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Q POLYNOMIAL OF A KNOT WITH SMALL DEGREE

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Abstract

We decide the Q polynomials of knots with degree up to four.

1. Introduction

The Q polynomial is an invariant of the isotopy type of an unoriented link, which was introduced by Brandt, Lickorish, Millett [1] and Ho [3]. The Q polynomial $Q(K) = Q(K;x) \in \mathbb{Z}[x]$ of a knot K is of the form

$$Q(K;x) = \sum_{k=0}^{n} c_k x^k, \tag{1}$$

where c_0 is an odd integer and $c_1, \ldots, c_n$ are even integers. The degree of $Q(K;x)$ is less than the crossing number of K [1, Property 8]; cf. [8, 9, 12]. In this note we show the following.

Theorem 1.1. *If the degree of the Q polynomial of a knot is up to four, then it is the Q polynomial of a knot with crossing number up to five or the granny (or square) knot, which is equal to one of* $Q(U)$, $Q(3_1)$, $Q(4_1)$, $Q(5_1)$, $Q(5_2)$, $Q(3_1\#3_1)(= Q(3_1)^2)$*; see Table 1.*

We use the knot names in [10].

Remark 1.2. Miyazawa [11] discovered two 16-crossing knots with trivial Q polynomial, which have nontrivial Jones polynomials. By connected sums we can construct infinitely many knots sharing the same Q polynomial as that of a particular knot.

In [6] we have shown an analogous result for the knot Jones polynomial [4]. Note that Kauffman's F polynomial [7] specializes to the Q and Jones polynomials.

Knots	Q Polynomial
U	1
3_1	$-3+2x+2x^2$
4_1	$-3-2x+4x^2+2x^3$
5_1, 13n_1636	$5-2x-6x^2+2x^3+2x^4$
5_2, 13n_3082	$1-4x-2x^2+4x^3+2x^4$
$3_1\#3_1$, $3_1!\#3_1$, 13n_2561	$9-12x-8x^2+8x^3+4x^4$

Table 1. Q polynomials of knots with degree up to four.

This note is organized as follows. In Sect. 2 we review the Q polynomial and give some restrictions a knot Q polynomial satisfies. Using them we prove Theorem 1.1. In Sect. 3 we consider potential knot Q polynomials with degree ≥ 5.

2. Q polynomial

The Q polynomial $Q(L) = Q(L;x) \in \mathbb{Z}[x^{\pm 1}]$ [1, 3] is an invariant of the isotopy type of an unoriented link L defined by

$$Q(U) = 1, \tag{2}$$

$$Q(L_+) + Q(L_-) = x\left(Q(L_0) + Q(L_\infty)\right), \tag{3}$$

where U is the unknot and $(L_+, L_-, L_0, L_\infty)$ is an *unoriented skein quadruple*, which is an ordered set of four unoriented links that are identical except near one point where they are as in Fig. 1.

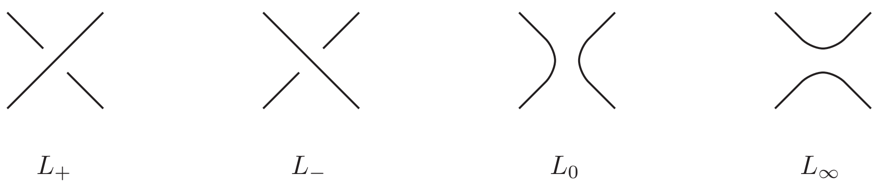

Figure 1. Unoriented skein quadruple.

For the Q polynomial of a knot K, $Q(x)$, we have the following evaluations:

$$Q(1) = 1, \tag{4}$$

$$Q(-2) = 1, \tag{5}$$

$$Q(2) = \delta^2, \tag{6}$$

$$Q(-1) = (-3)^d, \tag{7}$$

$$Q((\sqrt{5}-1)/2) = \pm(\sqrt{5})^e, \tag{8}$$

where δ is the determinant of K, that is, the order of $H_1(\Sigma_2(K);\mathbb{Z})$, $\Sigma_2(K)$ the double covering space of S^3 branched over K, $d = \dim H_1(\Sigma_2(K);\mathbb{Z}_3)$, and $e = \dim H_1(\Sigma_2(K);\mathbb{Z}_5)$. Equations (4)–(7) are proved in [1], and Eq. (8) in [5]. Then from Eqs. (7) and (8) we obtain

$$\delta \equiv 0 \pmod{3^d}, \tag{9}$$

$$\delta \equiv 0 \pmod{5^e}. \tag{10}$$

Proof of Theorem 1.1. Let $Q(x) = b_0 + 2b_1x + 2b_2x^2 + 2b_3x^3 + 2b_4x^4$, where b_0, b_1, b_2, b_3, b_4 are integers. Then using Eqs. (4)–(8), we obtain

$$b_0 + 2b_1 + 2b_2 + 2b_3 + 2b_4 = 1, \tag{11}$$

$$b_0 - 4b_1 + 8b_2 - 16b_3 + 32b_4 = 1, \tag{12}$$

$$b_0 + 4b_1 + 8b_2 + 16b_3 + 32b_4 = \delta^2, \tag{13}$$

$$b_0 - 2b_1 + 2b_2 - 2b_3 + 2b_4 = (-3)^d, \tag{14}$$

$$(b_0 - b_1 + 3b_2 - 4b_3 + 7b_4) + \sqrt{5}(b_1 - b_2 + 2b_3 - 3b_4) = \pm(\sqrt{5})^e. \tag{15}$$

From Eq. (15), we have

$$b_0 - b_1 + 3b_2 - 4b_3 + 7b_4 = \pm 5^{e/2}, \tag{16}$$

$$b_1 - b_2 + 2b_3 - 3b_4 = 0, \tag{17}$$

if $e \equiv 0 \pmod 2$, and

$$b_0 - b_1 + 3b_2 - 4b_3 + 7b_4 = 0, \tag{18}$$

$$b_1 - b_2 + 2b_3 - 3b_4 = \pm 5^{(e-1)/2}, \tag{19}$$

if $e \equiv 1 \pmod 2$. Then, using Eqs. (11), (12), (14), (16)–(19), we obtain

$$(b_0, 2b_1, 2b_2, 2b_3, 2b_4) = \begin{cases} \left(D, \dfrac{-1-3D+4E_0}{2}, -D+E_0, 1+D-2E_0, \dfrac{1+D-2E_0}{2}\right) & \text{if } e \equiv 0 \pmod 2, \\ \left(D+4E_1, \dfrac{-1-3D}{2}, -D-5E_1, 1+D, \dfrac{1+D+2E_1}{2}\right) & \text{if } e \equiv 1 \pmod 2, \end{cases} \tag{20}$$

where $D = (-3)^d$, $E_0 = \pm 5^{e/2}$, and $E_1 = \pm 5^{(e-1)/2}$. Substituting Eq. (20) into Eq. (13), we obtain

$$\delta^2 = \begin{cases} 15 + 10D - 24E_0 & \text{if } e \equiv 0 \pmod 2, \\ 15 + 10D & \text{if } e \equiv 1 \pmod 2. \end{cases} \tag{21}$$

First, we consider the case $e \equiv 1 \pmod 2$. From Eq. (21) using Eq. (9), we have $15 \equiv 0 \pmod{3^d}$, which implies $d = 0$ or 1. If $d = 0$, then $D = 1$ and

$\delta^2 = 25$, and so by Eq. (10) we have $e = 1$. Then $E_1 = \pm 1$, giving $Q(4_1)$ and $Q(5_1)$. If $d = 1$, then $\delta^2 = -15$, a contradiction.

Next, we consider the case $e \equiv 0 \pmod 2$. Suppose $e = 0$. Then $E_0 = \pm 1$. If $E_0 = 1$, then from Eq. (21) using Eq. (9), we have $\delta^2 = 10(-3)^d - 9 \equiv 0 \pmod{3^d}$, which implies $d = 0, 1, 2$. If $d = 0$, then $D = 1$, giving $Q(U)$. If $d = 1$, then $\delta^2 = -39 < 0$, a contradiction. If $d = 2$, then $D = 9$, giving $Q(3_1\#3_1)$. If $E_0 = -1$, then from Eq. (21) using Eq. (9), we have $\delta^2 = 10(-3)^d + 39 \equiv 0 \pmod{3^d}$, which implies $d = 0, 1$. If $d = 0$, then $D = 1$, giving $Q(5_2)$. If $d = 1$, then $D = -3$, giving $Q(3_1)$.

Now, we consider the case $e \equiv 0 \pmod 2$ and $e \geq 2$. We put $f = e/2 \in \mathbb{N}$, and so $E_0 = \pm 5^f$. Suppose $d = 0$. Then, $\delta^2 = 25 - 24(\pm 5^f) > 0$, and so $E_0 = -5^f$. From Eq. (21) using Eq. (10), we have $\delta^2 = 25 + 24 \cdot 5^f \equiv 0 \pmod{5^{4f}}$. Clearly, $f \neq 1$, and so $f \geq 2$. Then, since $\delta^2 \leq (1 + 24)5^f = 5^{f+2} < 5^{4f}$, there does not exist a positive integer f satisfying $\delta^2 = 25 + 24 \cdot 5^f \equiv 0 \pmod{5^{4f}}$. Thus, $d \geq 1$. Then, Eq. (21) with $e \equiv 0 \pmod 2$ together with Eqs. (9) and (10) is written as

$$0 < 15 + 10(-3)^d - 24(\pm 5^f) \equiv 0 \pmod{3^{2d} \cdot 5^{4f}}, \tag{22}$$

where $d, f \in \mathbb{N}$. However, since $15 + 10(-3)^d - 24(\pm 5^f) \leq 15 + 10 \cdot 3^d + 24 \cdot 5^f$, by Lemma 2.1 below there do not exist positive integers d and f satisfying Eq. (22). Therefore, we cannot find any other Q polynomial of a knot with degree up to four. This completes the proof. □

Lemma 2.1. *If $d, f \in \mathbb{N}$, then $0 < 15 + 10 \cdot 3^d + 24 \cdot 5^f < 3^{2d} \cdot 5^{4f}$.*

Proof. Since $\left(5 \cdot 3^d - 8\right)\left(3 \cdot 5^f - 2\right) \geq (15 - 8)(15 - 2) = 91$, we have

$$10 \cdot 3^d + 24 \cdot 5^f \leq 15 \cdot 3^d \cdot 5^f - 75 = 3^{d+1} \cdot 5^{f+1} - 75.$$

Then $15 + 10 \cdot 3^d + 24 \cdot 5^f \leq 3^{d+1} \cdot 5^{f+1} - 60 < 3^{2d} \cdot 5^{4f}$, completing the proof. □

3. Polynomials with degree ≥ 5

For the polynomials with degree ≥ 5 satisfying Eqs. (4)–(8) we have the following.

Theorem 3.1. *For each integer $n \geq 5$ there exist infinitely many polynomials $Q(x) \in \mathbb{Z}[x]$ of degree n satisfying Eqs. (4)–(8) with δ an odd integer, $d = 0$, and $\pm(\sqrt{5})^e = 1$.*

Proof. First, we consider the case $n = 5$. The polynomial

$$\begin{aligned} Q(x) &= 1 + 2a(2 - x - 5x^2 + 3x^4 + x^5) \\ &= 1 + 2a(1 - x)(2 + x)(1 + x)(1 - x - x^2) \end{aligned} \tag{23}$$

satisfies Eqs. (4)–(8) with $\delta^2 = 1 + 120a$, $d = 0$, and $\pm(\sqrt{5})^e = 1$. So, if $a = (15b^2+11b+2)/2$, $b \in \mathbb{Z}$, then $Q(2) = (30b+11)^2$. This gives degree 5 polynomials satisfying Eqs. (4)–(8) with $\delta = |30b + 11|$, $d = 0$, and $\pm(\sqrt{5})^e = 1$.

Now, we consider the case $n \geq 6$. The polynomial

$$Q(x) = 1+2(a+x+x^2+\cdots+x^{n-6})(1-x)(2+x)(2-x)(1+x)(1-x-x^2), \tag{24}$$

$a \in \mathbb{Z}$, satisfies Eqs. (4)–(8) with $\delta = 1$, $d = 0$, and $\pm(\sqrt{5})^e = 1$. Note that for $n = 6$ the degree of $Q(x)$ is 6 if $a \neq 0$ and for $n \geq 7$ the degree of $Q(x)$ is n. $\square$

Remark 3.2. The polynomials Eq. (23) with $a = 1$, 3 are

$$Q(6_2),\ Q(17\text{nh}0001459),$$

respectively; for the knot 17nh0001459, see [2].

Question 3.3. For each integer $n \geq 5$ do there exist infinitely many knot Q polynomials with degree n?

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