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A note on isothermic coordinate systems for spacelike surfaces with constant mean curvature in Lorentz-Minkowski space

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Dedicated to Professor Wayne Rossman on his sixtieth birthday

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Abstract

In this note, we use isothermic coordinate systems to explore global properties of space-like surfaces with constant mean curvature in the Lorentz-Minkowski three-space.

1. Introduction

The existence of special coordinate systems often plays a crucial role in proving global properties in surface theory. A well-known example is the proof of the Hilbert theorem by using a Tschebyscheff coordinate system, and it states that a surface in the Euclidean 3-space $\mathbb{E}^3$ with constant negative Gaussian curvature must have singular points. For further details, see [8, Chapter IX]. As another example, in [17, Chapter 5], a proof of the Bernstein theorem by using a special isothermal coordinate system is described, and it asserts that any entire minimal graph in $\mathbb{E}^3$ must be a plane. Recently, Hasanis, Savas-Halilaj, Vlachos [9] characterized complex analytic curves within the class of entire minimal graphs in the Euclidean 4-space $\mathbb{E}^4$ via their Jacobians also by using this coordinate system.

First we prove the existence (Theorem 3.2) of isothermic coordinate systems for space-like surfaces in the Lorentz-Minkowski 3-space $\mathbb{L}^3$ with non-zero constant mean curvature (abbrev. CMC). Theorem 3.2 is already a known result, see for example, [3], [11, Proposition 1.4], [10]. We then demonstrate its application to global properties. Space-like surfaces in $\mathbb{L}^3$ with non-zero CMC have interesting global properties. See [7, 19, 20] for example. Recently, Dorfmeister, Inoguchi

and Kobayashi [6] showed the relationship between minimal surfaces in the 3-dimensional Heisenberg group and space-like surfaces in $\mathbb{L}^3$ with non-zero CMC by the Sym-formula.

We here provide a much simpler proof of some uniqueness theorems (Corollary 3.5, Theorem 3.6) by using the isothermic coordinate systems for complete space-like surfaces in $\mathbb{L}^3$ with non-zero CMC, as previously proved by Milnor [16] and Yamada [22].

2. Preliminaries

In this section, we briefly review fundamental notions of space-like surfaces in $\mathbb{L}^3$, analogous to those in the theory of surfaces in $\mathbb{E}^3$, using notation consistent with that of [13].

2.1. Basic theory. Let $\mathbb{L}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_L)$ be the Lorentz-Minkowski 3-space with Lorentz metric

$$\langle \mathbf{a}, \mathbf{b} \rangle_L := a_1 b_1 + a_2 b_2 - a_3 b_3,$$

and vector product

$$\mathbf{a} \times_L \mathbf{b} := (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, -(a_1 b_2 - a_2 b_1)),$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. We can easily show that

$$\langle \mathbf{a} \times_L \mathbf{b}, \mathbf{a} \rangle_L = \langle \mathbf{a} \times_L \mathbf{b}, \mathbf{b} \rangle_L = 0 \quad \text{and} \quad \langle \mathbf{a} \times_L \mathbf{b}, \mathbf{c} \rangle_L = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}),$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{L}^3$. Here, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are considered as column vectors and $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the determinant of the 3×3 -matrix $(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

Let D be a domain in the (u, v) -plane. A surface $X: D \rightarrow \mathbb{L}^3$ is called *space-like* if the induced metric, that is, the first fundamental form I_X of X on the surface, is positive definite, and from now on we assume a surface is space-like. Set

$$(1) \quad E = \langle X_u, X_u \rangle_L, \quad F = \langle X_u, X_v \rangle_L, \quad G = \langle X_v, X_v \rangle_L.$$

The *first fundamental form* is then

$$(2) \quad I_X = E du^2 + 2F dudv + G dv^2,$$

and the following three statements are equivalent:

- (a) $X: D \rightarrow \mathbb{L}^3$ is a space-like surface.
- (b) $(EG - F^2)(p) > 0$ for any point $p = (u, v) \in D$.
- (c) $X_u \times_L X_v$ is time-like on D , that is, $\langle X_u \times_L X_v, X_u \times_L X_v \rangle_L < 0$ for any point $p = (u, v) \in D$.

A surface $X: D \rightarrow \mathbb{L}^3$ is *complete* if I_X is complete.

Set

$$(3) \quad \mathbf{n} = -\frac{X_u \times_L X_v}{\sqrt{-\langle X_u \times_L X_v, X_u \times_L X_v \rangle_L}}$$

to be the unit *normal vector field* of X . Define

$$(4) \quad L = -\langle X_{uu}, \mathbf{n} \rangle_L, \quad M = -\langle X_{uv}, \mathbf{n} \rangle_L, \quad N = -\langle X_{vv}, \mathbf{n} \rangle_L.$$

The quadratic form

$$(5) \quad \mathbb{I}_X = L du^2 + 2M dudv + N dv^2$$

is called the *second fundamental form* of X .

Let κ_1 and κ_2 be the principal curvatures of X , that is, the eigenvalues of the Weingarten matrix

$$A := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Then κ_1 and κ_2 are real-valued functions on D . When the two principal curvatures are equal, we say that such a point on the surface is an *umbilic point*. Moreover, the *Gaussian curvature* K and the *mean curvature* H of X are written as

$$K = -\kappa_1 \kappa_2 = -\frac{LN - M^2}{EG - F^2}, \quad H = \frac{\kappa_1 + \kappa_2}{2} = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

This definition of the Gaussian curvature is chosen so as to be consistent with Gauss's Theorema Egregium.

We now give two typical examples of complete space-like surfaces in $\mathbb{L}^3$ with non-zero CMC.

Example 2.1. The surface $X: \mathbb{R}^2 \rightarrow \mathbb{L}^3$ given by

$$(6) \quad X(u, v) = (r \sinh u, rv, r \cosh u), \quad (u, v) \in \mathbb{R}^2, \quad r > 0$$

is called a *hyperbolic cylinder*. Then the unit normal vector field is

$$\mathbf{n}(u, v) = (\sinh u, 0, \cosh u)$$

and the fundamental forms are

$$(7) \quad \mathbb{I}_X = r^2 (du^2 + dv^2), \quad \mathbb{II}_X = r du^2$$

with

$$K = 0, \quad H = \frac{1}{2r}.$$

Example 2.2. The surface $X: \mathbb{R}^2 \rightarrow \mathbb{L}^3$ given by

$$(8) \quad X(u, v) = (r \sinh u \cos v, r \sinh u \sin v, r \cosh u), \quad (u, v) \in \mathbb{R}^2, \quad r > 0$$

is called a *pseudosphere*, with

$$\mathbf{n}(u, v) = (\sinh u \cos v, \sinh u \sin v, \cosh u),$$

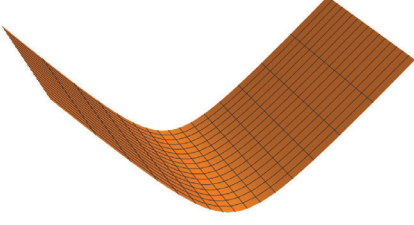


FIGURE 1. Hyperbolic cylinder

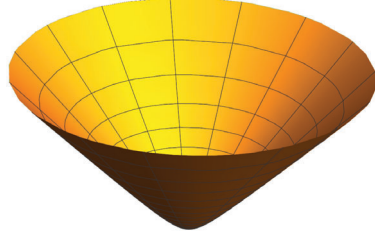


FIGURE 2. Pseudosphere

$$I_X = r^2 du^2 + r^2 \sinh^2 u dv^2, \quad \mathbb{I}_X = r du^2 + r \sinh^2 u dv^2, \quad K = -\frac{1}{r^2}, \quad H = \frac{1}{r}.$$

We next explain the integrability conditions of space-like surfaces in $\mathbb{L}^3$. Here, we assume that $i, j, k, \dots$ are indices varying from 1 to 2, and that $u_1 = u$ and $u_2 = v$. We set

$$X_i = \frac{\partial X}{\partial u_i}, \quad X_{ij} = \frac{\partial^2 X}{\partial u_i \partial u_j}, \quad \mathbf{n}_i = \frac{\partial \mathbf{n}}{\partial u_i},$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

and $(g^{ij}) = (g_{ij})^{-1}$. With Christoffel symbols

$$\Gamma_{ij}^k := \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{li}}{\partial u_j} + \frac{\partial g_{lj}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l} \right),$$

the Gauss and Codazzi equations are

$$(9) \quad \sum_t g_{tm} \left(\frac{\partial \Gamma_{ij}^t}{\partial u_k} - \frac{\partial \Gamma_{ik}^t}{\partial u_j} + \sum_s (\Gamma_{ij}^s \Gamma_{sk}^t - \Gamma_{ik}^s \Gamma_{sj}^t) \right) = -(h_{ij} h_{km} - h_{ik} h_{jm}),$$

$$(10) \quad \frac{\partial h_{ij}}{\partial u_k} - \frac{\partial h_{ik}}{\partial u_j} + \sum_t (\Gamma_{ij}^t h_{tk} - \Gamma_{ik}^t h_{tj}) = 0.$$

respectively. For a space-like surface in $\mathbb{L}^3$, the following fundamental theorem of surface theory holds.

Fact 2.3. *Let g_{ij} be the coefficients of a Riemannian metric on a simply connected domain D , and let h_{ij} ($1 \leq i, j \leq 2$) be smooth functions with $h_{ij} = h_{ji}$ on D satisfying the differential equations (9) and (10). Then there exists a space-like surface $X: D \rightarrow \mathbb{L}^3$ such that*

$$I_X = \sum_{i,j} g_{ij} du_i du_j, \quad \mathbb{I}_X = \sum_{i,j} h_{ij} du_i du_j.$$

Furthermore, such a surface $X(D)$ is unique up to isometries of $\mathbb{L}^3$.

2.2. Moving frames. Let $X: D \rightarrow \mathbb{L}^3$ be a space-like surface. We take a triple of vector fields $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{L}^3$ which is a Lorentz orthonormal basis, that is,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = \delta_{ij}, \quad \langle \mathbf{e}_i, \mathbf{e}_3 \rangle_L = 0, \quad \langle \mathbf{e}_3, \mathbf{e}_3 \rangle_L = -1 \quad (i, j = 1, 2),$$

and assume that $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent, and $\mathbf{e}_3$ is perpendicular, to the surface. We call this triple a *moving frame* of X .

We take the exterior derivative

$$(11) \quad dX = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

where ω_i ($i = 1, 2$) are 1-forms on D , which form the dual basis of $\mathbf{e}_i$ considered as tangent vector fields on D , and $I_X = \omega_1^2 + \omega_2^2$.

The exterior derivative of $\mathbf{e}_A$ ($A = 1, 2, 3$)

$$(12) \quad d\mathbf{e}_A = \sum_B \omega_{AB} \mathbf{e}_B$$

satisfies, since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal,

$$\begin{cases} \omega_{AB} = \omega_{BA}, & (A, B) = (1, 3), (2, 3), (3, 1), (3, 2), \\ \omega_{AB} = -\omega_{BA}, & \text{otherwise.} \end{cases}$$

Restricting ω_{AB} to $X(D)$, we can represent ω_{i3} as

$$(13) \quad \omega_{i3} = \sum_{j=1}^2 b_{ij} \omega_j, \quad (i = 1, 2).$$

Taking the exterior derivative of (11) and using $d(dX) = 0$, we have

$$(14) \quad d\omega_i = \sum_{j=1}^2 \omega_{ij} \wedge \omega_j, \quad \sum_{k=1}^2 \omega_k \wedge \omega_{k3} = 0.$$

The Gaussian curvature K satisfies

$$(15) \quad d\omega_{12} = -K\omega_1 \wedge \omega_2,$$

and the second fundamental form and mean curvature are

$$\mathbb{I}_X = b_{11}\omega_1^2 + 2b_{12}\omega_1\omega_2 + b_{22}\omega_2^2, \quad H = (b_{11} + b_{22})/2.$$

Taking the exterior derivative of (12), we have

$$(16) \quad d\omega_{12} = \omega_{13} \wedge \omega_{32},$$

$$(17) \quad d\omega_{13} = \omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = \omega_{21} \wedge \omega_{13}.$$

Then (16) is equivalent to $K = -(b_{11}b_{22} - b_{12}^2)$ by (13) and (15). This is the Gauss equation.

We assume that each point of $X(D)$ is not umbilic. Then the principal curvatures κ_1 and κ_2 differ from each other at each point of $X(D)$. We may assume $\kappa_1 > \kappa_2$. Let the tangent vector fields $\mathbf{e}_1$ and $\mathbf{e}_2$ be the principal curvature vectors at each point. With respect to $\mathbf{e}_1$ and $\mathbf{e}_2$, the symmetric matrix (b_{ij}) is diagonalized. Thus we obtain $\omega_{13} = \kappa_1 \omega_1$ and $\omega_{23} = \kappa_2 \omega_2$. Then (17) is equivalent to

$$(18) \quad d\kappa_1 \wedge \omega_1 + (\kappa_1 - \kappa_2) \omega_{12} \wedge \omega_2 = 0,$$

$$(19) \quad d\kappa_2 \wedge \omega_2 + (\kappa_1 - \kappa_2) \omega_{12} \wedge \omega_1 = 0.$$

This is the Codazzi equation for an umbilic-free space-like surface, in the theory of moving frames.

2.3. Isothermal coordinate systems. Let $X: D \rightarrow \mathbb{L}^3$ be a space-like surface. With suitable choice of parametrization, the first fundamental form becomes

$$(20) \quad I_X = \lambda^2(u, v)(du^2 + dv^2)$$

for some positive smooth function $\lambda(u, v)$, and we call the coordinates (u, v) an *isothermal coordinate system*. Since $X_{uvv} = X_{vvu}$, $X_{vvu} = X_{vuv}$, the Gauss and Codazzi equations are

$$(21) \quad -\frac{1}{\lambda^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \lambda = -\frac{LN - M^2}{\lambda^4},$$

$$(22) \quad \frac{\partial L}{\partial v} - \frac{\partial M}{\partial u} - \frac{\partial \log \lambda}{\partial v} (L + N) = 0, \quad \frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} + \frac{\partial \log \lambda}{\partial u} (L + N) = 0.$$

As $H = (L + N)/2\lambda^2$, this implies

$$(23) \quad \frac{1}{2} \frac{\partial(L - N)}{\partial v} - \frac{\partial M}{\partial u} = -\lambda^2 \frac{\partial H}{\partial v}, \quad \frac{1}{2} \frac{\partial(L - N)}{\partial u} + \frac{\partial M}{\partial v} = \lambda^2 \frac{\partial H}{\partial u}.$$

We set $z = u + iv$ and consider the domain D of X as a subset of $\mathbb{C}$. We define a complex-valued function on D by

$$(24) \quad \phi(z) := \frac{L - N}{2} - iM.$$

The complex quadratic differential $\phi(z)dz^2$ is called the *Hopf differential* of X . By (23), $\phi(z)$ is holomorphic if and only if H is constant. Then we can obtain

$$(25) \quad |\phi(z)| = \frac{\lambda^2}{2} |\kappa_1 - \kappa_2|.$$

Thus a space-like surface in $\mathbb{L}^3$ with CMC is either entirely umbilic or has isolated umbilic points, because $\phi(z)$ is holomorphic.

3. Main results

We first recall the definition of isothermic coordinate systems. Let $X: D \rightarrow \mathbb{L}^3$ be a space-like surface. The coordinates (u, v) are called a *curvature line coordinate system* if the second fundamental form of X satisfies

$$(26) \quad \mathbb{I}_X = L du^2 + N dv^2.$$

When the coordinates (u, v) satisfy both (20) and (26), (u, v) are called an *isothermic coordinate system*.

Example 3.1. For the hyperbolic cylinder defined by (6), by (7) the coordinates (u, v) are an isothermic coordinate system.

We next show the existence of an isothermic coordinate system for umbilic-free space-like surfaces in $\mathbb{L}^3$ with non-zero CMC.

Theorem 3.2. *For a simply-connected domain D and a space-like surface $X: D \rightarrow \mathbb{L}^3$ with CMC- H ($H > 0$), assume that every $p \in D$ is non-umbilic. Then there exists an isothermic coordinate system (u, v) on D satisfying*

$$(27) \quad \mathbb{I}_X = e^\omega (du^2 + dv^2), \quad \mathbb{II}_X = 2He^{\frac{\omega}{2}} \left(\cosh \frac{\omega}{2} du^2 + \sinh \frac{\omega}{2} dv^2 \right),$$

where $\omega = \omega(u, v)$ satisfies

$$(28) \quad \omega_{uu} + \omega_{vv} - 4H^2 \sinh \omega = 0.$$

Conversely, for given $H > 0$ and solution $\omega(u, v)$ of (28), there exists a space-like surface with CMC- H whose fundamental forms are given by (27), which is unique up to isometries of $\mathbb{L}^3$.

Proof. Assume κ_1 and κ_2 of X satisfy $\kappa_1 > \kappa_2$ on D and $\mathbf{e}_i$ ($i = 1, 2$) is a unit vector field determining the principal direction corresponding to κ_i . Then $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal. Let ω_i be the dual 1-form of $\mathbf{e}_i$. By (14), (18) and (19), we obtain

$$(29) \quad \begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 = -\frac{1}{\kappa_1 - \kappa_2} (d\kappa_1 \wedge \omega_1), \\ d\omega_2 &= \omega_{21} \wedge \omega_1 = \frac{1}{\kappa_1 - \kappa_2} (d\kappa_2 \wedge \omega_2). \end{aligned}$$

We consider the 1-form θ_i defined by

$$\theta_i = \sqrt{\frac{\kappa_1 - \kappa_2}{2}} \omega_i, \quad (i = 1, 2).$$

By (29), we have

$$(30) \quad d\theta_1 = d\sqrt{\kappa_1 - H} \wedge \omega_1 - \frac{1}{2\sqrt{\kappa_1 - H}} (d\kappa_1 \wedge \omega_1).$$

On the other hand, we obtain

$$d\sqrt{\kappa_1 - H} = \frac{1}{2\sqrt{\kappa_1 - H}} d\kappa_1,$$

because $H = (\kappa_1 + \kappa_2)/2$ is constant. Thus θ_1 is closed. Similarly, θ_2 is closed. By the Poincaré lemma, there exists a function u_i on D such that $\theta_i = du_i$ ($i = 1, 2$). If we choose τ so that

$$e^{-\tau} = \frac{\kappa_1 - \kappa_2}{2},$$

we have $\omega_i = e^{(\tau/2)} du_i$ ($i = 1, 2$) and

$$\mathrm{I}_X = e^\tau (du_1^2 + du_2^2), \quad \mathrm{II}_X = (He^\tau + 1)du_1^2 + (He^\tau - 1)du_2^2.$$

Moreover, if we choose ω, u, v so that $e^\tau = e^\omega/H$, $u_1 = \sqrt{H}u$, $u_2 = \sqrt{H}v$, we obtain (27) and (28).

The converse follows by Fact 2.3, because the Codazzi equation holds automatically for (27), and (28) is the Gauss equation (21). $\square$

We will apply Theorem 3.2 to complete space-like CMC surfaces in $\mathbb{L}^3$. Let Σ be a connected and oriented 2-manifold. It is well-known that every complete space-like surface $X: \Sigma \rightarrow \mathbb{L}^3$ is entire (e.g. see [1, Proposition 9.3]). In particular, Σ is conformally equivalent to either the entire plane $\mathbb{R}^2$ or the unit disk $\mathbb{D}$, because Σ is simply-connected and non-compact. On the other hand, Yamada [22, Proposition 3.1] proved the following result.

Fact 3.3. *The only solution of (28) on the entire plane $\mathbb{R}^2$ is the trivial solution $\omega \equiv 0$.*

Remark 3.4. Consider the ordinary differential equation

$$(31) \quad \omega''(r) + \frac{1}{r}\omega'(r) = 4H^2 \sinh \omega(r) \quad \text{for } r \geq 0,$$

where $' := d/dr$. Then the solution of (31) is a radially symmetric solution of (28) with $r = \sqrt{u^2 + v^2}$. The equation (31) is a Painlevé equation of the third kind.

By combining Theorem 3.2 and Fact 3.3, we can show the following uniqueness theorem.

Corollary 3.5. *Any complete non-umbilic space-like surface in $\mathbb{L}^3$ with non-zero CMC- H defined on the entire plane $\mathbb{R}^2$ must be a hyperbolic cylinder.*

Proof. By reversing the unit normal vector if necessary, we may assume $H > 0$. Moreover, we may assume $H = 1/2$ without loss of generality, because a homothety of the surface scales the mean curvature accordingly. Since the surface is non-umbilic and simply-connected, we can take an isothermic coordinate system

(u, v) as in Theorem 3.2 on the entire plane $\mathbb{R}^2$. By Fact 3.3, we have $\omega \equiv 0$. Thus

$$I_X = du^2 + dv^2, \quad II_X = du^2$$

and the corresponding space-like surface in $\mathbb{L}^3$ is a hyperbolic cylinder by Example 2.1 and Fact 2.3. $\square$

Milnor [16, Theorem 8] and Yamada [22, Theorem] proved the following uniqueness theorem:

Theorem 3.6. *Any complete space-like surface in $\mathbb{L}^3$ with non-zero CMC and principal curvatures κ_1 and κ_2 satisfying*

$$(32) \quad (\kappa_1 - \kappa_2)^2 \geq \varepsilon^2$$

for some positive number ε must be a hyperbolic cylinder.

The assumption (32) is necessary. In fact, there exist non-umbilic space-like surfaces defined on $\mathbb{D}$ in $\mathbb{L}^3$ with non-zero CMC on which

$$\lim_{(u,v) \rightarrow \partial\mathbb{D}} (\kappa_1 - \kappa_2)^2 = 0,$$

where $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$ ([22, Remark 1]).

Proof. By Corollary 3.5, we only need to consider $\Sigma = \mathbb{D}$. The following argument is based on [22, Section 4]. Suppose that there exists a complete space-like surface defined on $\mathbb{D}$ in $\mathbb{L}^3$ with non-zero CMC and principal curvatures κ_1, κ_2 satisfying (32). Then we can take an isothermic coordinate system (u, v) in Theorem 3.2 on $\mathbb{D}$. Since I_X is complete, we have

$$\lim_{(u,v) \rightarrow \partial\mathbb{D}} e^\omega = +\infty.$$

From (25) and (32), ϕ is a non-vanishing holomorphic function on $\mathbb{D}$ which satisfies

$$\lim_{(u,v) \rightarrow \partial\mathbb{D}} |\phi(z)| = +\infty.$$

Set $\psi = 1/\phi$. Then ψ is holomorphic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial\mathbb{D}$ with $\psi|_{\partial\mathbb{D}} = 0$. By the Cauchy expression, we obtain

$$\psi(0) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\psi(z)}{z} dz = 0.$$

This contradicts that ϕ is a non-vanishing holomorphic function on $\mathbb{D}$. $\square$

Finally, we give a corollary of Theorem 3.6. By using the argument for the existence of an isothermic coordinate system, Klotz and Osserman [15] proved that a complete CMC surface in $\mathbb{E}^3$ whose Gaussian curvature does not change sign is either a sphere, a minimal surface, or a right circular cylinder. In $\mathbb{L}^3$, as a corollary of Theorem 3.6, we can show the following uniqueness result.

Corollary 3.7. *If $X: \Sigma \rightarrow \mathbb{L}^3$ is a complete space-like CMC- H surface with non-negative Gaussian curvature, then it is a plane or hyperbolic cylinder.*

Before proving this, we call the Calabi-Bernstein theorem (Fact 3.8), first shown by Calabi [4]. For other proofs of this fact, see [2, 4, 12, 14, 18, 21] for example. We also note that a space-like surface in $\mathbb{L}^3$ whose mean curvature vanishes everywhere is called a maximal surface in $\mathbb{L}^3$.

Fact 3.8. *Any complete maximal surface in $\mathbb{L}^3$ must be a plane.*

Proof. The proof of Corollary 3.7 is straightforward. If $H = 0$, then $X(\Sigma)$ must be a plane by Fact 3.8. If $H \neq 0$, since $K = -\kappa_1\kappa_2 > 0$, we obtain $(\kappa_1 - \kappa_2)^2 = 4(H^2 + K) \geq 4H^2$. By virtue of Theorem 3.6, $X(\Sigma)$ must be a hyperbolic cylinder. $\square$

Treibergs [20] showed there exist many complete space-like surfaces in $\mathbb{L}^3$ with non-zero CMC other than hyperbolic cylinders. By Corollary 3.7, those surfaces must have at least one point $p \in \Sigma$ where $K(p) < 0$.

On the other hand, it is known that the Gaussian curvature of a complete space-like surface in $\mathbb{L}^3$ with non-zero CMC must be non-positive (see [20, Corollary on page 49], [5, Corollary on page 419] for example). Thus a complete space-like surfaces in $\mathbb{L}^3$ with non-zero CMC does not have any point $p \in \Sigma$ where $K(p) > 0$, and in fact, the conclusion of Corollary 3.7 includes only surfaces $H \equiv 0$.

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