u(x, 0) = u_0 & \text{on } \mathbf{R}^N, \end{cases}$$

where $m > 1$ and $u_0 \geq 0$. By a solution of (1.1) we mean a function $u(x, t)$ such that

$$u(x, t) \geq 0 \quad \text{in } \mathbf{R}^N \times (0, T),$$

$$\int_0^T \int_{\mathbf{R}^N} [u(x, t)^2 + |\nabla_x u(x, t)|^2] dx dt < \infty$$

and

$$\int_0^T \int_{\mathbf{R}^N} (u\phi_t - \nabla_x u^m \cdot \nabla_x \phi) dx dt + \int_{\mathbf{R}^N} u_0(x)\phi(x, 0) dx = 0$$

for any continuously differentiable function $\phi(x, t)$ with compact support in $\mathbf{R}^N \times [0, T)$. The problem (1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the survey of Kalashnikov [6]. The existence and the uniqueness of solutions of (1.1) are due to [8] and [9] under some assumption on u_0 .

We are concerned to the regularity property for u . The local Hölder continuity of u was shown by Caffarelli and Friedman [3]. Aronson and Bénilan [1] proved that Δu^m belongs to $L^1_{loc}(\mathbf{R}^N \times (0, T))$. The method of their proof is to obtain the inequality

$$u_t \geq -\frac{1}{t} \left(m - 1 + \frac{2}{N} \right)^{-1} \quad \text{in } \mathbf{R}^N \times (0, T),$$

which is in the distribution sence. Soon after Bénilan [2] proved that

$$\Delta u^m \in L^p_{loc}(\mathbf{R}^N \times (0, T)),$$

if $1 < p < 1 + 1/m$. Here p needs to be less than 2 in virtue of $m > 1$. When $p = 2$, there is the following result by one of the authors [4]:

$$\partial_{x_i} \partial_{x_j} u^m \in L^2(\mathbf{R}^N \times (0, T)), \quad i, j = 1, \dots, N,$$

if $1 < m < 3N/(3N - 2)$. When u is spherically symmetric under some assumption on u_0 , this result was improved in [5] as follows:

the condition $1 < m < 3N/(3N - 2)$ can be weakend with $1 < m < 3$, for $N = 1, 2, 3$.

We consider the well-known Barenblatt solution:

$$w(x, t) = (t + \tau)^{-k} \left(\left[a^2 - \frac{k(m-1)}{2Nm} \frac{|x|^2}{(t + \tau)^{2k/N}} \right]_+ \right)^{1/(m-1)},$$

where $a, \tau > 0$ and $k = (m - (N - 2)/N)^{-1}$. We rewrite w simply with $(t + \tau)^{-k}([g]_+)^{1/(m-1)}$. Then obviously

$$\partial_{x_i} \partial_{x_j} w^m = ([g]_+)^{(2-m)/(m-1)} P_{ij} + ([g]_+)^{1/(m-1)} Q_{ij},$$

where P_{ij} and Q_{ij} are smooth functions. Hence we see that

$$\partial_{x_i} \partial_{x_j} w^m \in L^p(\mathbf{R}^N \times (0, T)), \quad \text{if } p(2 - m)/(m - 1) > -1.$$

Let u be the spherically symmetric solution of (1.1). Then from the above we conjecture that for any given $p > 2$

$$\partial_{x_i} \partial_{x_j} u^m \in L^p(\mathbf{R}^N \times (0, T)), \quad i, j = 1, \dots, N,$$

if m is close to 1. In this paper our first aim is to verify this conjecture (see Theorem 1). Secondly we give precisely the admissible value of m in Theorem 1, when $N = 1$ (see Theorem 2). The tool is to prepare some L^p -estimates with a weight by the integration by parts.

2. Results

Throughout this paper we suppose that $p \geq 2$. And for the case of $N \geq 2$ we set

$$(2.1) \quad q = N - 2 + p - \delta,$$

where δ is a number with

$$(2.2) \quad 0 < \delta < p - 1.$$

We rewrite (2.1) as follows;

$$(2.3) \quad N - 2 + p - q = \delta > 0.$$

For the case of $N = 1$ we set $q = p$. From the uniqueness we see that the solution u of (1.1) is spherically symmetric, if u_0 is so.

Our first aim is to prove

Theorem 1. *Let $N \geq 2$. Let u_0 be a spherically symmetric function such that $u_0 \in C_0^4(\mathbf{R}^N)$ and $u_0 \geq 0$ in $\mathbf{R}^N$. Then there exists a constant $m_0 > 1$, depending only on N and p , such that if $1 < m < m_0$, any solution u of (1.1) satisfies*

$$\partial_{x_i} \partial_{x_j} u^m \cdot |x|^{(p-q-2)/p} \in L^p(\mathbf{R}^N \times (0, T)) \quad i, j = 1, \dots, N.$$

In particular, we consider the case of $N = 1$, where we do not assume that u is spherically symmetric. We define

$$I(p) = 1 + \frac{4(p-1)}{p(p-1)^2 + 4},$$

which is used in the following theorem. For example $I(2) = 5/3$, $I(3) = 3/2$, $I(4) = 13/10$.

Then we have

Theorem 2. *Let $N = 1$. Let u_0 belong to $C_0^4(\mathbf{R}^1)$ such that $u_0 \geq 0$ in $\mathbf{R}^1$. Then the solution u of (1.1) satisfies*

$$\partial_x^2 u^m \in L^p(\mathbf{R} \times (0, T)),$$

if $1 < m < I(p)$.

3. Preliminaries

Throughout this paper we put $\alpha = 1 - 1/m > 0$. We set $v = u^m$ in (1.1). Then (1.1) becomes

$$(1.1') \quad \begin{cases} v^{-\alpha} v_t = m \Delta v & \text{in } \mathbf{R}^N \times (0, T) \\ v(x, 0) = u_0^m(x) & \text{on } \mathbf{R}^N. \end{cases}$$

By the usual way the approximating solution of (1.1') is considered. Let u_0 belong to $C_0^3(\mathbf{R}^N)$. Let $\eta > 0$ and n be a natural number such that $\text{supp } u_0 \subset B_n(O)$, where $B_n(O)$ is the open sphere with center O and radius n . We define

$$M = \sup_{\mathbf{R}^N} u_0 > 0, \quad Q_T^{(n)} = B_n(O) \times (0, T).$$

The following fact is due to the famous book [7]:

Let $0 < \beta < 1$. Then there exists a unique $v \in H^{2+\beta, 1+\beta/2}(\overline{Q_T^{(n)}})$ of

$$(3.1) \quad \begin{cases} v^{-\alpha} v_t = m \Delta v & \text{in } Q_T^{(n)} \\ v(x, 0) = (u_0(x) + \eta)^m & \text{on } B_n(O) \\ v(x, t) = \eta^m & \text{on } \partial B_n(O) \times (0, T), \end{cases}$$

where the space $H^{2+\beta, 1+\beta/2}(\overline{Q_T^{(n)}})$ with norm $|\cdot|_{\frac{(2+\beta)}{Q_T^{(n)}}}$ is referred to [7]. It is known that

$$\eta^m \leq v \leq (M + \eta)^m \quad \text{in } Q_T^{(n)}$$

and for some β with $0 < \beta < 1$

$$|v|_{\frac{(2+\beta)}{Q_T^{(n)}}} \leq C \quad (< \infty),$$

where β and C are independent of n .

If $u_0 \in C_0^4(\mathbf{R}^N)$, the solution of (3.1) satisfies

$$v \in H^{3+\beta, (3+\beta)/2}(\overline{Q_T^{(n)}}) \quad \text{and} \quad |v|_{\frac{(3+\beta)}{Q_T^{(n)}}} \leq C \quad (< \infty),$$

where C dose not depend on n . The following is known:

Taking $n \rightarrow \infty$ and $\eta \rightarrow +0$ afterwards, we can obtain the solution v of (1.1'). If we set $u = v^{1/m}$, then u is the solution of (1.1).

From now on let u_0 be in $C_0^4(\mathbf{R}^N)$. So, the solution v of (3.1) belongs to $H^{3+\beta, (3+\beta)/2}(\overline{Q_T^{(n)}})$.

If v is the spherically symmetric solution of (3.1), then v satisfies

$$(3.2) \quad \begin{cases} v^{-\alpha} v_t = m(v_{rr} + (N-1)r^{-1}v_r) & \text{in } Q_T^{(n)} \\ v(\cdot, 0) = (u_0(\cdot) + \eta)^m & \text{on } B_n(O) \\ v(n, t) = \eta^m & \text{on } (0, T). \end{cases}$$

Now we prepare the following lemmas. The next lemma is due to [5]. But we repeat its proof by way of precaution.

Lemma 3.1 ([5]). *For the solution v of (3.1) it holds that*

$$|\nabla v| \leq Cn^{-1} \quad \text{on } \partial B_n(O) \times (0, T),$$

where C is a positive constant independent of n and η .

Proof. Let x_0 be any fixed point on $\partial B_n(O)$ and v be the outer normal of $\partial B_n(O)$ at x_0 . We take a linear function $\pi(x)$ such that

$$\pi(x^0) = \eta^m, \quad v(x, 0) \leq \pi(x) \quad \text{on } B_n(O)$$

and

$$\left| \frac{\partial \pi(x^0)}{\partial v} \right| \leq Cn^{-1},$$

where C dose not depend on n , η and x^0 . Obviously

$$\pi_t - m\Delta\pi = 0 \quad \text{in } Q_T^{(n)},$$

$$v \leq \pi \quad \text{on } \partial B_n(O) \times (0, T) \text{ and } B_n(O) \times \{t = 0\}.$$

By the maximum principle we have

$$v \leq \pi \quad \text{in } Q_T^{(n)}.$$

Hence

$$0 \leq v(x, t) - v(x^0, t) \leq \pi(x) - \pi(x^0) \quad \text{in } Q_T^{(n)},$$

which implies immediately that

$$\left| \frac{\partial v(x^0, t)}{\partial v} \right| \leq \left| \frac{\partial \pi(x^0)}{\partial v} \right| \leq Cn^{-1}.$$

The tangential derivative of v at x^0 equals 0. Thus we have finished the proof of Lemma 3.1. Q.E.D.

Hereafter let $N \geq 2$, if not saying.

Lemma 3.2. *For the solution v of (3.2) it holds that*

$$|v_{rr}| \leq Cn^{-2} \quad \text{on } \partial B_n(O) \times (0, T),$$

where C is a positive constant independent of n and η .

Proof. From (3.2)

$$v_{rr} = \frac{1}{m} v^{-\alpha} v_t - (N-1)r^{-1}v_r.$$

We see that $v_t = 0$ on $\partial B_n(o) \times (0, T)$. Hence we obtain the required inequality from Lemma 3.1. Q.E.D.

Since v_{rr} is unifomly bounded in $B_n(O) \times (0, T)$ with respect to n , and $v_r(0, t) = 0$ we have the following lemma by using the mean value theorem:

Lemma 3.3. *For the solution v of (3.2) it holds that*

$$|v_r| \leq Cr \quad \text{in } B_{1/2}(O) \times (0, T),$$

where C dose not depend on n .

From (2.1) and (2.2)

$$(3.3) \quad N - 2 - p - q = \delta - 2p < 0$$

and

$$(3.4) \quad N + 1 + (p - q) - 2p = 3 - 2p + \delta < 0.$$

Lemma 3.4. *It holds that*

$$1 - \frac{p}{(p-1)(q+2-N)^2} (q(q+1) - (N-1)(p+q)) = \frac{\delta(Np - \delta)}{(p-1)(q+2-N)^2}$$

Proof. If we set

$$H = (p-1)(q+2-N)^2 - p(q(q+1) - (N-1)(p+q)),$$

then

$$\begin{aligned} H &= (p-1)(p-\delta)^2 - p\{(p + (N-2-\delta))(p + (N-1-\delta)) \\ &\quad - (N-1)(2p + (N-2-\delta))\}. \end{aligned}$$

Obviously

$$\begin{aligned} (p-1)(p-\delta)^2 &= p^3 - (1+2\delta)p^2 + \delta(2+\delta)p - \delta^2, \\ (p + (N-2-\delta))(p + (N-1-\delta)) - (N-1)(2p + (N-2-\delta)) \\ &= p^2 - (1+2\delta)p - \delta(N-2-\delta). \end{aligned}$$

Hence

$$H = \delta(Np - \delta).$$

This means the required equality.

Q.E.D.

From now on we denote by $(\cdot, \cdot)$ the inner product of $L^2(0, n)$. And we write often by $o(1)$ simply the notation of Landau, $o(1)$ ($n \rightarrow \infty$). We use often the integration by parts without saying.

Lemma 3.5. *For the solution v of (3.2) it holds that*

$$(|v_r|^p, r^{N-3-q}) \leq \left(\frac{p}{p-\delta}\right)^2 (|v_r|^{p-2} v_{rr}^2, r^{N-1-q}) + o(1) \quad (n \rightarrow \infty).$$

Proof. Obviously

$$\begin{aligned} (|v_r|^p, r^{N-3-q}) &= \frac{1}{N-2-q} (|v_r|^p, (r^{N-2-q})_r) \\ &= -\frac{p}{N-2-q} (|v_r|^{p-2} v_r, v_{rr} r^{N-2-q}) \\ &\quad + \frac{1}{N-2-q} [|v_r|^p r^{N-2-q}]_{r=0}^{r=n}. \end{aligned}$$

From (2.3) and Lemma 3.3

$$\lim_{r \rightarrow 0} (|v_r|^p r^{N-2-q})(r, t) = 0.$$

And from (3.3) and Lemma 3.1

$$\lim_{r \rightarrow \infty} (|v_r|^p r^{N-2-q})(n, t) = 0.$$

Hence using (3.3) again and Cauchy's inequality, we have

$$\begin{aligned} (|v_r|^p, r^{N-3-q}) &= \frac{p}{p-\delta} (|v_r|^{p-2} v_r, v_{rr} r^{N-2-q}) + o(1) \\ &\leq \frac{p}{p-\delta} \left(\varepsilon (|v_r|^p, r^{N-3-q}) + \frac{1}{4\varepsilon} (|v_r|^{p-2} v_{rr}^2, r^{N-1-q}) \right) + o(1). \end{aligned}$$

If we put $\varepsilon = (p-\delta)/(2p)$ particularly, the required inequality is obtained.
Q.E.D.

We set $\kappa = p - q$ and $\psi_\eta = (u_0 + \eta)^m$ for the initial function u_0 in our theorems. Then we have

Lemma 3.6. *Let u_0 be the function in Theorem 1. Then*

$$\begin{aligned} &((\psi_\eta)^{-\alpha} |(\psi_\eta)_{rr}|^p, r^{N-1+\kappa}) \\ &\leq C(M+1)^{(p-1)(m-1)} [(|u_0)_{rr}|^p, r^{N-1+\kappa}) + (|(u_0)_r|^p, r^{N-1+\kappa})], \end{aligned}$$

where C does not depend on η . If u_0 is the function in Theorem 2, then

$$((\psi_\eta)^{-\alpha} |(\psi_\eta)_{rr}|^p) \leq C(M+1)^{(p-1)(m-1)} (1, |(u_0)_{rr}|^p),$$

where $(\cdot, \cdot)$ means the $L^2(-n, n)$ inner product.

Proof. We prove the first half. We put $l = N - 1 + \kappa$, $s = m - 1 + p(2 - m)$ and $w = u_0 + \eta$. Obviously

$$(\psi_\eta)^{-\alpha} |(\psi_\eta)_{rr}|^p \leq C(w^{(p-1)(m-1)} |w_{rr}|^p + w^{-s} |w_r|^{2p}).$$

So, it is enough to estimate $(w^{-s}, |w_r|^{2p} r^l)$ only. By integration by parts

$$\begin{aligned} (w^{-s}, |w_r|^{2p} r^l) &= (w^{-s}, |w_r|^{2p-2} w_r w_r r^l) \\ &= s(w^{-s}, |w_r|^{2p} r^l) - (2p-1)(w^{1-s}, |w_r|^{2p-2} w_{rr} r^l) \\ &\quad - l(w^{1-s}, |w_r|^{2p-2} w_r r^{l-1}) + [w^{1-s} |w_r|^{2p-2} w_r r^l]_{r=0}^{r=n}. \end{aligned}$$

The last term vanishes from (2.3). Hence

$$(s-1)(w^{-s}, |w_r|^{2p} r^l) \leq (2p-1)(w^{1-s}, |w_r|^{2p-2} |w_{rr}| r^l) + l(w^{1-s}, |w_r|^{2p-1} r^{l-1}).$$

By Hölder's inequality

$$\begin{aligned} (w^{1-s}, |w_r|^{2p-2} |w_{rr}|^l r^l) &\leq \varepsilon (w^{-s}, |w_r|^{2p} r^l) + C(\varepsilon) (w^{p-s}, |w_{rr}|^p r^l), \\ (w^{1-s}, |w_r|^{2p-1} r^{l-1}) &\leq \varepsilon (w^{-s}, |w_r|^{2p} r^l) + C(\varepsilon) (w^{p-s}, |w_r|^p r^{l-p}). \end{aligned}$$

Noting that $p - s = (p - 1)(m - 1)$, we obtain the first half from the above and the inequality

$$(3.5) \quad (|w_r|^p, r^{l-p}) \leq C(|(u_0)_{rr}|^p, r^{N-1+\kappa}).$$

By the way of precaution we verify (3.5) at the end of this section, see *Remark*. Secondly let u_0 be the function in Theorem 2. Since $l = 0$, it follows from the above that

$$\begin{aligned} (s - 1)(w^{-s}, |w_r|^{2p}) &\leq (2p - 1)(w^{1-s} |w_r|^{2p-2}, |w_{rr}|) \\ &\leq \varepsilon (w^{-s}, |w_r|^{2p}) + C(\varepsilon) (w^{p-s}, |w_r|^p r^{l-p}). \end{aligned}$$

Thus the second estimate has been obtained. More easily we can prove that

$$(3.6) \quad ((\psi_\eta)^{-\alpha} |(\psi_\eta)_r|^p, r^{N-1-q}) \leq C(M + 1)^{(p-1)(m-1)} (|(u_0)_r|^p, r^{N-1-q})$$

for the function u_0 in Theorem 1.

Q.E.D.

Remark. The function w in the proof of Lemma 3.6 satisfies $w_r = O(r)$ ($r \rightarrow 0$), since w is spherically symmetric.

In general, let $f(r)$ be in $C^1[0, \infty)$ such that $f(0) = 0$ and $f(r) = 0$ for sufficiently large r . Let $a > 1$, $\varepsilon > 0$ and $a \neq \varepsilon$. Then it holds that

$$(3.7) \quad \int_0^\infty |f(r)|^a r^{\varepsilon-a-1} dr \leq C \int_0^\infty |f'(r)|^a r^{\varepsilon-1} dr,$$

where C depends only on a and ε .

The inequality (3.5) is obtained immediately from (3.7). So, it is enough to prove (3.7). By the mean value theorem, $f(r) = O(r)$ ($r \rightarrow 0$). By integration by parts

$$\begin{aligned} \int_0^\infty |f(r)|^a r^{\varepsilon-a-1} dr &= \frac{1}{\varepsilon - a} \int_0^\infty |f(r)|^a (r^{\varepsilon-a})' dr = \frac{1}{\varepsilon - a} [|f(r)|^a r^{\varepsilon-a}]_0^\infty \\ &\quad - \frac{a}{\varepsilon - a} \int_0^\infty |f(r)|^{a-2} f(r) f'(r) r^{\varepsilon-a} dr. \end{aligned}$$

Hence

$$\int_0^\infty |f|^a r^{\varepsilon-a-1} dr \leq C \int_0^\infty |f|^{a-1} |f'| r^{\varepsilon-a} dr$$

(by Hölder's inequality)

$$\leq C \left(\int_0^\infty |f|^a r^{\varepsilon-a-1} dr \right)^{(a-1)/a} \cdot \left(\int_0^\infty |f'|^a r^{\varepsilon-1} dr \right)^{1/a},$$

which means (3.7).

4. Lemma 4.1 and its proof

From now on we denote by $((,))$ the integral $\int_0^T (,) dt$.

Lemma 4.1. *For the solution v of (3.2) it holds that*

$$\begin{aligned} (4.1) \quad & ((|v_r|^{p-2}, v_{rr}^2 r^{N-1-q})) + \frac{1}{p(p-1)} ((N-1)(p+q) - q(q+1)) (|v_r|^p, r^{N-3-q}) \\ &= \frac{\alpha}{p(p+1)} ((v^{-2} |v_r|^{p+2}, r^{N-1-q})) \\ &+ \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1} v_r, |v_r|^p r^{N-2-q})) \\ &- \frac{1}{p(p-1)m} [(v^{-\alpha}, |v_r|^p r^{N-1-q})]_{t=0}^{t=T} \\ &+ o(1) \quad (n \rightarrow \infty). \end{aligned}$$

Proof. From (3.2)

$$(v^{-\alpha} v_t, |v_r|^{p-2} v_{rr} r^{N-1-q}) = m(|v_r|^{p-2}, v_{rr}^2 r^{N-1-q}) + m(N-1)(|v_r|^{p-2} v_r, v_{rr} r^{N-2-q}).$$

Obviously

$$\begin{aligned} (|v_r|^{p-2} v_r, v_{rr} r^{N-2-q}) &= \frac{1}{p} ((|v_r|^p)_r, r^{N-2-q}) \\ &= -\frac{N-2-q}{p} (|v_r|^p, r^{N-3-q}) + \frac{1}{p} [|v_r|^p r^{N-2-q}]_{r=0}^{r=n}. \end{aligned}$$

As previously

$$\lim_{r \rightarrow +0} (|v_r|^p r^{N-2-q})(r, t) = 0, \quad \lim_{n \rightarrow \infty} (|v_r|^p r^{N-2-q})(n, t) = 0.$$

Hence we obtain

$$\begin{aligned} (4.2) \quad & (v^{-\alpha} v_t, |v_r|^{p-2} v_{rr} r^{N-1-q}) = m(|v_r|^{p-2}, v_{rr}^2 r^{N-1-q}) \\ &- \frac{m}{p} (N-1)(N-2-q) (|v_r|^p, r^{N-3-q}) \\ &+ o(1). \end{aligned}$$

On the other hand

$$\begin{aligned}
(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}) &= \frac{1}{p-1}(v^{-\alpha}v_t, (|v_r|^{p-2}v_r)_r r^{N-1-q}) \\
&= -\frac{1}{p-1}((v^{-\alpha}v_t)_r, |v_r|^{p-2}v_{rr}r^{N-1-q}) \\
&\quad - \frac{N-1-q}{p-1}(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-2-q}) \\
&\quad + \frac{1}{p-1}[v^{-\alpha}v_t |v_r|^{p-2}v_{rr}r^{N-1-q}]_{r=0}^{r=n}.
\end{aligned}$$

As previously

$$\lim_{r \rightarrow +0} (|v_r|^{p-2}v_{rr}r^{N-1-q})(r, t) = 0.$$

Since $(v^{-\alpha}v_t)_r = v^{-\alpha}v_{tr} - \alpha v^{-\alpha-1}v_r v_t$, we have

$$\begin{aligned}
(4.3) \quad (v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}) &= \frac{1}{p-1}(v^{-\alpha}v_t, (|v_r|^{p-2}v_r)_r r^{N-1-q}) \\
&= -\frac{1}{p-1}(v^{-\alpha}v_{tr}, |v_r|^{p-2}v_{rr}r^{N-1-q}) \\
&\quad + \frac{\alpha}{p-1}(v^{-\alpha-1}v_t v_r, |v_r|^{p-2}v_{rr}r^{N-1-q}) \\
&\quad - \frac{N-1-q}{p-1}(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-2-q}) \\
&\quad + \frac{1}{p-1}(v^{-\alpha}v_t |v_r|^{p-2}v_{rr}r^{N-1-q})(n, t).
\end{aligned}$$

Here the last term vanishes by virtue of (3.2).

We can write

$$-\frac{1}{p-1}(v^{-\alpha}v_{tr}, |v_r|^{p-2}v_{rr}r^{N-1-q}) = -\frac{1}{p(p-1)}(v^{-\alpha}, (|v_r|^p)_t r^{N-1-q}).$$

Hence

$$\begin{aligned}
-\frac{1}{p-1}((v^{-\alpha}v_{tr}, |v_r|^{p-2}v_{rr}r^{N-1-q})) &= -\frac{\alpha}{p(p-1)}((v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q})) \\
&\quad - \frac{1}{p(p-1)}[(v^{-\alpha}, |v_r|^p r^{N-1-q})]_{t=0}^{t=T}.
\end{aligned}$$

Combining this with (4.3), we obtain

$$(4.4) \quad \begin{aligned} ((v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q})) &= \frac{\alpha}{p}((v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q})) \\ &\quad - \frac{N-1-q}{p-1}((v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-2-q})) \\ &\quad - \frac{1}{p(p-1)}[(v^{-\alpha}, |v_r|^p r^{N-1-q})]_{t=0}^{t=T}. \end{aligned}$$

First we estimate the first term on the right-hand side of (4.4). From (3.2)

$$(v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}) = m(v^{-1}v_{rr}, |v_r|^p r^{N-1-q}) + m(N-1)(v^{-1}v_r, |v_r|^p r^{N-2-q}).$$

And

$$\begin{aligned} (v^{-1}v_{rr}, |v_r|^p r^{N-1-q}) &= \frac{1}{p+1}(v^{-1}, (|v_r|^p v_r)_r r^{N-1-q}) \\ &= \frac{1}{p+1}(v^{-2}, |v_r|^{p+2} r^{N-1-q}) \\ &\quad - \frac{N-1-q}{p+1}(v^{-1}v_r, |v_r|^p r^{N-2-q}) \\ &\quad + \frac{1}{p+1}[v^{-1}v_r |v_r|^p r^{N-1-q}]_{r=0}^{r=n}. \end{aligned}$$

As previously

$$\lim_{r \rightarrow +0} (v_r |v_r|^p r^{N-1-q})(r, t) = 0, \quad \lim_{n \rightarrow \infty} (v_r |v_r|^p r^{N-1-q})(n, t) = 0.$$

So from the above

$$(4.5) \quad \begin{aligned} (v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}) &= \frac{m}{p+1}(v^{-2}, |v_r|^{p+2} r^{N-1-q}) \\ &\quad + \frac{m((N-1)p+q)}{p+1}(v^{-1}v_r, |v_r|^p r^{N-2-q}) \\ &\quad + o(1). \end{aligned}$$

Next we estimate the second term on the right-hand side of (4.4). From (3.2) again

$$(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-2-q}) = m(v_{rr}v_r |v_r|^{p-2}, r^{N-2-q}) + m(N-1)(|v_r|^p, r^{N-3-q}).$$

And

$$\begin{aligned}
(v_{rr}v_r|v_r|^{p-2}, r^{N-2-q}) &= \frac{1}{p} ((|v_r|^p)_r, r^{N-2-q}) \\
&= -\frac{N-2-q}{p} (|v_r|^p, r^{N-3-q}) \\
&\quad + \frac{1}{p} [|v_r|^p r^{N-2-q}]_{r=0}^{r=n}.
\end{aligned}$$

As previously the last term equals $o(1)$. Hence we have

$$\begin{aligned}
(4.6) \quad (v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-2-q}) &= \frac{m}{p} ((p-1)(N-2) + p + q) (|v_r|^p, r^{N-3-q}) \\
&\quad + o(1).
\end{aligned}$$

Combining (4.4) and (4.5) with (4.6), we obtain

$$\begin{aligned}
(4.7) \quad ((v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q})) &= \frac{m\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) \\
&\quad + \frac{m\alpha}{p(p+1)} ((N-1)p + q) ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \\
&\quad - \frac{1}{p(p-1)} [(v^{-\alpha}, |v_r|^p r^{N-1-q})]_{t=0}^{t=T} \\
&\quad - \frac{m(N-1-q)}{p(p-1)} ((p-1)(N-2) + p + q) \\
&\quad \cdot ((|v_r|^p, r^{N-3-q})) \\
&\quad + o(1) \quad (n \rightarrow \infty).
\end{aligned}$$

Here we note that

$$\begin{aligned}
&(N-1-q)((p-1)(N-1) + q + 1) - (p-1)(N-1)(N-2-q) \\
&= (N-1)(p+q) - q(q+1).
\end{aligned}$$

Therefore the required equality follows from (4.2) and (4.7).

Q.E.D.

5. The first L^p estimate

We investigate the inequality deduced from Lemma 4.1.

We consider the first case: $(N-1)(p+q) - q(q+1) \geq 0$. From Lemma 3.5

$$\begin{aligned}
((|v_r|^{p-2}, v_{rr}^2 r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) &\leq \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) ((|v_r|^{p-2} v_{rr}^2, r^{N-1-q})) \\
&\quad + o(1).
\end{aligned}$$

And using Lemma 4.1, we have

$$\begin{aligned}
(5.1) \quad &((|v_r|^{p-2}, v_{rr}^2 r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) \\
&\leq \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) \frac{\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) \\
&\quad + \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1) \quad (n \rightarrow \infty),
\end{aligned}$$

where C is a positive constant depending only on p .

We consider the second case: $(N-1)(p+q) - q(q+1) < 0$. We multiply the both sides of the inequality in Lemma 3.5, by $[(N-1)(p+q) - q(q+1)]/[p(p-1)]$.

$$\begin{aligned}
(5.2) \quad &-\left(\frac{p}{p-\delta}\right)^2 \frac{1}{p(p-1)} (q(q+1) - (N-1)(p+q)) ((|v_r|^{p-2} v_{rr}^2, r^{N-1-q})) \\
&\leq \frac{1}{p(p-1)} ((N-1)(p+q) - q(q+1)) ((|v_r|^p, r^{N-3-q})).
\end{aligned}$$

Next adding $((|v_r|^{p-2} v_{rr}^2, r^{N-1-q}))$ to the both sides of (5.2), we obtain from Lemmas 3.4, 4.1 and (5.2) again

$$\begin{aligned}
&\left(1 - \frac{p}{(p-1)(p-\delta)^2} (q(q+1) - (N-1)(p+q))\right) ((|v_r|^{p-2} v_{rr}^2, r^{N-1-q})) \\
&= \frac{\delta(Np-\delta)}{(p-1)(p-\delta)^2} ((|v_r|^{p-2} v_{rr}^2, r^{N-1-q})) \\
&\leq ((|v_r|^{p-2} v_{rr}^2, r^{N-1-q})) + \frac{(N-1)(p+q) - q(q+1)}{p(p-1)} ((|v_r|^p, r^{N-3-q})) \\
&\leq \frac{\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) + \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1).
\end{aligned}$$

Hence

$$\begin{aligned}
((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) &\leq \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)} \frac{\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) \\
&\quad + \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)} \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1).
\end{aligned}$$

From the above and Lemma 3.5 it follows that

$$\begin{aligned}
&((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) \\
&\leq \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) ((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) \\
&\leq \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)} \frac{\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) \\
&\quad + \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)} \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1).
\end{aligned}$$

This means that

$$\begin{aligned}
(5.3) \quad &((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) \\
&\leq \left(1 + \left(\frac{p}{p-\delta}\right)^2\right) \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)} \\
&\quad \times \left\{ \frac{\alpha}{p(p+1)} ((v^{-2}|v_r|^{p+2}, r^{N-1-q})) \right. \\
&\quad \left. + \frac{\alpha((N-1)p+q)}{p(p+1)} ((v^{-1}v_r, |v_r|^p r^{N-2-q})) \right\} \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1),
\end{aligned}$$

where C is the positive constant as in (5.1).

Combining (5.1) and (5.3) with the above, we obtain

$$\begin{aligned}
(5.4) \quad &((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) \\
&\leq C\alpha\{((v^{-2}|v_r|^{p+2}, r^{N-1-q})) + ((v^{-1}, |v_r|^{p+1} r^{N-2-q}))\} \\
&\quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} \\
&\quad + o(1).
\end{aligned}$$

We estimate the right-hand side of (5.4). First we have by integration by parts

$$\begin{aligned} (v^{-2}|v_r|^{p+2}, r^{N-1-q}) &= (v^{-2}|v_r|^p v_r, v_r r^{N-1-q}) \\ &= 2(v^{-2}|v_r|^{p+2}, r^{N-1-q}) - (p+1)(v^{-1}v_{rr}|v_r|^p, r^{N-1-q}) \\ &\quad - (N-1-q)(v^{-1}v_r|v_r|^p, r^{N-2-q}) + [v^{-1}v_r|v_r|^p r^{N-1-q}]_{r=0}^{r=n}. \end{aligned}$$

As previously

$$\lim_{r \rightarrow +0} (|v_r|^{p+1} r^{N-1-q})(r, t) = 0, \quad \lim_{n \rightarrow \infty} (v^{-1}v_r|v_r|^p r^{N-1-q})(n, t) = 0.$$

Hence

$$\begin{aligned} (5.5) \quad (v^{-2}|v_r|^{p+2}, r^{N-1-q}) &= (p+1)(v^{-1}v_{rr}|v_r|^p, r^{N-1-q}) \\ &\quad + (N-1-q)(v^{-1}v_r|v_r|^p, r^{N-2-q}) \\ &\quad + o(1) \quad (n \rightarrow \infty). \end{aligned}$$

We use the following Cauchy's inequalities;

$$\begin{aligned} |(v^{-1}v_{rr}|v_r|^p, r^{N-1-q})| &\leq \varepsilon(v^{-2}|v_r|^{p+2}, r^{N-1-q}) + \frac{1}{4\varepsilon}(|v_r|^{p-2}v_{rr}^2, r^{N-1-q}), \\ (v^{-1}|v_r|^{p+1}, r^{N-2-q}) &\leq \varepsilon(v^{-2}|v_r|^{p+2}, r^{N-1-q}) + \frac{1}{4\varepsilon}(|v_r|^p, r^{N-3-q}). \end{aligned}$$

We see that $N-1-q = 1-p+\delta < 0$ from (2.3). Since $(p+1) + |N-1-q| = 2p-\delta$, it follows from the above and (5.5) that

$$\begin{aligned} (1 - \varepsilon(2p - \delta))(v^{-2}|v_r|^{p+2}, r^{N-1-q}) &\leq \frac{p+1}{4\varepsilon}(|v_r|^{p-2}v_{rr}^2, r^{N-1-q}) \\ &\quad + \frac{2p-\delta}{4\varepsilon}(|v_r|^p, r^{N-3-q}) + o(1). \end{aligned}$$

So setting $\varepsilon = \frac{1}{2(2p-\delta)}$, we have

$$\begin{aligned} (v^{-2}|v_r|^{p+2}, r^{N-1-q}) &\leq (p+1)(2p-\delta)(|v_r|^{p-2}v_{rr}^2, r^{N-1-q}) + (2p-\delta)^2(|v_r|^p, r^{N-3-q}) \\ &\quad + o(1). \end{aligned}$$

Hence we can estimate the first term on the right-hand side of (5.4). The second term can be also estimated from the above.

Combining these inequalities with (5.4), we conclude the following

Proposition 5.1. *There is a positive constant α_0 such that for the solution v of (3.2) it holds that if $0 < \alpha < \alpha_0$,*

$$((|v_r|^{p-2}v_{rr}^2, r^{N-1-q})) + ((|v_r|^p, r^{N-3-q})) \leq C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1) \\ (n \rightarrow \infty),$$

where α_0 and C depend only on N and p , but not on n and η .

6. Lemma 6.1 and its proof

We put $\kappa = p - q$ as in Lemma 3.6 for the sake of simplicity.

Lemma 6.1. *Let v be the solution of (3.2). Then*

$$(6.1) \quad \frac{\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) ((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa})) \\ + \frac{N-1+\kappa}{p-1} \left(N-1 - \frac{N-2+\kappa}{p} \right) ((|v_{rr}|^p, r^{N-3+\kappa})) \\ - (N-1)((v_r|v_{rr}|^{p-2}, v_{rrr}r^{N-3+\kappa})) + (N-1)((v_{rr}|v_{rr}|^{p-2}, v_{rrr}r^{N-2+\kappa})) \\ = \frac{3-p}{p-1} \alpha ((v^{-1}v_r|v_{rr}|^{p-2}, v_{rr}v_{rrr}r^{N-1+\kappa})) - \frac{\alpha\kappa}{p} ((v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa})) \\ - \alpha(N-1)((v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa})) \\ + \frac{\alpha^2}{p-1} (N-1)((v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa})) \\ - \frac{\alpha}{p-1} (N-1)(N+\kappa)((v^{-1}v_r^2, |v_r|^{p-2}v_{rr}r^{N-3+\kappa})) \\ + (N-1) \frac{N-1+\kappa}{p-1} ((v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa})) \\ - \frac{1}{p(p-1)m} [(v^{-\alpha}|v_{rr}|^p, r^{N-1+\kappa})]_{t=0}^{t=T} \\ + o(1) \quad (n \rightarrow \infty).$$

Proof. First by integration by parts

$$-(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) = -\alpha(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ + (v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ - [v^{-\alpha}v_t|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}]_{r=0}^{r=n}.$$

Noting that $v_t = 0$ for $r = n$. We have from (2.3)

$$(6.2) \quad -(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) = -\alpha(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ + (v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}).$$

From (3.2)

$$(6.3) \quad -\alpha(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) = -\alpha m(v^{-1}v_rv_{rr}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ - \alpha m(N-1)(v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}).$$

Combining (2.3) with (3.4), we have

$$(6.4) \quad (v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) = \frac{1}{p-1}(v^{-\alpha}v_{rt}, (|v_{rr}|^{p-2}v_{rr})_r r^{N-1+\kappa}) \\ = \frac{\alpha}{p-1}(v^{-\alpha-1}v_rv_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ - \frac{1}{p-1}(v^{-\alpha}v_{trr}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ - \frac{N-1+\kappa}{p-1}(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}) \\ + o(1).$$

In (6.2) we use the function $(|v_{rr}|^{p-2}, v_{rrr}r^{N-1+\kappa})_r$ in spite of $v \in H^{3+\beta, (3+\beta)/2}$ but it is not contradictory we may take an approximating smooth sequence of v .

We estimate the second term on the right-hand side of (6.4). We can write

$$(v^{-\alpha}v_{trr}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) = \frac{1}{p}(v^{-\alpha}, (|v_{rr}|^p)_t r^{N-1+\kappa}).$$

Since $(v^{-\alpha}v_t)_r = v^{-\alpha}v_{rt} - \alpha v^{-\alpha-1}v_tv_r$, it follows from (3.2) that

$$v^{-\alpha}v_{tr} = mv_{rrr} + m(N-1)(v_r r^{-1})_r + \alpha m v^{-1}v_r v_{rr} + \alpha m(N-1)r^{-1}v^{-1}v_r^2.$$

Hence (6.4) becomes

$$(6.5) \quad (v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ = \frac{\alpha m}{p-1}(v^{-1}v_rv_{rrr}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ + \frac{\alpha m}{p-1}(N-1)(v^{-1}v_r(v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ + \frac{\alpha^2 m}{p-1}(v^{-2}v_r^2, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}) \\ + \frac{\alpha^2 m}{p-1}(N-1)(v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa})$$

$$\begin{aligned}
& -\frac{1}{p(p-1)}(v^{-\alpha}, (|v_{rr}|^p)_t r^{N-1+\kappa}) \\
& -\frac{m}{p-1}(N-1+\kappa)(v_{rrr}|v_{rr}|^{p-2}, v_{rr}r^{N-2+\kappa}) \\
& -\frac{m}{p-1}(N-1+\kappa)(N-1)((v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr}r^{N-2+\kappa}) \\
& -\frac{\alpha m}{p-1}(N-1+\kappa)(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}) \\
& -\frac{\alpha m}{p-1}(N-1+\kappa)(N-1)(v^{-1}v_r^2, |v_{rr}|^{p-2} v_{rr}r^{N-3+\kappa}) \\
& + o(1).
\end{aligned}$$

Combining (6.2) and (6.3) with (6.5), we obtain

$$\begin{aligned}
(6.6) \quad & -(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) \\
& = \frac{2-p}{p-1}\alpha m(v^{-1}v_r v_{rrr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}) \\
& \quad - \alpha m(N-1)(v^{-1}v_r^2 v_{rrr}, |v_{rr}|^{p-2}r^{N-2+\kappa}) \\
& \quad + \frac{\alpha m}{p-1}(N-1)(v^{-1}v_r(v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}) \\
& \quad + \frac{\alpha^2 m}{p-1}(v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa}) \\
& \quad + \frac{\alpha^2 m}{p-1}(N-1)(v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}) \\
& \quad - \frac{1}{p(p-1)}(v^{-\alpha}, (|v_{rr}|^p)_t r^{N-1+\kappa}) \\
& \quad - \frac{m}{p-1}(N-1+\kappa)(v_{rrr}|v_{rr}|^{p-2}, v_{rr}r^{N-2+\kappa}) \\
& \quad - \frac{m}{p-1}(N-1+\kappa)(N-1)((v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}) \\
& \quad - \frac{\alpha m}{p-1}(N-1+\kappa)(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}) \\
& \quad - \frac{\alpha m}{p-1}(N-1+\kappa)(N-1)(v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa}) \\
& \quad + o(1).
\end{aligned}$$

On the third and eighth terms on the right-hand side of (6.6), we write $(v_r r^{-1})_r = v_{rr} r^{-1} - v_r r^{-2}$, that is

$$\begin{aligned} (v^{-1} v_r (v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa}) &= (v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa}) - (v^{-1} v_r^2 v_{rr}, |v_{rr}|^{p-2} r^{N-3+\kappa}), \\ ((v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa}) &= (|v_{rr}|^p, r^{N-3+\kappa}) - (v_r |v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa}). \end{aligned}$$

The integration over the sixth term equals

$$\begin{aligned} ((v^{-\alpha}, (|v_{rr}|^p)_t, r^{N-1+\kappa})) &= \alpha((v^{-\alpha-1} v_t, |v_{rr}|^p r^{N-1+\kappa})) + [(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa})]_{t=0}^{t=T} \\ &= \alpha m((v^{-1} v_{rr}, |v_{rr}|^p r^{N-1+\kappa})) \\ &\quad + \alpha m(N-1)((v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa})) \\ &\quad + [(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa})]_{t=0}^{t=T} \quad (\text{from (3.2)}). \end{aligned}$$

The seventh term is

$$\begin{aligned} (6.7) \quad (|v_{rr}|^{p-2} v_{rr}, v_{rrr} r^{N-2+\kappa}) &= \frac{1}{p} ((|v_{rr}|^p)_r, r^{N-2+\kappa}) \\ &= -\frac{1}{p} (N-2+\kappa)(|v_{rr}|^p, r^{N-3+\kappa}) + o(1) \\ &\quad (\text{as previously}). \end{aligned}$$

From the above (6.6) becomes

$$\begin{aligned} (6.8) \quad &-((v^{-\alpha} v_t, (|v_{rr}|^{p-2} v_{rrr} r^{N-1+\kappa})_r)) \\ &= \frac{2-p}{p-1} \alpha m((v^{-1} v_r v_{rrr}, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa})) \\ &\quad - \alpha m(N-1)((v^{-1} v_r^2 v_{rrr}, |v_{rr}|^{p-2} r^{N-2+\kappa})) \\ &\quad + \frac{\alpha^2 m}{p-1} ((v^{-2} v_r^2, |v_{rr}|^p r^{N-1+\kappa})) \\ &\quad + \frac{\alpha^2 m}{p-1} (N-1)((v^{-2} v_r^3, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa})) \\ &\quad - \frac{1}{p(p-1)} [(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa})]_{t=0}^{t=T} \\ &\quad - \frac{\alpha m}{p-1} \left(\frac{N-1}{p} + \kappa \right) ((v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa})) \\ &\quad - \frac{\alpha m}{p-1} (N-1)(N+\kappa)((v^{-1} v_r^2, |v_{rr}|^{p-2} v_{rr} r^{N-3+\kappa})) \end{aligned}$$

$$\begin{aligned}
& -\frac{m}{p-1}(N-1+\kappa)\left(N-1-\frac{1}{p}(N-2+\kappa)\right)((|v_{rr}|^p, r^{N-3+\kappa})) \\
& +\frac{m}{p-1}(N-1)(N-1+\kappa)((v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa})) \\
& -\frac{\alpha m}{p(p-1)}((v^{-1}v_{rr}, |v_{rr}|^p r^{N-1+\kappa})) \\
& +o(1).
\end{aligned}$$

By integration by parts the last term equals

$$\begin{aligned}
-(v^{-1}v_{rr}, |v_{rr}|^p r^{N-1+\kappa}) &= -(v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa}) \\
&+ p(v^{-1}v_r v_{rr}, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa}) \\
&+ (N-1+\kappa)(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}) \\
&+ o(1).
\end{aligned}$$

Thus (6.8) is rewritten as follows:

$$\begin{aligned}
(6.9) \quad & -\frac{1}{m}((v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa})_r)) \\
&= \frac{3-p}{p-1}\alpha((v^{-1}v_r v_{rrr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa})) \\
&\quad -\alpha(N-1)((v^{-1}v_r^2 v_{rrr}, |v_{rr}|^{p-2}r^{N-2+\kappa})) \\
&\quad +\frac{\alpha}{p-1}\left(\alpha-\frac{1}{p}\right)((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) \\
&\quad +\frac{\alpha^2}{p-1}(N-1)((v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa})) \\
&\quad -\frac{1}{p(p-1)m}[(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa})]_{t=0}^{t=T} \\
&\quad -\frac{\alpha}{p}\kappa((v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa})) \\
&\quad -\frac{\alpha}{p-1}(N-1)(N+\kappa)((v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa})) \\
&\quad -\frac{1}{p-1}(N-1+\kappa)\left(N-1-\frac{1}{p}(N-2+\kappa)\right)((|v_{rr}|^p, r^{N-3+\kappa}))
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p-1}(N-1)(N-1+\kappa)((v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa})) \\
& + o(1).
\end{aligned}$$

On the other hand from (3.2)

$$\begin{aligned}
-(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) &= -m(v_{rr}, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) \\
&\quad - m(N-1)(v_r r^{-1}, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r) \\
&= m(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}) \\
&\quad + m(N-1)(v_{rr}|v_{rr}|^{p-2}, v_{rrr}r^{N-2+\kappa}) \\
&\quad - m(N-1)(v_r|v_{rr}|^{p-2}, v_{rrr}r^{N-3+\kappa}) \\
&\quad + o(1).
\end{aligned}$$

Combining this with (6.9), we conclude Lemma 6.1.

Q.E.D.

7. The second L^p estimate

We estimate the term $((|v_{rr}|^p, r^{N-3+\kappa}))$ on the left-hand of (6.1). First by integration by parts

$$\begin{aligned}
(|v_{rr}|^p, r^{N-3+\kappa}) &= (|v_{rr}|^{p-2}v_{rr}, v_{rr}r^{N-3+\kappa}) \\
&= -(p-1)(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}) \\
&\quad - (N-3+\kappa)(|v_{rr}|^{p-2}v_{rr}, v_r r^{N-4+\kappa}) \\
&\quad + [|v_{rr}|^{p-2}v_{rr}v_r r^{N-3+\kappa}]_{r=0}^{r=n}.
\end{aligned}$$

As previously

$$\lim_{r \rightarrow 0} (|v_r|^{p-2}v_{rr}v_r r^{N-3+\kappa})(r, t) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (|v_r|^{p-2}v_{rr}v_r r^{N-3+\kappa})(n, t) = 0,$$

because $N-2-q = \delta - p < 0$. Hence

$$\begin{aligned}
(7.1) \quad (|v_{rr}|^p, r^{N-3+\kappa}) &= -(p-1)(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}) \\
&\quad - (N-3+\kappa)(|v_{rr}|^{p-2}v_{rr}, v_r r^{N-4+\kappa}) \\
&\quad + o(1).
\end{aligned}$$

By Cauchy's inequality

$$|(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa})| \leq \varepsilon(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}) + C(\varepsilon^{-1})(|v_{rr}|^{p-2}, v_r^2 r^{N-5+\kappa})$$

$$\varepsilon > 0.$$

On the other hand by Hölder's inequality

$$|(|v_{rr}|^{p-2}, v_r^2 r^{N-5+\kappa})| \leq \varepsilon'(|v_{rr}|^p, r^{N-3+\kappa}) + C(\varepsilon'^{-1})(|v_r|^p, r^{N-3-q}) \quad \varepsilon' > 0.$$

The above inequalities are trivial for $p = 2$. From these inequalities

$$(7.2) \quad |(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa})| \leq \varepsilon(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}) + C(\varepsilon^{-1})$$

$$\times \{\varepsilon'(|v_{rr}|^p, r^{N-3+\kappa}) + C(\varepsilon'^{-1})(|v_r|^p, r^{N-3-q})\}.$$

Further

$$|(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-4+\kappa})| \leq \varepsilon(|v_{rr}|^p, r^{N-3+\kappa}) + C(\varepsilon^{-1})(|v_r|^p, r^{N-3-q}).$$

From the above and (7.1) we see that

$$(1 - C(\varepsilon^{-1})\varepsilon'(p-1) - \varepsilon|N-3+\kappa|)(|v_{rr}|^p, r^{N-3+\kappa}) \leq \varepsilon(p-1)(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa})$$

$$+ C(\varepsilon^{-1}, \varepsilon'^{-1})(|v_r|^p, r^{N-3-q})$$

$$+ o(1).$$

Taking $\varepsilon > 0$ as sufficiently small, we fix it. Afterwards we take $\varepsilon' > 0$ anew. Then

$$1 - C(\varepsilon^{-1})\varepsilon'(p-1) - \varepsilon|N-3+\kappa| > 1/2.$$

Here we use Proposition 5.1. Then the following lemma holds:

Lemma 7.1. *For the solution v of (3.2) and any $\delta > 0$ it holds that*

$$((|v_{rr}|^p, r^{N-3+\kappa})) \leq \delta((|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}))$$

$$+ C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}$$

$$+ o(1) \quad (n \rightarrow \infty),$$

where C depends on N , p and δ^{-1} , but not on n and η .

We use (7.2) and the inequality

$$|(v_{rr}|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-2+\kappa})| \leq \varepsilon(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}) + C(\varepsilon^{-1})(|v_{rr}|^p, r^{N-3+\kappa}).$$

Then for some $A > 0$ it follows from Lemmas 6.1 and 7.1 that

$$\begin{aligned}
(7.3) \quad & \frac{\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) ((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + (1 - A\delta)((|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa})) \\
& \leq \frac{3-p}{p-1} \alpha((v^{-1}v_r |v_{rr}|^{p-2}, v_{rr}v_{rrr} r^{N-1+\kappa})) \\
& \quad - \alpha(N-1)((v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rrr} r^{N-2+\kappa})) \\
& \quad - \frac{\alpha\kappa}{p}((v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa})) + \frac{\alpha^2}{p-1}(N-1)((v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr} r^{N-2+\kappa})) \\
& \quad - \frac{\alpha}{p-1}(N-1)(N+\kappa)((v^{-1}v_r^2, |v_r|^{p-2}v_{rr} r^{N-3+\kappa})) \\
& \quad + \frac{1+\delta}{p-1}(N-1)((v_r |v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa})) \\
& \quad + \frac{1}{p(p-1)m} (v^{-\alpha} |v_{rr}|^p, r^{N-1+\kappa})_{t=0} \\
& \quad + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} \\
& \quad + o(1).
\end{aligned}$$

Here we put $\delta = 1/(2A)$ and take $\alpha > 0$ as $\alpha < 1/(2p)$. Using again Lemma 7.1, we have from (7.3)

$$\begin{aligned}
(7.4) \quad & \alpha((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa})) + ((|v_{rr}|^p, r^{N-3+\kappa})) \\
& \leq C|((v_r |v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa}))| \\
& \quad + C\alpha\{|((v^{-1}v_r |v_{rr}|^{p-2}, v_{rr}v_{rrr} r^{N-1+\kappa}))| + |((v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}))|\} \\
& \quad + |((v^{-1}v_r^2, |v_{rr}|^{p-2}v_{rrr} r^{N-2+\kappa}))| + |((v^{-1}v_r^2, |v_r|^{p-2}v_{rr} r^{N-3+\kappa}))|\} \\
& \quad + C\alpha^2|((v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr} r^{N-2+\kappa}))| \\
& \quad + C\{(v^{-\alpha} |v_{rr}|^p, r^{N-1+\kappa})_{t=0} + (v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}\} \\
& \quad + o(1).
\end{aligned}$$

We shall prove

Proposition 7.2. *Let v be the solution of (3.2). Then there are two positive constants α_1 and C such that if $0 < \alpha < \alpha_1$, it holds that*

$$\begin{aligned}
& \alpha((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa})) + ((|v_{rr}|^p, r^{N-3+\kappa})) \\
& \leq C\{(v^{-\alpha} |v_{rr}|^p, r^{N-1+\kappa})_{t=0} + (v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}\} + o(1) \quad (n \rightarrow \infty),
\end{aligned}$$

where α_1 and C depend on N and p , but not on n and η . For the case of $N = 1$, the last term $(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}$ disappears.

Proof. We set the right-hand side of (7.4) as follows:

$$CI_1 + C\alpha \sum_{k=2}^5 I_k + C\alpha^2 I_6 + C\{J_1 + J_2\} + o(1).$$

We estimate each term I_i , $i = 1, \dots, 6$. For these we apply Hölder's inequality:

$$\begin{aligned} (7.5) \quad I_1 &= |((v_r |v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa}))| \\ &\leq \varepsilon(|v_{rr}|^p, r^{N-3+\kappa}) + C(\varepsilon^{-1})(|v_r|^p, r^{N-3-q}) \\ &\leq \varepsilon(|v_{rr}|^p, r^{N-3+\kappa}) + C(v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0} + o(1) \\ &\quad (\text{by Proposition 5.1}), \\ I_2 &= |((v^{-1} v_r |v_{rr}|^{p-2}, v_{rr} v_{rrr} r^{N-1+\kappa}))| \\ &\leq \varepsilon((v^{-2} v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + C(\varepsilon^{-1})(|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}), \\ I_3 &= |((v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa}))| \\ &\leq \varepsilon((v^{-2} v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + C(\varepsilon^{-1})(|v_{rr}|^p, r^{N-3+\kappa}), \\ 2I_4 &= 2|((v^{-1} v_r^2, |v_{rr}|^{p-2} v_{rrr} r^{N-2+\kappa}))| \\ &\leq (|v_{rr}|^{p-2}, v^{-2} v_r^4 r^{N-3+\kappa}) + (|v_{rr}|^{p-2}, v_{rrr}^2 r^{N-1+\kappa}). \end{aligned}$$

The first term on the right-hand side of (7.5) becomes

$$(7.6) \quad (|v_{rr}|^{p-2}, v^{-2} v_r^4 r^{N-3+\kappa}) \leq \varepsilon(v^{-2} v_r^2, |v_{rr}|^p r^{N-1+\kappa}) + C(\varepsilon^{-1})(v^{-2} |v_r|^{p+2}, r^{N-1-q}).$$

On the other hand by integration by parts

$$\begin{aligned} (v^{-2} |v_r|^{p+2}, r^{N-1-q}) &= (v^{-2} |v_r|^p v_r, v_r r^{N-1-q}) \\ &= 2(v^{-2} |v_r|^{p+2}, r^{N-1-q}) \\ &\quad - (p+1)(v^{-1} |v_r|^p v_{rr}, r^{N-1-q}) \\ &\quad - (N-1-q)(v^{-1} |v_r|^p v_r, r^{N-2-q}) \\ &\quad + [v^{-1} |v_r|^p v_r r^{N-1-q}]_{r=0}^{r=n}. \end{aligned}$$

As previously

$$\lim_{r \rightarrow +0} (v^{-1} |v_r|^p v_r r^{N-1-q})(r, t) = 0, \quad \lim_{n \rightarrow \infty} (v^{-1} |v_r|^p v_r r^{N-1-q})(n, t) = 0.$$

Hence

$$\begin{aligned} (v^{-2}|v_r|^{p+2}, r^{N-1-q}) &= (p+1)(v^{-1}|v_r|^p v_{rr}, r^{N-1-q}) \\ &\quad + (N-1-q)(v^{-1}|v_r|^p v_r, r^{N-2-q}) \\ &\quad + o(1). \end{aligned}$$

Using the inequality

$$|(v^{-1}|v_r|^p v_{rr}, r^{N-1-q})| \leq \varepsilon'(v^{-2}|v_r|^{p+2}, r^{N-1-q}) + C(\varepsilon'^{-1})(|v_r|^{p-2} v_{rr}^2, r^{N-1-q}),$$

we have

$$(7.7) \quad (v^{-2}|v_r|^{p+2}, r^{N-1-q}) \leq C\{|v_r|^{p-2} v_{rr}^2, r^{N-1-q}\} + (v^{-1}|v_r|^{p+1}, r^{N-2-q}) + o(1).$$

By Hölder's inequality

$$(|v_r|^{p-2} v_{rr}^2, r^{N-1-q}) \leq C\{(|v_{rr}|^p, r^{N-3+\kappa}) + (|v_r|^p, r^{N-3-q})\}$$

and

$$|(v^{-1}|v_r|^{p+1}, r^{N-2-q})| \leq \varepsilon(v^{-2}|v_r|^{p+2}, r^{N-1-q}) + C(\varepsilon^{-1})(|v_r|^p, r^{N-3-q}).$$

Then it follows from the above and (7.7) that

$$(7.8) \quad (v^{-2}|v_r|^{p+2}, r^{N-1-q}) \leq C\{(|v_{rr}|^p, r^{N-3+\kappa}) + (|v_r|^p, r^{N-3-q})\} + o(1).$$

Combining (7.5), (7.6) and (7.8) with Proposition 5.1, we obtain

$$\begin{aligned} 2I_4 &\leq \varepsilon((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((|v_{rr}|^{p-2}, v_{rr}^2 r^{N-1+\kappa})) \\ &\quad + C\{((|v_{rr}|^p, r^{N-3+\kappa})) + (v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}\} + o(1). \end{aligned}$$

Next we estimate I_5 . First

$$\begin{aligned} I_5 &= |((v^{-1}v_r^2, |v_{rr}|^{p-2} v_{rr} r^{N-3+\kappa}))| \\ &\leq C\{((v^{-2}v_r^4, |v_{rr}|^{p-2} r^{N-3+\kappa})) + ((|v_{rr}|^p, r^{N-3+\kappa}))\} + o(1). \end{aligned}$$

From (7.6) and (7.8)

$$\begin{aligned} |((v^{-1}v_r^2, |v_{rr}|^{p-2} v_{rr} r^{N-3+\kappa}))| &\leq \varepsilon((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) \\ &\quad + C\{((|v_{rr}|^p, r^{N-3+\kappa})) + ((|v_r|^p, r^{N-3-q}))\}. \end{aligned}$$

Hence by Proposition 5.1

$$I_5 \leq \varepsilon((v^{-2}v_r^2, |v_{rr}|^p r^{N-3+\kappa})) + C\{((|v_{rr}|^p, r^{N-3+\kappa})) + (v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}\} + o(1).$$

Lastly we estimate I_6 . We have

$$\begin{aligned}
I_6 &= |((v^{-2}v_r^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}))| \\
&\leq C\{((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((v^{-2}v_r^4, |v_{rr}|^{p-2}r^{N-3+\kappa}))\}.
\end{aligned}$$

From (7.6) and (7.8) we obtain

$$I_6 \leq C\{((v^{-2}v_r^2, |v_{rr}|^p r^{N-1+\kappa})) + ((|v_{rr}|^p, r^{N-3+\kappa})) + (v^{-\alpha}, |v_r|^p r^{N-1-q})_{t=0}\} + o(1).$$

We have finished the proof of Proposition 7.2 from (7.4) and the above estimates of each term I_i , $i = 1, \dots, 6$. Q.E.D.

8. Proof of results

Let D be a bounded domain in $\mathbf{R}^{N+1}$ ($= \mathbf{R}_x^N \times \mathbf{R}_t^1$). For any function u on $\bar{D}$ we define its norms as follows:

$$|u|_{\bar{D}}^{(2)} = \sum_{2s+|\alpha| \leq 2} \sup_{\bar{D}} |\partial_t^s \partial_x^\alpha u|.$$

And we define

$$\begin{aligned}
H^{2,1}(\bar{D}) &= \{u \mid u \text{ is continuous in } \bar{D}, \text{ together with all derivaties of the} \\
&\quad \text{form } \partial_t^s \partial_x^\alpha u \text{ for } 2s + |\alpha| \leq 2\}.
\end{aligned}$$

Then $H^{2,1}(\bar{D})$ is a Banach space with its norm $|\cdot|_{\bar{D}}^{(2)}$.

We denote by v_n^η the solution of (3.2). For the time being let $\eta > 0$ be sufficiently small and fixed. The following fact is known:

There exist a subsequence $\{v_{n_v}^\eta\}$ of $\{v_n^\eta\}$ and a function v^η such that

$$v_{n_v}^\eta \rightarrow v^\eta \quad \text{in } H^{2,1}(\bar{D}) \quad (v \rightarrow \infty).$$

for any bounded subdomain D in $\mathbf{R}^N \times (0, T)$.

Hence $\partial_t^s \partial_x^\alpha v^\eta \in C(\mathbf{R}^N \times [0, T])$ for $2s + |\alpha| \leq 2$ and v^η satisfies

$$(8.1) \quad \begin{cases} (v^\eta)^{-\alpha} (v^\eta)_t = m((v^\eta)_{rr} + (N-1)r^{-1}(v^\eta)_r) & \text{in } \mathbf{R}^N \times (0, T) \\ v^\eta(\cdot, 0) = (\psi^\eta)^m & \text{on } \mathbf{R}^N, \end{cases}$$

where $\psi^\eta = u_0 + \eta$.

We set $u^\eta = (v^\eta)^{1/m}$ and $0 < \eta < \eta'$. Then it is known that

$$0 \leq u^\eta(x, t) \leq u^{\eta'}(x, t) \leq M + 1,$$

and there exists a function u such that

$$u^\eta(x, t) \rightarrow u(x, t) \quad \text{a.e } \mathbf{R}^N \times (0, T) \quad (\eta \rightarrow +0),$$

where u is the required solution of (1.1).

In general it holds that for any function f

$$|\partial_{x_i} \partial_{x_j} f| \leq C(|f_{rr}| + r^{-1}|f_r|).$$

Hence it follows from (3.6), Lemma 3.6, Propositions 5.1 and 7.2 that

$$\int_0^T \int_0^n |\partial_{x_i} \partial_{x_j} v_n^\eta|^p r^{N-3+\kappa} dr dt \leq C + o(1), \quad i, j = 1, \dots, N,$$

where C is independent of n and η . Letting $n \rightarrow \infty$, we have

$$\int_0^T \int_0^\infty |\partial_{x_i} \partial_{x_j} v^\eta|^p r^{N-3+\kappa} dr dt \leq C, \quad i, j = 1, \dots, N.$$

We denote by $L_q^p(\mathbf{R}^N \times (0, T))$ the Banach space with its norm $(\int_0^T \int_0^\infty |f|^p r^{N-3+p-q} dr dt)^{1/p}$. Then from the above there exist a function v on $\mathbf{R}^N \times (0, T)$ and a subsequence $\{\eta_\nu\}$ of $\{\eta\}$ such that $\eta_\nu \rightarrow 0$ as $\nu \rightarrow \infty$. And they satisfy

$$\partial_{x_i} \partial_{x_j} v^{\eta_\nu} \text{ converges weakly to } \partial_{x_i} \partial_{x_j} v \text{ in } L_q^p(\mathbf{R}^N \times (0, T)).$$

We easily see that $v = u^m$. Thus Theorem 1 has been proved.

Lastly we prove Theorem 2.

When $N = 1$, we easily see that the solution v dose not need to be spherically symmetric in the proof of Lemma 6.1. Let v be the solution of (3.1) and let us proceed in parallel with the previous argument. But we set $\kappa = 0$, namely $p = q$. The inner product $(\cdot, \cdot)$ means that of $L^2(-n, n)$.

From the proof of Lemma 6.1 we have

$$\begin{aligned} & \frac{\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) ((v^{-2} v_x^2, |v_{xx}|^p)) + ((|v_{xx}|^{p-2}, v_{xxx}^2)) \\ &= \frac{3-p}{p-1} \alpha ((v^{-1} v_x |v_{xx}|^{p-2}, v_{xx} v_{xxx})) - \frac{1}{mp(p-1)} [(v^{-\alpha}, |v_{xx}|^p)]_{t=0}^{t=T} + o(1). \end{aligned}$$

Here we put

$$X = |v_{xx}|^{(p-2)/2} |v_{xxx}|, \quad Y = v^{-1} |v_x| |v_{xx}|^{p/2}.$$

Then

$$\begin{aligned} & ((1, X^2)) + \frac{\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) ((1, Y^2)) - \frac{3-p}{p-1} \alpha ((X, Y)) \\ & \leq -\frac{1}{mp(p-1)} [(v^{-\alpha}, |v_{xx}|^p)]_{t=0}^{t=T} + o(1). \end{aligned}$$

Hence it is enough to hold that

$$X^2 + \frac{\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) Y^2 - \frac{3-p}{p-1} \alpha XY \geq 0,$$

for any two variables X and Y . For this sake the following is sufficient:

$$(8.2) \quad \left(\frac{3-p}{p-1} \right)^2 \alpha^2 - \frac{4\alpha}{p-1} \left(\frac{1}{p} - \alpha \right) < 0,$$

which is equivalent to

$$\alpha < \frac{4(p-1)}{p(p^2 - 2p + 5)}.$$

This is the same as $m < I(p)$, where $I(p)$ is the quantity in Section 2. Therefore we have

$$(v^{-\alpha}, |v_{xx}|^p)_{t=T} \leq (v^{-\alpha}, |v_{xx}|^p)_{t=0} + o(1).$$

Replacing T with t on the left-hand side, we integrate the both sides on $0 \leq t \leq T$. Then

$$((v^{-\alpha}, |v_{xx}|^p)) \leq T(v^{-\alpha}, |v_{xx}|^p)_{t=0} + o(1),$$

which means

$$((1, |v_{xx}|^p)) \leq T(M+1)^{m-1}(v^{-\alpha}, |v_{xx}|^p)_{t=0} + o(1).$$

Using Lemma 3.6, we proceed in parallel with the proof of Theorem 1. The remained part is similar. Thus we complete the proof of Theorem 2.

References

- [1] Aronson, D. G. and B enilan, P., R egularit e des solutions de l' equation des milieus poreux dans $\mathbb{R}^N$, C. R. Acad. Sci. Paris, S er. A, **288** (1979), 103–105.
- [2] B enilan, P., A strong regularity L^p for solution of the porous media equation, Research Notes in Mathematics Pitman London, **89**, 39–58.
- [3] Caffarelli, L. A. and Friedman, A., Regularity of the free boundary of a gas flow in n -dimensional porous medium, Indiana Univ. Math. J., **29** (1980), 361–369.
- [4] Hayasida, K., On the regularity properties for solutions of the Cauchy problem for the porous media equation, Proc. Amer. Math. Soc., **107** (1989), 107–112.
- [5] Hayasida, K., Barenblatt solution and spherically symmetric solution of the porous media equation, Applicable Analysis, **69** (1998), 387–407.
- [6] Kalashnikov, A. S., Some problem of the qualitative theory of non-linear degenerate second-order parabolic equation, Russian Math. Surveys, **42** (1987), 169–222.
- [7] Ladyzhenskaja, O. A., Solonnikov, V. A. and Ural'ceva, N. N., *Linear and quasilinear equations of parabolic type*, Transl. Math. Monogr. Amer. Math. Soc. Providence, R.I., **23** (1968).

- [8] Oleinik, O. A., Kalashnikov, A. A. and Chzou, Yui-Lin, The Cauchy problem and boundary problems for equations of the type of unsteady filtrations, *Izv. Akad. Nauk SSSR, Ser. Math.*, **22** (1958), 667–704.
- [9] Sabinina, E. S., On the Cauchy problem for the equation of nonstationary gas filtration in several space variables, *Soviet Math.*, **2** (1961), 166–199.

nuna adreso:

Kazuya Hayasida
Fukui University of Technology
Fukui-City 910-8505
Japan

Masatomo Kobayashi
NEC Communication Systems, LTD
Abiko 270-1198
Japan

(Ricevita la 5-an de junio, 2000)
(Reviziita la 22-an de marto, 2001)