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Theoretical Foundations

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Designing an Investment Trust: Theoretical Foundations

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Abstract

This paper deals with a dynamic portfolio with dividends. The portfolio consists of a risky asset and a safe one. Investors initially buy the portfolio and enjoy their consumption based on the dividends over time. During their lifetime, they reinvest the funds after the dividends are paid. In short, investors adopt a strategy of buying and holding. They decide on a portfolio and the dividend rate to maximize their utility in every period, in a series of separate decisions. In this situation, there seems to be a tradeoff between dividends and reinvestment, to the possible detriment of the portfolio. Thus, we focus on establishing an optimal path for dividends.

JEL Classification: G-11; G-12; G-23; G-32.
Keywords: CRRA type utility function, Dividend, Ito's lemma, Portfolio choice.

1. Motivations

Human beings inevitably age, which naturally drives them into hard working conditions. They retire from their jobs, then eke out livings relying on pensions. In this situation, the current small pensions in Japanese society matter.
Since the scale of pensions generally depends on the economy, small pensions are inevitable if the economy deteriorates, as the Japanese economy has been doing for decades.

We consider investment trusts as an alternative to the pensions: Investors invest in multiple assets with returns; a portion of a portfolio, dividends, is returned to the investors in every period. After the dividends are paid, the remains are reinvested. In short, since dividends are a financial resource for the old as an alternative to pensions, the more, the better. However, there seems to be a tradeoff between dividends and reinvestment. This conjecture naturally occurs when one doubts that more dividends lead to less reinvestment, which leads to a small portfolio, facing possible collapse. Thus, we aim to demonstrate whether optimal dividends can be determined or not. Our model is naturally a dynamic one explaining the movements of the value of a portfolio over time. Merton (1969) is useful in this regard.

We organize this paper as follows: in the next chapter, a portfolio is defined, and in the 3rd chapter, a basic model framework is introduced; the 4th chapter describes budget constraints, then in the 5th and 6th chapters, an objective function and an optimal value function are respectively explained; subsequently, the 7th and 8th chapters describe the structure and fluctuations of risky asset prices; and the 9th and 10th chapters are respectively devoted to a separation theorem and the dynamic structure of the portfolio; then, numerical examples and concluding remarks follow.

2. Portfolio

Investors invest in two assets, namely a risky asset and a safe one. We assume one unit regarding the above assets respectively, so we analyze the value movements of the portfolio only by the price movements of the assets, then consider total returns in terms of these price movements.
As a result, the total return from the portfolio equals the sum of the total returns from the risky and safe assets.

3. The model

We define symbols as follows:
$X_t^1 =$ the value (price) of the risky asset in t period.
X_t^2 = the value (price) of the safe asset in t period.
W_t = the value (price) of the portfolio in t period.
Then,

$$W_t = X_t^1 + X_t^2$$

prevails.

Additional definitions:
α_t= the ratio of the value of the risky asset to that of the portfolio in t period, which is described as

$$\alpha_t^1 = \frac{X_t^1}{W_t}$$

κ_t=dividends in t period.
t = discrete time; t = 0,1,2,⋯T.
T = end time.

At this stage, we need to clarify that investors offer a fund W_t, which is devoted to current dividends and investment in assets. That is, the fund after dividends becomes a financial resource for assets in the next period. In this situation, the investors make decisions regarding dividends, κ_t. and the portfolio, α_t. Moreover, the value of the fund for assets is determined by the markets in the next period.

4. Budget constraints

Concerning the risky asset, the total return in terms of a period is described as a random variable R.

$$R_t = \frac{X_t^1}{X_{t-1}^1}$$

prevails. On the other hand, regarding the safe asset, R_f, a constant, shows the total return in terms of a period.
Next, let us analyze the portfolio. The portfolio fund in t period is $W_t - \kappa_t$, which is exposed to two different kinds of risks, namely R_t and R_f. However, the risk to the portfolio is a weighted average of R_t and R_f. The weight is α_t. Reflecting these risks, the total return from the portfolio can be denoted as

$$R_f(1 - \alpha_t) + R_{t+1}\alpha_t.$$

Thus, we can show the result of the portfolio in t+1 period:

$$(W_t - \kappa_t)\big(R_f(1 - \alpha_t) + R_{t+1}\alpha_t\big),$$

which equals W_{t+1}. Finally, we obtain a budget constraint, a stochastic difference equation regarding total returns from the portfolio,

$$W_{t+1} = (W_t - \kappa_t)\big(R_f(1 - \alpha_t) + R_{t+1}\alpha_t\big). \qquad (1)$$

5. An objective function and system

Investors have two characteristics, as buyers of mutual funds and as consumers. Of course, consumers aim to maximize their utility. In this situation, we assume that their consumption is based on the dividends from a portfolio. So, we can say that a utility function has an argument expressed as dividends, and that it belongs to a constant relative risk-aversion type,
which leads to the following utility function:

$$U(\kappa) = \frac{\kappa^{1-\gamma}}{1-\gamma} \qquad (2)$$

where γ is a positive constant. In our context, κ is supposed to be random. So to value $U(\kappa)$ is inadequate. Instead, consider the expectation of an instant utility, then reevaluate the instant utility by the rate of time preference β, and thereafter add the discounted utility.

The above procedures yield

$$E\left(\sum_{t=0}^{T} \beta^t \ U_t(\kappa_t) + B(W_T, T)\right), \qquad (3)$$

where E means an expectation operator, β a time discount rate, and B a salvage function.

Next, (3) should be maximized regarding κ and α. Thus, (3) can be expressed as

$$Max_{(\kappa,\alpha)} E\left(\sum_{t=0}^{T} \beta^t \ U_t(\kappa_t) + B(W_T, T)\right). \qquad (4)$$

As a result, our system can be described: an objective function

$$Max_{(\kappa,\alpha)} E\left(\sum_{t=0}^{T} \beta^t \ U_t(\kappa_t) + B(W_T, T)\right),$$

(2) and (1) with the followings:

$$0 \leq \kappa_t < W_t \text{ and } W_0 > 0..$$

Here, W_0 shows an initial fund. Furthermore, $\beta^t U_t(\kappa_t)$ in the above objective function shows the utility in t period which is evaluated at 0 period.

6. Optimal value function

We define the following optimal function based on Bellman's optimal principle:

$$max_{\{\kappa_t,\alpha_t\}_{t=0}^{T-1}} E_0(t)\left[\ \sum_{t=0}^{T-1} \beta^t U(\kappa_{t)} + \beta^T B(W_T, T)\right] \qquad (5)$$

Noticing that (5) is a function with arguments, a state variable and time, we can express (5) in terms of an optimal value function such that

V_t(W)= $max_{(0\leq\kappa\leq W,\alpha)}[\frac{\kappa^{1-\gamma}}{1-\gamma} + \beta\, \mathrm{E}_t\ [V_{t+1}[((W-\kappa)\ R\{\tilde{\alpha}\ ))]]$. (6)

Here, V_t(W) shows an optimal value function in t period. In addition, the total return from the portfolio is noted below:

$$R(\tilde{\alpha}\ ) = R_f(1-\alpha_t) + R_{t+1}\alpha_t.$$

Moreover, a boundary condition is defined as

$$V_T(W) = S(W) = B\ \frac{W^{1-\gamma}}{1-\gamma},$$

which features the same style as the utility function regarding κ_t. (See (2)). This implies that investors enjoy their final consumption from end dividends.

7. The system

At this stage, we need to intensively elaborate our system. Specifically, our aim here is to derive the optimal value function based on (6).

First, let us consider a stricture of the optimal value function V(W). Let us focus on the right side of (6). The state relationship is obviously linear regarding W and κ, while the utility function is a homogeneous function of degree (1−γ).

As a result, we can conclude that V(W) is a homogeneous function of degree (1−γ). Therefore, we can describe the value function as

$$V_t(W) = A_t\ \frac{W^{1-\gamma}}{1-\gamma}, \qquad (7)$$

where A_t is a scaler regarding t.

Substituting (7) into (6) yields

$$A_t \ \frac{W^{1-\gamma}}{1-\gamma} = underset(\kappa \in [0,1], \alpha)\text{Max}\left[\frac{\kappa^{1-\gamma}}{1-\gamma} + \beta A_{t+1}\frac{(W-\kappa)^{1-\gamma}}{1-\gamma}E\left(\widetilde{R^{1-\gamma}}\right)\right]$$

Transforming the above, we obtain

$$A_t = underset(\theta \in [0,1], \alpha) \text{ Max } [\ \theta^{1-\gamma} + \beta A_{t+1}(1-\theta)^{1-\gamma} m_t(\alpha)]. \qquad (8)$$

Here, $\theta = \frac{\kappa}{W}$ and

$$m_t(\alpha) = E\left(\widetilde{R^{1-\gamma}}\right) = E\left(\ \left(R_f(1-\alpha_t) + R_{t+1}\alpha_t\right)^{1-\gamma}\ \right). \qquad (9)$$

θ shows the ratio of dividends in a portfolio, and (9) the expectation in t period of the total return from a portfolio multiplied by (1-γ) in (t+1) period. Investors aim to maximize (9). (8) requires that A_t be determined when adequate θ and α are selected.

Moreover, when investors select an optimal portfolio and a dividend ratio, both can be individually determined. This is called the separation theorem. Below, we intensively investigate these mechanisms.

7-2 Selection of α and θ

First, let us discuss α. Investors can maximize (9) only regarding α, because function $m_t(\alpha)$ has an argument α. A necessary condition for the maximization of (9) requires

$$E\left(\left(R_{t+1} - R_f\right)\widetilde{R}(\alpha)^{-\gamma}\right) = 0. \qquad (10)$$

In other words, α is determined satisfying (10).

Second, let us analyze θ. Given α determined by (10), investors can select θ by maximizing (8) regarding θ. The necessary condition requires

$$(1-\gamma)\theta^{-\gamma} - \beta(1-\gamma)(1-\theta)^{-\gamma} m_t\, A_{t+1} = 0, \qquad (11)$$

which eventually leads to

$$\theta_t^* = \frac{1}{1+(\beta A_{t+1} m_t)^{1/\gamma}} \ . \qquad (12)$$

Based on the facts that we have analyzed so far, we know how control variables, α and θ are determined. Next, we need to elucidate a new related thema, how a state variable W is determined, if possible. We derive a structural form of the value function based on (7), which is a key concept related to W. So, we proceed to discuss A_t.

7-3 Determination of A_t

Substituting (12) into (8) and arranging the results, we obtain the required conclusion as

$$A_t = \left(1 + \left(\beta A_{t+1} m_{t(}\alpha_t^*)\right)^{1/\gamma}\right)^{\gamma}, \qquad (13)$$

$$A_T = B$$

$$t = 0, 1, \cdots, T-1.$$

Based on (13), we can specifically seek A_t. Denoting $\left(\beta m(\alpha_t^*)\right)^{1/\gamma}$ = h and

$A_t^{1/\gamma} = G_t$, (13) can be expressed such that

$$G_t = hG_{t+1} + 1,$$

$$G_T = B^{1/\gamma}.$$

Solving the above recurrence formula concerning G, we specifically obtain A_t as

$$A_t = \left(B^{1/\gamma}(\mathrm{h})^{T-t} + \frac{1-(\mathrm{h})^{T-t}}{1-h}\right)^{\gamma},$$

$$= \left(B^{1/\gamma}\left((\beta m)^{1/\gamma}\right)^{T-t} + \frac{1-\left((\beta m)^{1/\gamma}\right)^{T-t}}{1-(\beta m)^{1/\gamma}}\right)^{\gamma}, \qquad (14)$$

$$\mathrm{t} = 1, 2, \cdots, T.$$

Here, A_t is obviously a scalar coefficient which moves with time and conveys lots of intriguing information about our context. m shows $m_t(\alpha_t)$.

However, for (14) to hold, the following needs to be satisfied:
The recurrence relationship $G_t = hG_{t+1} + 1$

is satisfied with the convergence condition h < 1. G_t is solved backward from T, i.e., as a terminal term, so h < 1 is clearly required for $A_t$to reach a convergence point. Thus, condition $\left(\beta m(\alpha_t^*)\right)^{1/\gamma}$<1 has to prevail, which leads to

$$\beta\, m(\alpha^*) < 1 \qquad (15)$$

$m(\alpha^*)$ shows the expectation of $\tilde{R}$ raised to the power of (1-γ) in a portfolio, as defined in (9), which is maximized regarding α_t. So, (15) refers to the expectation discounted by β.

At this stage, we should pay attention to the economic meanings of A_t. Recall the value function (7):

$$V_t(W) = A_t \, \frac{W^{1-\gamma}}{1-\gamma}.$$

A marginal value of W is deduced as

$$V_t'(W) = A_t W^{-\gamma}.$$

In this context, A_t is an indication of the degree to which investors are serious regarding the future. When A_t is large, they are induced to cut current consumption, because the marginal value of the future is high. On the other hand, when A_t is small, they tend to highly evaluate current consumption.
In order to grasp our system concretely, we need to investigate the movements of total returns and the mutual relationships between the risky and safe assets.

8. Fluctuations of risky asset prices

Since the fluctuations of risky asset prices are random, we assume that the prices follow a geometrical Brownian motion. Thus,

$$R_{t+1} = \frac{X_{t+1}}{X_t} = \exp\left(\Delta\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\Delta}\,\sigma\epsilon_{t+1}\right) \qquad (16)$$

prevails. Below, an upper script on X_t^1 is omitted.

Δ shows a time period, ϵ_{t+1}, a random variable followed by N(0,1), μ and σ, a constant, respectively.

According to (16), we can say that R_{t+1} is a random variable followed by a lognormal distribution with a mean and a variance. From (16), we obtain the following:

$$E(R_{t+1}) = e^{\Delta \mu},$$

which shows that μ is an expectation of R_{t+1} in terms of a logarithm.

8-2. Determination of α

Based on our analysis in (9), we know that α is determined to maximize $m_t(\alpha)$, if R_{t+1} is given. Now that we can consider R_{t+1} by means of (16), we can proceed to analyze how α is determined. For simplicity, we rewrite (9) in this way:

$$m(\alpha) = E(\widetilde{R^k}) = E\Big(\big(\alpha(R - R_f) + R_f\big)^k \Big),$$

$$k = 1 - \gamma,$$

$$\tilde{R} = R_f + \alpha \Delta R = \alpha(R - R_f) + R_f,$$

and $\quad \Delta R = R - R_f.$

ΔR shows the difference of total returns between the risky and safe assets, a key concept.

At this stage, notice that the function $\widetilde{R^k}$ has an argument, R, which follows the geographical Brownian motion described in (16), so Taylor series expansions around R_f are described as

$$\widetilde{R^k} = R^k{}_f + \mathrm{k}\, \mathrm{R}^{k-1}{}_f\, \alpha\, \Delta R + \tfrac{1}{2} k(k-1) R^{k-2}{}_f\, \alpha^2 (\Delta R)^2,$$

which Ito’s lemma implies.

Then, in terms of an expectation

$$E(\widetilde{R^k}) = m(\alpha) = R_f^k + \mathrm{k}\, \mathrm{R}_f^{k-1}\, \alpha\, \mathrm{E}(\Delta R) + \tfrac{1}{2}\alpha^2\, k\,(k-1) E(\Delta R)^2 R_f^{k-2},$$

prevails, which is our key resource for seeking α. Based on the above relationship, α is determined in the sense that the expectation of $(\widetilde{R^k})$ is maximized. The following prevails, if there is an inner point regarding α:

$$m'(\alpha) = k\, E(\Delta R)\, \mathrm{k}\mathrm{R}_f^{k-1} + \alpha\ \mathrm{k}\,(k-1)\ E(\Delta R)^2 R_f^{k-2} = 0,$$

which results in

$$\alpha^* = \frac{R_f\left(E(R)-R_f\right)}{\gamma\, E(\Delta R)^2}. \qquad (17)$$

$$\simeq \frac{R_f\left(E(R)-R_f\right)}{\gamma\, Var(R)}. \qquad (17)'$$

where $(17)'$ holds if $E(R) \simeq R_F$, i.e., the distribution of the return from the risky asset on average equals the return from the safe one. Additionally, (17) has to t satisfy $0 \leqslant \alpha^* \leqslant 1$, because we do not suppose short selling.

From the logic so far, it is clear that the portfolio is chosen regardless of the determination of the ratio of dividends in it. Based on (17), we obtain an interesting conclusion, namely that the determination of α^* is time-independent, as found in Merton (1969).

This feature leads to the strategy of rebalancing a portfolio. (17) and $(17)'$ feature the following: (a) the greater the expectation regarding total returns from a risky asset is, the larger α^* becomes, and vice versa. This implies that funds naturally move back and forth between the two assets according to anticipated returns from them. However, this story is true if E(R) > R_f. (b) the larger Var(R) is, the smaller α^* is, i.e., large variance in the risky asset drives investors to reduce funds allotted to it, and vice versa. (c) the smaller γ is, the larger α^* is, i.e., when the degree of risk aversion is small, investors allocate more founds to the risky asset than to the safe one, and vice versa.

8-3. Determination of $\boldsymbol{\theta}$

We have induced θ in 7-2, but our analysis stops just short of showing the arguments determining θ. Let us deepen our logic. We induced θ in (12) as

$$\theta_t^* = \frac{1}{1+(\beta A_{t+1} m_t)^{1/\gamma}}.$$

The above relationship implies that if β、α、 B, γ and A_{t+1} are given, θ_t^* is uniquely determined. Additionally, since A_{t+1} can be determined by (14), the preconditions are fully satisfied.

θ_t^* has the following features: it changes as time passes, especially depending on current maturity; and it shows a preference for the future, for the value of the portfolio at the end period, and a degree of a risk aversion.

At this stage, we consider a situation with no time difference. We denote $A_{t+1} = A_t = A$. Then, from (8),

$$A = \theta^{1-\gamma} + A\,\beta m(1-\theta)^{1-\gamma}$$

prevails. Moreover, based on (11) we obtain

$$\left(\frac{1-\theta}{\theta}\right)^{\gamma} = \beta A m_t,$$

which can be expressed as

$$\frac{1-\theta}{\theta} = (\beta\, A\, m_t)^{1/\gamma}.$$

Defining $(\beta\, A\, m_t)^{1/\gamma} = s$, and substituting it into the right side in (8), the right side becomes $(s+1)^{\gamma}$, which equals A. So, finally the following holds:

$$A = (s+1)^{\gamma}.$$

Substituting $A = (s+1)^{\gamma}$ into the definition of s, we obtain

$$s = \frac{(\beta m)^{1/\gamma}}{1-(\beta m)^{1/\gamma}},$$

which can be transformed as follows:

$$\theta_t^* = 1 - \left(\beta m(\alpha_t^*)\right)^{1/\gamma}. \qquad (18)$$

(18) is simpler and more understandable than (12), because A_{t+1} disappears. In this situation, m shows the maximum of the expectation of the total returns with the power of $(1-\gamma)$ in the portfolio, so the value discounted in one term must be less than 1. Otherwise, θ_t^* goes beyond 1. Moreover, (18) implies that θ_t^* is decided when $\propto$ is given. In 8-2, we have shown how α_t^* is

determined, which shows that $E(R)$, $E(\Delta R)^2, R_f, and$ γ determine α**,** while θ_t^* does so in (18). In (18), θ_t^* and $\left(\beta m(\alpha_t^*)\right)^{1/\gamma}$ are disproportional.

In fact, $\left(\beta m(\alpha_t^*)\right)^{1/\gamma}$ shows an investment ratio in the portfolio in t period, i.e., $\frac{W_t - \kappa_t}{W_t}$, which can appear in (18) as

$$\theta_t^* + \left(\beta m(\alpha_t^*)\right)^{1/\gamma} = 1.$$

The above relationship implies that the dividends and the investment ratios are respectively independent across periods. Remember the fact that α is determined independently from periods. This assertion is a crucial result, since the optimal dividends ratio, which we are looking for in the framework of a finite time horizon, can certainly be found in an infinite time horizon.

9. Why are the weight and dividend ratio separately decided?

Investors face two dilemmas. one is the problem of how to increase returns from the portfolio; the other is how to enjoy consumptions from dividends. The former refers to a problem concerning the amount of investment. The selection depends heavily on anticipation regarding future returns, and this anticipation naturally relies on how investors consider risks. On the other hand, the latter refers to how much investors enjoy current consumption.

We have postulated a key concept about utility, i.e., a constant relative risk-aversion type utility function. This concept means that investors take unchanged stances toward risks regardless of their wealth, which drives them to be indifferent to how much current money to keep and how much to invest. This is why the separation theorem is deduced.

10. The dynamic structure of the portfolio

What we have discussed implies that our system randomly fluctuates. This causes the portfolio to vary at random, because risky asset prices randomly occur. As a result, the value of the portfolio and dividends are not uniquely determined. However, the dividend ratios and the weights, which act as control variables in our system, can be uniquely determined.

When a dividends ratio is uniquely selected, the dividend is randomly determined, because the dividend is the result of the product of the dividend rate and the value of the portfolio.

Nevertheless, we can say that an optimal dividend ratio can be deduced along with a principle of maximization concerning utility, regardless of the value of the portfolio.

11. Numerical Examples

Let us confirm our dynamic system numerically. We assume the followings:
$\mu = 0.06$; $\sigma = 0.15$; $\Delta = 1$; $\gamma = 5$; $\beta = 0.9$; B=1; $R_f = 1.02$;

We subsequently discuss three key factors namely α, θ, and A.
First, α is determined to maximize $m(\alpha)$. Since $m(\alpha)$ is expressed as shown in 8-2, it appears in a quadratic form as

$$m(\alpha) = 0.9238 - 0.15157\ \alpha + 0.24336\ \alpha^2,$$

which reveals the maximum, $m(1)$=1.01564. This shows that investors wholly select risky assets, and that no fund is allocated to safe assets. Second, as demonstrated in (14), A_t can be expressed as

$$A_t = \left(B^{\frac{1}{\gamma}} \left(\frac{1}{h}\right)^{t-T} + \frac{1-\left(\frac{1}{h}\right)^{t-T}}{1-h} \right)^{\gamma}.$$

Initially, we need to confirm $h = (\beta m(\alpha^*))^{1/\gamma}$. Then, h = 0.982192, due to $m(1)$=1.01564. Assuming that T=6, we can obtain A_t as

$$\{12877.7., 6226.35, 2615.26, 895.8, 222.24, 30.6005, 1.\}$$

As t passes, we can confirm that A_t gradually decrease.

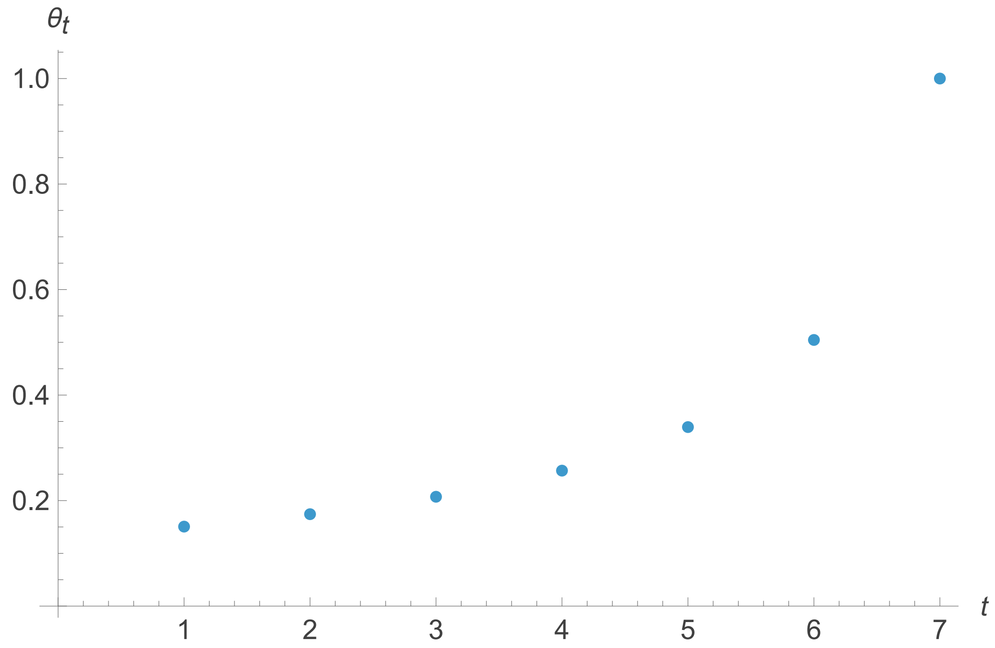
Third, θ_t^* is determined in (12), which can be transformed as

$$\theta_t^* = \frac{1}{1+h\left(B^{\frac{1}{\gamma}}\left(\frac{1}{h}\right)^{t-T+1}+\frac{1-\left(\frac{1}{h}\right)^{t-T+1}}{1-h}\right)},$$

which refers to the following list:

$$\{0.150672,\ 0.17424,\ 0.207251,\ 0.256778,\ 0.33934,\ 0.504492,\ 1.\}.$$

Here is the above list in graph form.



Graph 1.

Based on Graph 1, we acknowledge that θ_t^* increases as time passes, which implies a situation in which current maturity is decreasing. However, what we are looking for is a maximum point regarding θ_t^*, i. e., a constant θ through various periods, if possible. But the optimum dividend ratios changes over time, as we have demonstrated. The only possibility then is to consider a case with no time-difference.

12. Concluding remarks

We have demonstrated fundamental structures of an investment trust, emphasizing dividends from a portfolio. The portfolio naturally faces risks and driving forces from outside. The value of the portfolio randomly fluctuates, resulting in a series of random dividends over time. But dividend rates can be uniquely determined in processes to maximize utilities. Investors are supposed to adopt a basic strategy *Buy and Hold*.

Concrete time paths of W, A, α and θ are controlled by combinations among several parameters, such as μ, σ, γ, β, B and R_f, and especially, an investor's attitude towards risks can influence the time paths.

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