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(Citation)

Funkcialaj Ekvacioj, 45(2):209-222

(Issue Date)

2002-08

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100501205>

(URL)

<https://hdl.handle.net/20.500.14094/0100501205>



Forced Vibrations of Abstract Wave Equations

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1. Introduction

In this paper, we investigate the existence of periodic solutions for certain abstract wave equations. We are motivated by the papers of K. Ben-Naoum and J. Mawhin [1], and P. J. McKenna [8], where existence results of periodic solutions are proved for one-dimensional wave equations when the ratio between the space length and the period was irrational. Related equations are also studied by M. Yamaguchi in [10]. We proceeded in this direction in the paper [4]. We studied the equation

$$(1.1) \quad u_{tt} + Au = \varepsilon f(u, t),$$

where A is a self-adjoint, unbounded linear operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$, ε is a small parameter and f is T -periodic in $t \in \mathbf{R}$. By a T -periodic solution of (1.1) we mean a weak solution specified below. The following results are proved in [4] under additional assumptions on A, f .

Theorem 1.1 ([4]). *Assume there exists a constant $c > 0$ such that*

$$(1.2) \quad \left| \alpha^2 - \frac{m^2}{\lambda_i} \right| \geq \frac{c}{\lambda_i} \quad \forall m \in \mathbf{N}, \forall \lambda_i > 0,$$

where $\alpha = T/2\pi$. Then (1.1) has a weak T -periodic solution for any ε small.

Theorem 1.2 ([4]). *Assume*

$$\sum_{\lambda_i > 0} \frac{1}{\sqrt{\lambda_i}} < \infty.$$

Then the Lebesgue measure of the set of all positive α not satisfying (1.2) is zero.

We also studied the case when $0 < \dim \ker A < \infty$. Finally we considered the example

$$\begin{aligned}
 (1.3) \quad & u_{tt} - u_{xx} - n^2 u = \varepsilon f(u, t) \\
 & u(t + T, \cdot) = u(t, \cdot) \quad \forall t \in S^T \\
 & u(t, 0) = u(t, \pi) = 0 \quad \forall t \in S^T,
 \end{aligned}$$

where $f : \mathbf{R} \times S^T \rightarrow \mathbf{R}$ is C^1 -smooth and globally Lipschitz in $u, n \in \mathbf{N}$. Here $S^T = \mathbf{R}/[0, T]$ is the circle. The following result is proved in [4].

Theorem 1.3 ([4]). *The equation (1.3) has a weak T -periodic solution, provided that it holds*

$$(1.4) \quad \inf_{i, m \in \mathbf{N}, i > n} |i^2 - n^2 - \omega^2 m^2| > 0,$$

where $\omega = 1/\alpha$, and there is a $z \in \mathbf{R}$ such that

$$\begin{aligned}
 & \int_0^T \int_0^\pi f(z \cdot \sin nx, t) \sin nx \, dx dt = 0 \\
 & \int_0^T \int_0^\pi \frac{\partial f}{\partial u}(z \cdot \sin nx, t) \sin^2 nx \, dx dt \neq 0.
 \end{aligned}$$

The purpose of this paper is two-fold. We firstly release the parameter ε in (1.1), so we consider the equation

$$(1.5) \quad u_{tt} + Au = f(u, t).$$

We have used in [4] the Banach fixed point theorem. To get our results in this paper, we apply the Leray-Schauder fixed point theorem. For this reason, we need more precision condition than (1.2), see (2.2) below. We also study the resonant case when $0 < \dim \ker A < \infty$. We derive a Landesman-Lazer type result [7]. Finally, we present a forced beam equation as an example.

We secondly investigate more correctly and thoroughly the condition (1.4) than in [4]. By using some results of the number theory [2, 3, 6], we derive several conditions for ω when (1.4) is either satisfied or not.

2. Nonresonant cases

Let X be a Banach space continuously embedded into a Hilbert space Y with an inner product $\langle \cdot, \cdot \rangle$ and with the corresponding norm $|\cdot|$. We assume that $A : X \rightarrow Y$ is bounded and the eigenvectors $\{u_j\}_1^\infty$ of A corresponding to $\{\lambda_j\}_1^\infty$ form an orthonormal basis of Y . Moreover we suppose that the linear

hull of $\{u_j\}_1^\infty$ is dense in X . Furthermore, $f : Y \times S^T \rightarrow Y$ is continuous satisfying

$$(2.1) \quad |f(y, t)| \leq c_1(|y|^\sigma + 1) \quad \forall y \in Y, \forall t \in S^T$$

for constants $1 > \sigma \geq 0$, $c_1 > 0$.

In this section we study the case when

$$(n) \quad \lambda_i \neq 0 \quad \forall i.$$

A **weak T -periodic solution** of (1.5) is some $u \in L^2(S^T, Y)$ satisfying

$$(W) \quad \int_0^T \langle u(t), v_{tt}(t) + Av(t) \rangle dt = \int_0^T \langle f(u(t), t), v(t) \rangle dt, \quad \forall v \in C^2(S^T, X).$$

We mean the integrability in the sense of Bochner [5]. We note that (2.1) implies the continuity of the Nemytskii operator $u \rightarrow f(u(t), t)$ from $L^2(S^T, Y)$ to $L^2(S^T, Y)$.

Lemma 2.1. *Assume the following condition*

$$(2.2) \quad \left| \alpha^2 - \frac{m^2}{\lambda_i} \right| \geq \frac{c}{\lambda_i^p}, \quad \forall m \in N, \forall \lambda_i > 0,$$

where p, c are positive constants such that $0 < p < 1$. Then the equation

$$(2.3) \quad \int_0^T \langle u(t), v_{tt}(t) + Av(t) \rangle dt = \int_0^T \langle h(t), v(t) \rangle dt, \quad \forall v \in C^2(S^T, X)$$

has a unique solution $u = Lh \in L^2(S^T, Y)$ for any $h \in L^2(S^T, Y)$. Moreover, $L : L^2(S^T, Y) \rightarrow L^2(S^T, Y)$ is compact.

Proof. By our assumptions, the Hilbert space $L^2(S^T, Y)$ has the orthogonal basis

$$(2.4) \quad \left\{ \sin m \frac{2\pi t}{T} \cdot u_j, \cos m \frac{2\pi t}{T} \cdot u_j \right\} \subset C^2(S^T, X)$$

$$m = 0, 1, 2, \dots \quad j = 1, 2, \dots$$

We expand u (formally) and h really in the basis (2.4) to get

$$u(t) = \sum_{m,j} \left(u_{mj}^1 \sin m \frac{2\pi t}{T} + u_{mj}^2 \cos m \frac{2\pi t}{T} \right) u_j$$

$$h(t) = \sum_{m,j} \left(h_{mj}^1 \sin m \frac{2\pi t}{T} + h_{mj}^2 \cos m \frac{2\pi t}{T} \right) u_j.$$

Of course, we take $u_{0j}^1 = 0$ and $h_{0j}^1 = 0$. We have

$$\sum_j T(h_{0,j}^2)^2 + \sum_{m \neq 0,j} (T/2)((h_{mj}^1)^2 + (h_{mj}^2)^2) < \infty.$$

If u is a solution of (2.3), then we take $v(t) = \sin m \frac{2\pi t}{T} \cdot u_j$ and $v(t) = \cos m \frac{2\pi t}{T} \cdot u_j$ to get

$$u_{mj}^i = \frac{\alpha^2}{\alpha^2 \lambda_j - m^2} h_{mj}^i, \quad i = 1, 2.$$

Hence if (2.3) has a solution $u \in L^2(S^T, Y)$, then it is unique and it should be given by

$$(2.5) \quad u(t) = \sum_{m,j} \frac{\alpha^2}{\alpha^2 \lambda_j - m^2} \left(h_{mj}^1 \sin m \frac{2\pi t}{T} + h_{mj}^2 \cos m \frac{2\pi t}{T} \right) u_j.$$

The assumption (2.2) gives a constant $c_2 > 0$ such that $|u_{mj}^i| \leq c_2 |h_{mj}^i|$, $i = 1, 2$. Hence

$$\begin{aligned} & \sum_j T(u_{0,j}^2)^2 + \sum_{m \neq 0,j} (T/2)((u_{mj}^1)^2 + (u_{mj}^2)^2) \\ & \leq c_2^2 \left(\sum_j T(h_{0,j}^2)^2 + \sum_{m \neq 0,j} (T/2)((h_{mj}^1)^2 + (h_{mj}^2)^2) \right). \end{aligned}$$

This gives $u \in L^2(S^T, Y)$ given by (2.5) and $|u| \leq c_2 |h|$. So L is continuous. Moreover the assumption (2.2) implies that $|\alpha^2 \lambda_j - m^2| \rightarrow \infty$ as $|\lambda_j| + m \rightarrow \infty$. This gives the compactness of L .

Now we show that this u satisfies (2.3). Our assumptions give that the linear hull L_H of (2.4) is dense in $C^2(S^T, X)$: one can prove this by using the Δ -approximation method like in [9], see also Fejér's theorem [9]. So for any $v \in C^2(S^T, X)$ there is a sequence $v_j \in L_H$ such that $v_j \rightarrow v$ in $C^2(S^T, X)$. This gives $v_{jtt} \rightarrow v_{tt}$ and $v_j \rightarrow v$ in $C(S^T, X)$. Hence $Av_j \rightarrow Av$ in $C(S^T, Y)$. The equality (2.3) holds for any $v_j \in L_H$, and since $X \subset Y$ continuously, we take the limit $j \rightarrow \infty$ in (2.3) for $v = v_j$ to get the validity of (2.3) for any v . The proof is finished. $\square$

Theorem 2.2. *If the conditions (2.1), (2.2) and (n) are satisfied, then (1.5) has a weak T -periodic solution.*

Proof. For a $u \in L^2(S^T, Y)$, we put $F(u) = f(u(t), t) \in L^2(S^T, Y)$. Then (W) is equivalent to $u = LF(u)$. Lemma 2.1 gives the compactness of the operator $LF : L^2(S^T, Y) \rightarrow L^2(S^T, Y)$. Moreover, Lemma 2.1 and the condition (2.1) give

$$\begin{aligned}
|LF(u)|^2 &\leq c_2^2 \int_0^T c_1^2 (|u(t)|^\sigma + 1)^2 dt \leq 2c_1^2 c_2^2 \int_0^T (|u(t)|^{2\sigma} + 1) dt \\
&= 2c_1^2 c_2^2 T + 2c_1^2 c_2^2 \int_0^T |u(t)|^{2\sigma} dt.
\end{aligned}$$

Since $0 \leq \sigma < 1$, the Jensen inequality gives

$$\begin{aligned}
|LF(u)|^2 &\leq 2c_1^2 c_2^2 T + 2c_1^2 c_2^2 T^{1-\sigma} \left(\int_0^T |u(t)|^2 dt \right)^\sigma \\
&= 2c_1^2 c_2^2 T^{1-\sigma} |u|^{2\sigma} + 2c_1^2 c_2^2 T \leq 2c_1^2 c_2^2 (T^{(1-\sigma)/2} |u|^\sigma + \sqrt{T})^2.
\end{aligned}$$

Consequently we get

$$|LF(u)| \leq \sqrt{2} c_1 c_2 (T^{(1-\sigma)/2} |u|^\sigma + \sqrt{T}).$$

By taking $K_0 > 0$ such that $\sqrt{2} c_1 c_2 (T^{(1-\sigma)/2} K_0^\sigma + \sqrt{T}) = K_0$, we see that the ball $B_{K_0} = \{u \in L^2(S^T, Y) \mid |u| \leq K_0\}$ is mapped to itself by LF . Hence the Leray-Schauder fixed point theorem [7] finishes the proof. $\square$

By the same way as for Theorem 1.2 of [4], we get the following result.

Theorem 2.3. *Assume*

$$\sum_{\lambda_i > 0} \frac{1}{\lambda_i^{\rho-(1/2)}} < \infty.$$

Then the Lebesgue measure of the set of all positive α not satisfying (2.2) is zero.

Proof. We present the proof for the reader convenience. If (2.2) is false for some $\alpha \in (K, K+1)$, $K > 0$, then for any $d > 0$ small there exist $m, \lambda_i > 0$ such that

$$\left| \alpha^2 - \frac{m^2}{\lambda_i} \right| \leq \frac{d}{\lambda_i^\rho}.$$

This implies

$$\left| \alpha - \frac{m}{\sqrt{\lambda_i}} \right| \leq \frac{d}{K \lambda_i^\rho}.$$

Since $\alpha \in (K, K+1)$, we have $m^2/\lambda_i < (K+1)^2 + 1$ for d small. Thus

$$m \leq \sqrt{((K+1)^2 + 1)\lambda_i}.$$

Denote by $\mathcal{M}$ the set of all $\alpha \in (K, K+1)$ for which (2.2) does not hold. Then the Lebesgue measure $\mu(\mathcal{M})$ of $\mathcal{M}$ satisfies

$$\begin{aligned} \mu(\mathcal{M}) &\leq \sum_{\lambda_i > 0} \frac{2d}{\lambda_i^\rho K} \cdot \sqrt{((K+1)^2 + 1)\lambda_i} \\ &= \frac{2d}{K} \sqrt{((K+1)^2 + 1)} \sum_{\lambda_i > 0} \frac{1}{\lambda_i^{\rho-(1/2)}} = O(d). \end{aligned}$$

Since d is arbitrarily small, $\mu(\mathcal{M}) = 0$. The proof is finished. $\square$

This theorem again says nothing for the case

$$\begin{aligned} Au &= -u_{xx}, \quad u \in C^2([0, \pi], \mathbf{R}) \\ u(0) &= u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0. \end{aligned}$$

But taking

$$\begin{aligned} Au &= u_{xxxx}, \quad u \in C^4([0, \pi], \mathbf{R}) \\ u(0) &= u_{xx}(0) = u_{xxxx}(0) = u(\pi) = u_{xx}(\pi) = u_{xxxx}(\pi) = 0 \end{aligned}$$

we have $\lambda_i = i^4 \forall i$, and Theorem 2.3 holds for $\rho = 4/5$. The operator $u_{tt} + u_{xxxx}$ appears in the equation of a beam.

Finally we note that equations of the following form are studied by M. Yamaguchi in the paper [10]

$$(2.6) \quad u_{tt} + Au = \varepsilon f_1(u, t) + f_2(t)$$

for a small $\varepsilon \in \mathbf{R}$, $f_1 : Y \times S^T \rightarrow Y$, $f_2 : S^T \rightarrow Y$ continuous. Like above, we get the following result.

Theorem 2.4. *If the conditions (2.2), (n) are satisfied and the following condition*

$$|f(y, t)| \leq c_1(|y| + 1) \quad \forall y \in Y, \quad |y| \leq \max_{t \in S^T} |f_2(t)| + 1, \quad \forall t \in S^T$$

holds as well for a positive constant c_1 , then (2.6) has a weak T -periodic solution for any ε sufficiently small.

3. Resonant cases

Now we consider (1.5) under the condition $0 < \dim \ker A < \infty$. The setting of the problem (1.5) is the same as in Section 2. So we suppose (2.2), but (2.1) is assumed with $\sigma = 0$, i.e

$$(3.1) \quad |f(y, t)| \leq c_1 \quad \forall y \in Y, \forall t \in S^T$$

for a positive constant c_1 . By (W) any weak T -periodic solution of (1.5) satisfies

$$\int_0^T \langle f(u(t), t), u_p \rangle dt = 0, \quad \forall u_p \in \ker A.$$

Then Lemma 2.1 holds as well for this case but only for h such that

$$\int_0^T \langle h(t), u_p \rangle dt = 0, \quad \forall u_p \in \ker A.$$

Let $P : Y \rightarrow \ker A$ be the orthogonal projection. Then (1.5) has the form

$$(3.2) \quad \begin{aligned} w_{tt} + Aw &= Qf(w + u_p, \cdot) \\ 0 &= \tilde{P}f(w + u_p, \cdot), \end{aligned}$$

where

$$Q = I - \tilde{P}, \quad \tilde{P}h = \frac{1}{T} \int_0^T Ph(t) dt, \quad w \in \text{im } Q, u_p \in \ker A.$$

Note, these projections are defined for the following reason

$$\tilde{P}h = 0 \Leftrightarrow \int_0^T \langle h(t), u_p \rangle dt = 0, \quad \forall u_p \in \ker A.$$

Clearly the operator L of Lemma 2.1 is defined as $L : \text{im } Q \rightarrow L^2(S^T, Y)$. Let $l_i : L^2(S^T, Y) \rightarrow \mathbf{R}$, $i = 1, 2, \dots, i_0 = \dim \ker A$, be continuous linear functionals such that $\tilde{P}h = 0$ if and only if $l_i(h) = 0 \ \forall i = 1, 2, \dots, i_0$. We identify $\ker A \simeq \mathbf{R}^{i_0}$.

Theorem 3.1. *Let (2.2) and (3.1) be satisfied. Suppose that*

$$(3.3) \quad \liminf_{|u_p| \rightarrow \infty} \sum_{i=1}^{i_0} |l_i(f(Lw + u_p, \cdot))| > 0$$

uniformly for $w \in \text{im } Q$ from bounded sets, i.e. $\forall K_2 > 0, \exists K_3 > 0, \exists \delta > 0$ such that $\sum_{i=1}^{i_0} |l_i(f(Lw + u_p, \cdot))| > \delta \ \forall w \in \text{im } Q, |w| \leq K_2, \forall u_p \in \ker A, |u_p| \geq K_3$. Then (1.5) has a weak T -periodic solution provided that the Brouwer degree $\deg(F(u_p), B_{K_1}, 0) \neq 0$, where $F : \mathbf{R}^{i_0} \simeq \ker A \rightarrow \mathbf{R}^{i_0}$ and B_{K_1} are defined by

$$\begin{aligned} F(u_p) &= (l_1(f(u_p)), l_2(f(u_p)), \dots, l_{i_0}(f(u_p))) \\ B_{K_1} &= \{u_p \in \ker A \simeq \mathbf{R}^{i_0} \mid |u_p| \leq K_1\} \subset \mathbf{R}^{i_0} \end{aligned}$$

for a sufficiently large constant $K_1 > 0$. According to (3.3), this degree is well-defined.

Proof. This is a Landesman-Lazer type result [7], so we follow the standard way by using the homotopy

$$(3.4) \quad w_{tt} + Aw = \lambda Qf(w + u_p, \cdot), \quad 0 \leq \lambda \leq 1,$$

$$0 = l_i(f(\lambda LQf(w + u_p, \cdot) + u_p, \cdot)), \quad i = 1, 2, \dots, i_0.$$

For $\lambda = 1$ we get (3.2). The assumptions (2.2) and (3.1) imply the existence of a constant $K_0 > 0$ such that there is no solution of (3.4) such that $|w| \geq K_0$. (3.1) and (3.3) give a constant $K_1 > 0$ such that

$$(3.5) \quad \sum_{i=1}^{i_0} |l_i(f(\lambda LQf(w + u_p, \cdot) + u_p, \cdot))| > 0$$

for any $w \in \text{im } Q$, $|w| \leq K_0$, $0 \leq \lambda \leq 1$ and $u_p \in \ker A$, $|u_p| \geq K_1$. Let us put

$$\Omega = \{(w, u_p) \in \text{im } Q \times \ker A \mid |w| \leq K_0, |u_p| \leq K_1\}.$$

According to the choice of the constants K_0, K_1 , we can easily verify that (3.4) has no solution on $\partial\Omega$ for any $0 \leq \lambda \leq 1$. Consequently to compute the Leray-Schauder degree of (3.4), we take $\lambda = 0$ and we are lead to the mapping $u_p \rightarrow F(u_p)$ and to the Brouwer degree $\deg(F(u_p), B_{K_1}, 0) \neq 0$. This gives the solvability of (3.4) for $\lambda = 1$ in Ω . The proof is finished. $\square$

Example. Consider

$$(3.6) \quad \begin{aligned} u_{tt} + u_{xxxx} - n^4 u &= f(u) + g(t)h(x) \\ u(t + T, \cdot) &= u(t, \cdot) \quad \forall t \in S^T \\ u(t, 0) = u_{xx}(t, 0) &= u(t, \pi) = u_{xx}(t, \pi) = 0 \quad \forall t \in S^T, \end{aligned}$$

where $f : \mathbf{R} \rightarrow \mathbf{R}$, $g : S^T \rightarrow \mathbf{R}$, $h : [0, \pi] \rightarrow \mathbf{R}$ are continuous and $n \in \mathbf{N}$.

Now, we have for this case

$$\begin{aligned} Au &= u_{xxxx} - n^4 u, \quad \ker A = \{\sin nx\}, \quad \lambda_i = i^4 - n^4, \quad u_i = \sin ix \\ X &= \{u \in C^4([0, \pi], \mathbf{R}) \mid u(0) = u_{xx}(0) \\ &= u_{xxxx}(0) = u(\pi) = u_{xx}(\pi) = u_{xxxx}(\pi) = 0\}, \quad Y = L^2(0, \pi). \end{aligned}$$

Now we take $\rho = 4/5$. Then Theorem 2.3 holds and the condition (2.2) gives $|\alpha^2 \lambda_j - m^2| \geq c \lambda_j^{1-\rho} = c(j^4 - n^4)^{1/5} \sim j^{4/5}$ for any $j > n$. Consequently, if $u \in \text{im } Q$ then Lu is given by (2.5) and $Lu \in L^2(S^T, Z)$, where Z is defined by

$$Z = \left\{ u(x) = \sum_{j=1}^{\infty} a_j \sin jx \mid \sum_{j=1}^{\infty} j^{8/5} a_j^2 < \infty \right\}.$$

Z is a Hilbert space with the norm $|u|_Z = \sqrt{\sum_{j=1}^{\infty} j^{8/5} a_j^2}$. Hence $\left\{ \frac{1}{j^{4/5}} \sin jx \right\}_{j=1}^{\infty}$ is an orthonormal basis of Z . Moreover, for any $u \in Z$ we have

$$\begin{aligned} |u(x)| &\leq \sum_{j=1}^{\infty} |a_j| \leq \sqrt{\sum_{j=1}^{\infty} j^{8/5} a_j^2} \sqrt{\sum_{j=1}^{\infty} j^{-8/5}} \\ &\leq |u|_Z \sqrt{1 + \int_1^{\infty} x^{-8/5} dx} = |u|_Z \sqrt{5/3}. \end{aligned}$$

Hence $Z \subset C[0, \pi]$ and $\max_{x \in [0, \pi]} |u(x)| \leq \sqrt{5/3} |u|_Z \quad \forall u \in Z$. (2.5) implies the continuity of $L : \text{im } Q \rightarrow L^2(S^T, Z) \subset L^2(S^T, C[0, \pi])$ and so there is a constant $\bar{c} > 0$ such that

$$\int_0^T \max_{x \in [0, \pi]} |(Lu)(t, x)|^2 dt \leq \bar{c}^2 |u|^2 \quad \forall u \in L^2(S^T, Y).$$

We take $l_1(u) = \int_0^T \int_0^{\pi} u(t, x) \sin nx \, dx dt$. To get (3.1), we suppose

$$(3.7) \quad \sup_{u \in R} |f(u)| = \delta < \infty.$$

The assumption (3.3) has the form

$$(3.8) \quad \liminf_{|c| \rightarrow \infty} \left| \int_0^T \int_0^{\pi} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt \right| > 0$$

uniformly for

$$(3.9) \quad |w|^2 = \int_0^T \int_0^{\pi} w(t, x)^2 \, dx dt \leq K_3^2,$$

where $K_3 > 0$ is any fixed constant. Here $\tilde{f}(u, t) = f(u) + g(t)h(x)$. Since $|L| \leq \bar{c}$ as $L : \text{im } Q \rightarrow L^2(S^T, C[0, \pi])$, we see that (3.9) implies

$$(3.10) \quad \int_0^T \max_{x \in [0, \pi]} |(Lw)(t, x)|^2 dt \leq K_3^2 \bar{c}^2.$$

Let $\varepsilon > 0$ be small and put $I_i = [\frac{\pi}{n}i + \varepsilon, \frac{\pi}{n}(i+1) - \varepsilon]$, $i = 0, 1, \dots, n-1$. Then

$$\begin{aligned} &\int_0^T \int_0^{\pi} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt \\ &= \sum_{i=0}^{n-1} \int_0^T \int_{I_i} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt + O(\varepsilon). \end{aligned}$$

(3.10) gives a set $N_{\varepsilon, w} \subset [0, T] \simeq S^T$ such that the Lebesgue measure $\mu(N_{\varepsilon, w}) \leq \varepsilon$, and $\forall t \notin N_{\varepsilon, w}$ it holds that $\max_{x \in [0, \pi]} |(Lw)(t, x)| \leq K_3 \bar{c} / \sqrt{\varepsilon}$. Hence

$$(3.11) \quad \int_0^T \int_{I_i} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt \\ = \int_{S^T \setminus N_{\varepsilon, w}} \int_{I_i} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt + O(\varepsilon).$$

The function $\sin nx$ does not change the sign on I_i , hence $|(Lw)(t, x) + c \sin nx| \rightarrow \infty$ as $|c| \rightarrow \infty$ for $x \in I_i$, $t \notin N_{\varepsilon, w}$. More precisely, for any $K_4 > 0$, there is a $K_5 > 0$ such that $|(Lw)(t, x) + c \sin nx| \geq K_4$ for $|c| \geq K_5$, $x \in I_i$, $t \notin N_{\varepsilon, w}$. We note that K_5 is independent of w .

Let us suppose

$$(3.12) \quad \lim_{u \rightarrow \pm\infty} f(u) = f^\pm \in \mathbf{R}.$$

Now we take $K_4 > 0$ sufficiently large such that for any $|c| \geq K_5$ we get from (3.11) and (3.12)

$$\int_{S^T \setminus N_{\varepsilon, w}} \int_{I_i} \tilde{f}((Lw)(t, x) + c \sin nx, t) \sin nx \, dx dt \\ = \int_{S^T \setminus N_{\varepsilon, w}} \int_{I_i} (f^{\text{sign}(-1)^i c} + g(t)h(x)) \sin nx \, dx dt + O(\varepsilon) \\ = \int_0^T \int_{i\pi/n}^{(i+1)\pi/n} (f^{\text{sign}(-1)^i c} + g(t)h(x)) \sin nx \, dx dt + O(\varepsilon) \\ = (-1)^i \frac{2T}{n} f^{\text{sign}(-1)^i c} + \int_0^T g(t) dt \int_{i\pi/n}^{(i+1)\pi/n} f(x) \sin nx \, dx + O(\varepsilon).$$

Summarizing we see that (3.8) holds if

$$\frac{T}{n} \Delta_n^\pm + \int_0^T g(t) dt \int_0^\pi h(x) \sin nx \, dx \neq 0,$$

where

$$\Delta_n^\pm = \begin{cases} (n-1)(f^\pm - f^\mp) + 2f^\pm & \text{for an odd } n \\ n(f^\pm - f^\mp) & \text{for an even } n. \end{cases}$$

The mapping $F(u_p)$ of Theorem 3.1 has now the form

$$(3.13) \quad F(c) = \int_0^T \int_0^\pi (f(c \sin nx) + g(t)h(x)) \sin nx \, dx dt.$$

By using (3.12) like above for (3.11), we see from (3.13) that the Brouwer degree of Theorem 3.1 given now by $\deg(F(c), (-K_1, K_1), 0)$ is nonzero if it holds

(3.14)

$$\left(\frac{T}{n} \Delta_n^+ + \int_0^T g(t) dt \int_0^\pi h(x) \sin nx \, dx \right) \left(\frac{T}{n} \Delta_n^- + \int_0^T g(t) dt \int_0^\pi h(x) \sin nx \, dx \right) < 0.$$

On the other hand, the equality below (3.1) implies that (3.6) has no solution if

$$(3.15) \quad T\pi\delta < \left| \int_0^T g(t) dt \int_0^\pi h(x) \sin nx \, dx \right|.$$

Summarizing, we get the following result.

Theorem 3.2. *There is a subset $S \subset (0, \infty)$ with a zero Lebesgue measure such that for any $0 < T \notin S$, if (3.12) and (3.14) are satisfied then the problem (3.6) has a weak T -periodic solution. On the other hand, if (3.7) and (3.15) hold for a $T > 0$ then (3.6) has no weak T -periodic solution.*

4. Some remarks on the condition (1.4)

This section is devoted to results concerning the condition (1.4). There is a nice characterization of such ω for $n = 0$ in terms of continued fractions [1, 6]. We intend to derive similar results for $n \in \mathbb{N}$. This situation is different from $n = 0$.

Theorem 4.1. *Let $\omega = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$. The condition (1.4) holds if and only if any $n_2 \in \mathbb{N}$, $n_2 \mid n/(p, n)$ satisfies*

- (i) *If n_2 is odd then $p/(p, n)$ does not divide $(a^2 - b^2)/2$ for any $a > b$, $a, b \in \mathbb{N}$ such that $n_2 = ab$.*
- (ii) *If n_2 is even then $p/(p, n)$ does not divide $a^2 - b^2$ for any $a > b$, $a, b \in \mathbb{N}$ such that $n_2 = 2ab$.*

Here as usually (p, n) is the largest common divisor of p and n .

Proof. The condition (1.4) does not hold if and only if there are $i, m \in \mathbb{N}$, $i > n$ such that

$$(4.1) \quad \begin{aligned} i^2 &= \omega^2 m^2 + n^2 \\ q^2 i^2 &= p^2 m^2 + q^2 n^2. \end{aligned}$$

Hence $q \mid pm$ implies $q \mid m$, i.e. $m = rq$, $r \in \mathbb{N}$ and (4.1) gives

$$(4.2) \quad i^2 = p^2 r^2 + n^2.$$

After dividing (4.2) by $(p, n)^2$, we get $i_1^2 = p_1^2 r^2 + n_1^2$, $p_1 = p/(p, n)$, $n_1 = n/(p, n)$, $i_1 = i/(p, n)$. If $(r, n_1) > 1$ then we have $i_2^2 = p_1^2 r_1^2 + n_2^2$ and $(n_2, p_1 r_1) = 1$.

We have two possibilities:

If n_2 is odd, then $p_1 r_1$ is even and we get

$$\frac{i_2 + n_2}{2} \frac{i_2 - n_2}{2} = \left(\frac{p_1 r_1}{2}\right)^2.$$

Since $(p_1 r_1, n_2) = 1$, we get

$$\begin{aligned} i_2 + n_2 &= 2A^2, & i_2 - n_2 &= 2B^2, & p_1 r_1 &= 2AB \\ i_2 &= A^2 + B^2, & n_2 &= A^2 - B^2 = (A - B)(A + B). \end{aligned}$$

So $A = (a + b)/2$, $B = (a - b)/2$, $n_2 = ab$, $a > b \in \mathbb{N}$ and $p_1 r_1 = (a^2 - b^2)/2$. This proves (i).

If n_2 is even, then $p_1 r_1$ is odd and we get

$$\frac{i_2 + p_1 r_1}{2} \frac{i_2 - p_1 r_1}{2} = \left(\frac{n_2}{2}\right)^2.$$

Similarly like above we get $i_2 = A^2 + B^2$, $p_1 r_1 = A^2 - B^2$, $n_2 = 2AB$, $A > B \in \mathbb{N}$. This proves (ii). The proof is finished. $\square$

Corollary 4.2. *Let $\omega = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$ and let $n > 1$ be a prime number. The condition (1.4) holds for $n = 2$, and for $n > 2$ if and only if p does not divide $(n^2 - 1)/2$.*

Theorem 4.3. *If $\omega = \sqrt{p/q}$ is irrational for $p, q \in \mathbb{N}$, $(p, q) = 1$, then (1.4) does not hold for any $n \in \mathbb{N}$.*

Proof. Since $\sqrt{pq}$ is irrational, the Pellé equation $i_0^2 = pqm_0^2 + 1$ has a natural number solution (infinitely many). Then $i = i_0 n$, $m = m_0 q n$ satisfy $i^2 = \frac{p}{q} m^2 + n^2$, i.e. $i^2 = \omega^2 m^2 + n^2$. The proof is finished. $\square$

It is well-known (see [6]) that if ω has the continued fraction $\omega = [a_0, a_1, \dots]$ such that $a_k \leq M$ for some $M > 0$ and all $k \geq 1$, then it holds

$$|\omega - p/q| \geq \frac{1}{M+2} \frac{1}{q^2}$$

for any $p, q \in \mathbb{N}$. Hence

$$\left| \left(\frac{i}{m}\right)^2 - \omega^2 \right| = \left| \frac{i}{m} - \omega \right| \left| \frac{i}{m} + \omega \right| \geq \frac{\omega}{M+2} \frac{1}{m^2}$$

for any $i, m \in \mathbb{N}$. Consequently, we get the following result.

Theorem 4.4. *The condition (1.4) holds if $\omega = [a_0, a_1, \dots]$ is such that $a_k \leq M$ for some $M > 0$ and all $k \geq 1$, and $\omega > (M + 2)n^2$.*

Proof. For $i > n, m \in N$, according to the above result, we have

$$\begin{aligned} |i^2 - \omega^2 m^2 - n^2| &\geq |i^2 - \omega^2 m^2| - n^2 \\ &= m^2 \left| \left(\frac{i}{m} \right)^2 - \omega^2 \right| - n^2 \geq \frac{\omega}{M+2} - n^2 > 0. \end{aligned}$$

The proof is finished. $\square$

For instance, if $\omega = (2M - 1 + \sqrt{5})/2$, $M \in N$ then $\omega = [M, 1, 1, \dots]$ and we get the condition $(2M - 1 + \sqrt{5})/2 > 3n^2$.

Moreover, we can get further characterizations of ω to satisfy (1.4). We follow [2, 3]. Let

$$\mu(\alpha)^{-1} = \liminf_{q \rightarrow +\infty} |q(q\alpha - p)|,$$

where p and q are arbitrary integers. The set of values taken by $\mu(\alpha)$ as α varies is called the Lagrange spectrum. If $\alpha = [a_0, a_1, a_2, \dots]$ then

$$\mu(\alpha) = \limsup_{k \rightarrow +\infty} ([a_{k+1}, a_{k+2}, \dots] + [0, a_k, a_{k-1}, \dots, a_1]).$$

We know [6] that $\mu(\alpha) < +\infty$ if and only if α is irrational and all a_i are uniformly bounded.

According to [2], two real numbers θ and θ' are **equivalent** if there are integer numbers r, s, t, u such that

$$\theta = \frac{r\theta' + s}{t\theta' + u}, \quad ru - ts = \pm 1.$$

We know from [2] that $\mu(\theta) = \mu(\theta')$.

Let $\omega > 0$ be irrational such that $\mu(\omega) \neq +\infty$. Then for any $\epsilon > 0$ and for any $n < i, m \in N$, expect of a finite number, we have like above

$$|i^2 - \omega^2 m^2 - n^2| \geq |i - \omega m| |i + \omega m| - n^2 \geq \left(\frac{1}{\mu(\omega)} - \epsilon \right) \omega - n^2.$$

On the other hand, if $i_0^2 - \omega^2 m_0^2 - n^2 = 0$ for some $n < i_0, m_0 \in N$, then the proof of Theorem 4.3 gives infinitely many such i_0, m_0 .

Summarizing, we get the following result.

Theorem 4.5. *Let $\omega' > 0$ be irrational such that $\mu(\omega') \neq +\infty$. Then (1.4) holds for any $\omega > 0$ equivalent to ω' satisfying $\omega > \mu(\omega')n^2$.*

For instance, if we take $u=t=1$, $r=s+1$, $s \in \mathbf{Z}$, then $\omega = s + \omega' / (\omega' + 1)$ is equivalent to ω' , and Theorem 4.5 holds for any $s \in \mathbf{N}$ such that

$$s > n^2 \mu(\omega') - \frac{\omega'}{\omega' + 1}.$$

Finally, we note [2, 3] that $\mu(\alpha) \geq \sqrt{5}$ for any α . Moreover, the Lagrange spectrum on the interval $[\sqrt{5}, 3)$ is the set $\{\sqrt{9 - 4m^{-2}}\}$, where m is a positive integer such that

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2, \quad m_{1,2} \leq m$$

holds for some positive integers m_1 and m_2 . Then ω' is a root of the Markoff form F_m such that $\mu(\omega') = \sqrt{9 - 4m^{-2}}$. We also note that according to the Hall theorem [3], the Lagrange spectrum contains every number greater or equal to $\sqrt{21}$.

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(Ricevita la 19-an de marto, 2001)