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Ueda, Yoshihiro

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Nonlinear stability of equilibrium states for the viscous conservation laws with time delay

Yoshihiro Ueda[1]

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Abstract
In this article, we consider the nonlinear stability of the non-zero equilibrium state for the viscous scalar conservation laws with a delay effect in the one-dimensional whole space. The necessary and sufficient conditions for the linear stability are obtained in Ueda (Ueda, Y in Jpn. J. Ind. Appl. Math 42 2057–2072, 2025), and the situation strongly depends on equilibrium states. Based on the result of the linear stability, the purpose of this article is to derive the condition for the nonlinear stability. The method for deriving the linear stability is the analysis of the corresponding eigenvalue problem; however, it does not apply to the derivation of nonlinear stability. Thus, we apply the energy method to construct the a priori estimate and prove the existence of a global-in-time solution.

Keywords Viscous scalar conservation laws · Time delay · Asymptotic stability

Mathematics Subject Classification 35B35 · 35E15 · 35Q35

1 Introduction

We consider the scalar viscous conservation laws with a time delay in the one-dimensional whole space

$$\partial_t \rho - \nu \partial_x^2 \rho + \partial_x (\rho V(\rho_\tau)) = 0 \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}$. Here, $\rho = \rho(t, x)$ is an unknown function, ν is a positive constant, and $V = V(\rho)$ denotes a given smooth function that depends on ρ only. In this article, we use a notation that $\rho_\tau(t, x) = \rho(t - \tau, x)$ with a positive constant τ

✉ Yoshihiro Ueda
ueda@maritime.kobe-u.ac.jp

1 Faculty of Maritime Sciences, Kobe University, Kobe, Japan

called the delay parameter. For the equation (1.1), we assign an initial history

$$\rho(\theta, x) = \rho_0(\theta, x) \tag{1.2}$$

for $-\tau \leq \theta \leq 0$ and $x \in \mathbb{R}$. Here, $\rho_0 = \rho_0(\theta, x)$ is a given function.

Letting τ tends to zero formally, (1.1) is reduced to the scalar viscous conservation laws

$$\partial_t \rho - \nu \partial_x^2 \rho + \partial_x (\rho V(\rho)) = 0, \tag{1.3}$$

which is one of the important and fundamental physical models in fluid mechanics, and there are many known results for (1.3). For example, the explicit solution formula for the viscous Burgers equation is analyzed in [1, 6, 17]. Furthermore, the asymptotic behavior of its solution and the stability of traveling waves and rarefaction waves are also considered in [4, 11] etc.

One example of the physical models described by the viscous scalar conservation laws (1.3) is a traffic flow model. A simple model of traffic flow is described as (1.3) with

$$V(\rho) = V_{max} \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{1.4}$$

where the unknown function ρ denotes the traffic density, the positive constant ν is the diffusion coefficient, and the positive constants ρ_{max} and V_{max} denote the maximum density and the maximum speed as $\rho \to 0$, respectively (see Garavello et al. [2] for example). It is well-known that the viscous scalar conservation laws (1.3) is a strictly parabolic partial differential equation, so its solution has an infinite speed of propagation. In terms of traffic flow, this means that drivers and their vehicles are assumed to react instantly to changes in the density and its gradient. To improve this troubling feature, we modify the term without a time delay to the one with a time delay and introduce (1.1) with (1.4).

The scalar conservation laws with a time delay is a relatively new research topic, and there are few known results. Especially, the initial-boundary value problem is well studied. Here, we focus on the viscous Burgers equation whose nonlinear term has a time delay (see [5, 7, 15] for other cases). Liu [10] considered the asymptotic behavior of the solutions of the viscous Burgers equation under zero-Dirichlet boundary conditions. By using the energy method, the author showed the existence of a unique global solution in time. Furthermore, if the time delay is sufficiently small, Tang and Wang [14] proved the exponential stability of the solution to the viscous Burgers equation under zero-Dirichlet boundary conditions. Smaoui and Mekkaoui [12] also considered the viscous Burgers equation with a time delay under the periodic boundary condition. They proved the existence of the unique global solution and derived the exponential decay estimate. Furthermore, Tang [13] considered the viscous Burgers equation with a nonlocal delay effect under zero-Dirichlet boundary condition. In a similar way to Tang and Wang [14], the author proved that the solution is exponentially stable, provided that the delay parameter is sufficiently small. Compared with the result for the problem of the bounded domain, there are few results for the problem of the unbounded domain. Recently, Kubo and Ueda [8, 9] treated the Cauchy problem of (1.1) and obtained the nonlinear stability of the zero equilibrium state. Furthermore, they derived the condition for the time delay and the initial history to achieve the nonlinear stability. We remark that their result is not assigned the smallness for both the initial history and the time delay.

In this article, we study the one-dimensional Cauchy problem of the scalar conservation laws with a time delay (1.1), (1.2). Especially, we focus on the stability of non-zero equilibrium state for (1.1), (1.2). Indeed, the usual scalar conservation laws (1.3) make no difference to the structure of the stability around the equilibrium state. On the other hand, for the scalar conservation laws with a time delay, the structure of the stability around the equilibrium state varies greatly depending on the effect of the time delay. In fact, Ueda [16] succeeded in deriving the sharp criterion to get the linear stability of the non-zero equilibrium state.

Let $\rho_* \in \mathbb{R}$ be an equilibrium state, and we introduce a perturbation $u := \rho - \rho_*$. Then the equation (1.1) is rewritten as

$$\partial_t u - \nu \partial_x^2 u + \partial_x \left((\rho_* + u)V(\rho_* + u_\tau)\right) = 0. \tag{1.5}$$

To derive the linear stability for the nonlinear equation, Ueda [16] analyzed the corresponding eigenvalue problem of the linearized equation. The linearized equation of (1.5) is given by

$$\partial_t u - \nu \partial_x^2 u + V(\rho_*)\partial_x u + \rho_* V'(\rho_*)\partial_x u_\tau = 0. \tag{1.6}$$

To derive the corresponding eigenvalue problem, we apply the Fourier transform to (1.6) and get

$$\partial_t \hat{u} + \nu \xi^2 \hat{u} + V(\rho_*) i\xi \hat{u} + \rho_* V'(\rho_*) i\xi \hat{u}_\tau = 0, \tag{1.7}$$

where $\xi \in \mathbb{R}$ denotes Fourier variable. The equation (1.7) is regarded as an ordinary differential equation with time delay, and it is known that the decay property of solutions to (1.7) is characterized by the corresponding eigenvalue problem and the following characteristic equation (for details, see [3]).

$$\lambda + \nu \xi^2 + V(\rho_*) i\xi + \rho_* V'(\rho_*) i\xi e^{-\tau\lambda} = 0. \tag{1.8}$$

For the eigenvalue of this eigenvalue problem, Ueda [16] derived the following result.

Theorem 1.1 *(Ueda [16]) All the roots of* (1.8) *satisfy*

$$\limsup_{|\xi|\to 0} \mathrm{Re}\lambda(\xi) \le 0, \qquad \lim_{|\xi|\to\infty} \mathrm{Re}\lambda(\xi) = -\infty.$$

Furthermore, they have negative real parts for any $\xi \in \mathbb{R}\backslash\{0\}$ if and only if ν, τ, ρ_* and V satisfy the one of the following three conditions:

(i)
$$\frac{\tau}{\nu}\rho_* V'(\rho_*)(V(\rho_*) + \rho_* V'(\rho_*)) \le 1, \qquad \frac{\tau}{\nu}\rho_* V(\rho_*)V'(\rho_*) \ge -\frac{\pi}{2}. \tag{1.9}$$

(ii)

$$\frac{\tau}{\nu}\rho_* V'(\rho_*)(V(\rho_*)+\rho_* V'(\rho_*)) \le 1, \qquad -\frac{3\pi}{2} < \frac{\tau}{\nu}\rho_* V(\rho_*)V'(\rho_*) < -\frac{\pi}{2}, \quad (1.10)$$

and

$$\frac{\tau}{\nu}V(\rho_*)^2\left(\sqrt{\frac{\tau^2}{\nu^2}\rho_*^2 V(\rho_*)^2 V'(\rho_*)^2 - \frac{\pi^2}{4}} + \frac{\tau}{\nu}\rho_*^2 V'(\rho_*)^2 - 1\right) \le \frac{\pi^2}{2}.$$

(iii) *Inequalities* (1.10)*,*

$$\frac{\tau}{\nu}V(\rho_*)^2\left(\sqrt{\frac{\tau^2}{\nu^2}\rho_*^2 V(\rho_*)^2 V'(\rho_*)^2 - \frac{\pi^2}{4}} + \frac{\tau}{\nu}\rho_*^2 V'(\rho_*)^2 - 1\right) > \frac{\pi^2}{2},$$

and

$$\frac{\tau}{\nu}\rho_* V'(\rho_*)\left(\rho_* V'(\rho_*)\theta_* - V(\rho_*)\right)\sqrt{1-\theta_*^2} < \frac{3\pi}{2} - \mathrm{Arccos}(\sqrt{1-\theta_*^2}).$$

Here, θ_* *is defined by*

$$\theta_* := \frac{1}{4|\rho_* V'(\rho_*)|}\left(\sqrt{V(\rho_*)^2 + 8\left(\rho_*^2 V'(\rho_*)^2 + \frac{\nu}{\tau}\right)} - |V(\rho_*)|\right).$$

Theorem 1.1 tells us that the delay effect induces the instability phenomena. For example, if the delay parameter τ satisfies $\tau > -\nu\pi/(2\rho_* V(\rho_*)V'(\rho_*))$ with $\rho_* V(\rho_*)V'(\rho_*) < 0$, the equilibrium state ρ_* is unstable.

Inspired by Theorem 1.1, the purpose of this article is to derive the condition to get the nonlinear stability of the equilibrium state ρ_*. As we can see from Theorem 1.1, it is known that the delay effect may induce the instability phenomena. This fact gives us the question of the nonlinear stability for (1.5) with a large initial history. In this situation, Kubo and Ueda [8, 9] introduced the criterion for the size of the delay parameter and the initial history to derive the nonlinear stability of the zero equilibrium state. In this article, we will extend the result in Kubo and Ueda [8, 9] for the non-zero equilibrium state.

The initial history for (1.5) is also assigned that

$$u(\theta, x) = u_0(\theta, x), \quad (1.11)$$

where $u_0 := \rho_0 - \rho_*$. Then we derive the asymptotic stability result for (1.5), (1.11). To state the main theorems, we introduce notations. For the initial history (1.11), we introduce the useful norm that

$$I_0 := \left(\sup_{-\tau \le \theta \le 0} \|u_0(\theta)\|_{L^2}^2 + \nu \int_{-\tau}^{0} \|\partial_x u_0(\theta)\|_{L^2}^2 d\theta \right)^{1/2}.$$

We introduce the functions that

$$W(M) := 2\sqrt{2}\left(2V_1(M) + |\rho_*|V_2(M)\right), \qquad V_n(M) := \sup_{|v|\le M} |V^{(n)}(\rho_* + v)|$$

for $M > 0$. We remark that $W(M)$ is a monotone increasing function and appears in the main theorems. On the other hand, we introduce the constant c_τ that

$$c_\tau = \begin{cases} \dfrac{|\rho_* V'(\rho_*)|}{6(|V(\rho_*)| + |\rho_* V'(\rho_*)|)} \left(1 - \dfrac{\tau}{\nu}(|V(\rho_*)| + |\rho_* V'(\rho_*)|)|\rho_* V'(\rho_*)|\right), & \rho_* V'(\rho_*) \neq 0, \\ \dfrac{1}{3}\left(\dfrac{\tau}{\nu}|V(\rho_*)|^2 + 1\right)^{-1}, & \rho_* V'(\rho_*) = 0. \end{cases} \tag{1.12}$$

Especially, we note that

$$c_\tau \uparrow c_0 := \begin{cases} \dfrac{|\rho_* V'(\rho_*)|}{6(|V(\rho_*)| + |\rho_* V'(\rho_*)|)}, & \rho_* V'(\rho_*) \neq 0, \\ \dfrac{1}{3}, & \rho_* V'(\rho_*) = 0 \end{cases}$$

as $\tau/\nu \downarrow 0$.

The following main theorems describe the existence of a global-in-time solution when the product of the size of the delay parameter and one of the initial history is suitably small.

Theorem 1.2 *Suppose that $u_0 \in C([-\tau, 0]; H^1)$ and V is C^2 class function under consideration. Assume that*

$$\frac{\tau}{\nu}|\rho_* V'(\rho_*)|\left(|V(\rho_*)| + |\rho_* V'(\rho_*)|\right) < 1 \tag{1.13}$$

and

$$\sqrt{\frac{\tau}{\nu}} W\left(\sqrt{C_1} I_1\right)\sqrt{C_1} I_1 < c_\tau \tag{1.14}$$

are satisfied. Here, I_1 and C_1 are defined by

$$I_1 := \left(I_0^2 + \|\partial_x u_0(0)\|_{L^2}^2\right)^{1/2}, \quad C_1 := 41 + \frac{84}{\nu}\left\{\frac{1}{3\tau} + \frac{4}{\nu c_\tau}(V(\rho_*)^2 + 2\rho_*^2 V'(\rho_*)^2)\right\}. \tag{1.15}$$

Then, (1.5)*,* (1.11) *has a unique global-in-time solution* $u \in C([-\tau, \infty); H^1)$ *satisfying* $\partial_t u \in L^2(0, \infty; L^2)$*,* $\partial_x u \in L^2(0, \infty; H^1)$ *and the energy estimate*

$$\|u(t)\|_{H^1}^2 + \int_0^\infty \left(\frac{1}{\nu} \|\partial_t u(s)\|_{L^2}^2 + \frac{\nu c_\tau}{2} \|\partial_x u(s)\|_{H^1}^2 \right) ds \leq C_1 I_1^2 \tag{1.16}$$

for $t \geq 0$*. Furthermore, the solution* u *satisfies the asymptotic behavior* $\|u(t)\|_{L^\infty} \to 0$ *as* $t \to \infty$*.*

Remark 1.3 Since (1.15), we derive $C_1 \uparrow \infty$ as $\tau \downarrow 0$.

Remark 1.3 tells that it is not certain that I_1 can be taken large enough to satisfy the condition (1.14) for small τ. However, if V' is uniformly bounded, I_1 can be taken large enough to satisfy the condition (1.14) for small τ. Since V' is uniformly bounded, there exists a certain positive constant V_* such that $|V'(\rho)| \leq V_*$ for $\rho \in \mathbb{R}$. Then this situation is expressed by the following theorem.

Theorem 1.4 *Suppose the same assumption as in Theorem 1.2. Furthermore, assume that* V' *is uniformly bounded. Then* I_1 *and* C_1 *are replaced by*

$$I_1 := \left\{ (1 + I_0^4) I_0^2 + \sup_{-\tau \leq \theta \leq 0} \|\partial_x u_0(\theta)\|_{L^2}^2 \right\}^{1/2}, \qquad C_1 := 41 + \frac{\tilde{C}_1 (1 + \nu^2)}{\nu^4 c_\tau}, \tag{1.17}$$

where $\tilde{C}_1 := 168(2V(\rho_*)^2 + 2\rho_*^2 V_*^2 + 168^2 V_*^4)$*, and the same result in Theorem 1.2 is obtained.*

Remark 1.5 Since (1.17), we have

$$C_1 \downarrow 41 + \frac{\tilde{C}_1 (1 + \nu^2)}{\nu^4 c_0}$$

as $\tau/\nu \downarrow 0$, and $C_1 \downarrow 41$ as $\nu \uparrow \infty$. This fact means that I_1 can be taken large enough to satisfy the condition (1.14) for small τ in Theorem 1.4.

Remark 1.6 (i) The condition (1.13) in Theorem 1.2 is a sufficient condition for the condition (1.9) in Theorem 1.1. Namely, the linear stability is supposed under the conditions in Theorem 1.2. (ii) In the case $\rho_* = 0$, we obtain Theorems 1.2 and 1.4 for $V \in C^1$ (see [8, 9]). (iii) In the case $\rho_* = 0$, it is easy to check (1.13). Therefore, Theorems 1.2 and 1.4 are generalizations of the results in Kubo and Ueda [8, 9]. (vi) One of the most popular velocity function V in the traffic flow models is (1.4). In this case, we have $V_1 = V_{max}/\rho_{max}$ and $V_2 = 0$, and Theorem 1.4 is applicable to this model.

This article is organized as follows. In preparation for the proofs of the main theorems, we will show the key lemmas in Section 2. We construct the a priori estimate for our problem in Section 3. In section 4, we give the proofs of Theorems 1.2 and 1.4, which are based on the a priori estimate and the existence theorem of the local-in-time solution.

2 Preliminaries

In this section, we introduce some technical lemmas and key lemmas that play an important role in our proof. The following lemma is derived by Kubo and Ueda [8].

Lemma 2.1 *([8]) Let $\tau \geq 0$, $0 \leq \kappa \leq \tau$ and $f(t)$ be a continuous function such that $f(t) \geq 0$ for $t \in [\kappa - \tau, \infty)$. Then the following estimate is satisfied.*

$$\int_\kappa^t \int_{s-\tau}^s f(\sigma) d\sigma ds + \int_{\kappa-\tau}^\kappa (\kappa - s) f(s) ds \leq \tau \int_{\kappa-\tau}^t f(s) ds.$$

Next, we introduce key lemmas to derive the a priori estimate. For $r > 0$, we set

$$\begin{aligned}
S^r &:= \{(\phi, \psi) \in \mathbb{R}^2; (\phi + \psi)\psi < r, \phi \geq 0, \psi \geq 0\}, \\
S^r_\beta &:= \left\{(\phi, \psi) \in \mathbb{R}^2; \frac{\beta}{2(1-\beta)} \left(\phi + \frac{1}{2\beta}\psi\right)^2 + \frac{1}{2}\psi^2 < r, \phi \geq 0, \psi \geq 0\right\}, \\
S^r_{\beta,\gamma} &:= \left\{(\phi, \psi) \in \mathbb{R}^2; \frac{\beta}{2(1-\gamma)} \left(\phi + \frac{1}{2\beta}\psi\right)^2 + \frac{\beta}{2\gamma}\psi^2 < r, \phi \geq 0, \psi \geq 0\right\},
\end{aligned}$$

and $\Lambda := \{(\beta, \gamma) \in \mathbb{R}^2; \beta > 0, 0 < \gamma < 1\}$, $\tilde{\Lambda} := \{(\beta, \gamma) \in \mathbb{R}^2; 0 < \beta < 1/2, 0 < \gamma < 1/2\}$. Then these sets satisfy the following lemma.

Lemma 2.2 *For any $r > 0$, the following equalities are satisfied.*

$$S^r = \bigcup_{0<\beta<1} S^r_\beta = \bigcup_{0<\beta<1/2} S^r_\beta = \bigcup_{(\beta,\gamma)\in\Lambda} S^r_{\beta,\gamma} = \bigcup_{(\beta,\gamma)\in\tilde{\Lambda}} S^r_{\beta,\gamma}. \tag{2.1}$$

Proof Firstly, we derive that $S^r_{\beta,\gamma} \subset S^r$ for $(\beta, \gamma) \in \Lambda$. We show this fact by the contradiction argument. Suppose that there exists $(\beta_1, \gamma_1) \in \Lambda$ such that $S^r_{\beta_1,\gamma_1} \cap (S^r)^c \neq \emptyset$, where $(S^r)^c$ is the complement of S^r. This means that there exists $(\phi_0, \psi_0) \in S^r_{\beta_1,\gamma_1} \cap (S^r)^c$. Namely, (ϕ_0, ψ_0) should satisfy $\phi_0 \geq 0$ and

$$\frac{\beta_1}{2(1-\gamma_1)} \left(\phi_0 + \frac{1}{2\beta_1}\psi_0\right)^2 + \frac{\beta_1}{2\gamma_1}\psi_0^2 < r, \qquad (\phi_0 + \psi_0)\psi_0 \geq r. \tag{2.2}$$

If $\psi_0 \geq \sqrt{r}$, we have

$$\begin{aligned}
\frac{\beta_1}{2(1-\gamma_1)} \left(\phi_0 + \frac{1}{2\beta_1}\psi_0\right)^2 + \frac{\beta_1}{2\gamma_1}\psi_0^2 &\geq \frac{1}{8\beta_1(1-\gamma_1)}\psi_0^2 + \frac{\beta_1}{2\gamma_1}\psi_0^2 \\
&\geq \frac{r}{2\sqrt{\gamma_1(1-\gamma_1)}} \geq r,
\end{aligned}$$

and this is a contradiction. On the other hand, if $0 < \psi_0 < \sqrt{r}$, the second inequality in (2.2) is $\phi_0 \geq r/\psi_0 - \psi_0 > 0$, and this gives

$$\frac{\beta_1}{2(1-\gamma_1)}\left(\phi_0 + \frac{1}{2\beta_1}\psi_0\right)^2 + \frac{\beta_1}{2\gamma_1}\psi_0^2 - r \geq \frac{1}{2(1-\gamma_1)\psi_0^2}H_{\beta_1,\gamma_1}(\psi_0^2),$$

where

$$H_{\beta,\gamma}(\chi) := \left(\frac{1}{4\beta} + \frac{\beta-\gamma}{\gamma}\right)\chi^2 - r(1+2(\beta-\gamma))\chi + r^2\beta$$

for $(\beta, \gamma) \in \Lambda$. Then we estimate

$$\frac{1}{4\beta} + \frac{\beta-\gamma}{\gamma} \geq \frac{1}{\sqrt{\gamma}} - 1 > 0,$$

and the discriminant $\mathfrak{D}_{\beta,\gamma}$ of the function $H_{\beta,\gamma}$ is computed as

$$\mathfrak{D}_{\beta,\gamma} = r^2(1+2(\beta-\gamma))^2 - 4r^2\beta\left(\frac{1}{4\beta} + \frac{\beta-\gamma}{\gamma}\right) = -\frac{4r^2(1-\gamma)(\beta-\gamma)^2}{\gamma} \leq 0. \tag{2.3}$$

Namely, this yields $H_{\beta,\gamma}(\chi) \geq 0$ for $\chi \in \mathbb{R}$, and we arrive at

$$\frac{\beta_1}{2(1-\gamma_1)}\left(\phi_0 + \frac{1}{2\beta_1}\psi_0\right)^2 + \frac{\beta_1}{2\gamma_1}\psi_0^2 \geq r,$$

which is also a contradiction.

Secondly, we show that, for any $(\phi, \psi) \in S^r$, there exists $(\beta_2, \gamma_2) \in \tilde{\Lambda}$ such that $(\phi, \psi) \in S^r_{\beta_2,\gamma_2}$. It is trivial that $(0, 0) \in S^r_{\beta,\gamma}$. If $\phi > 0$ and $\psi = 0$, we can take $\beta_2 < 2r/(2r + \phi^2)$ and $\gamma_2 = \beta_2$, and this gives $(\phi, 0) \in S^r_{\beta_2,\gamma_2}$. If $\psi > 0$, $(\phi, \psi) \in S^r$ means $0 \leq \phi < r/\psi - \psi$ and $0 < \psi < \sqrt{r}$. Thus we estimate

$$\frac{\beta}{2(1-\gamma)}\left(\phi + \frac{1}{2\beta}\psi\right)^2 + \frac{\beta}{2\gamma}\psi^2 - r < \frac{1}{2(1-\gamma)\psi^2}H_{\beta,\gamma}(\psi^2).$$

Since (2.3), we choose β_2 and γ_2 such that $\beta_2 = \gamma_2$, and this yields

$$H_{\beta_2,\gamma_2}(\chi) = \frac{(\chi - 2r\beta_2)^2}{4\beta_2}.$$

Namely, taking $\beta_2 = \gamma_2 = \psi^2/(2r)$, we arrive at $H_{\beta_2,\gamma_2}(\psi^2) = 0$, and

$$\frac{\beta_2}{2(1-\gamma_2)}\left(\phi + \frac{1}{2\beta_2}\psi\right)^2 + \frac{\beta_2}{2\gamma_2}\psi^2 < r.$$

Consequently, these arguments give

$$\bigcup_{(\beta,\gamma)\in\Lambda} S^r_{\beta,\gamma} \subset S^r \subset \bigcup_{(\beta,\gamma)\in\tilde{\Lambda}} S^r_{\beta,\gamma}.$$

On the other hand, it is trivial that

$$\bigcup_{(\beta,\gamma)\in\tilde{\Lambda}} S^r_{\beta,\gamma} \subset \bigcup_{(\beta,\gamma)\in\Lambda} S^r_{\beta,\gamma},$$

and the last two equalities in (2.1) are proved. By using the same argument, we also obtain the first two equalities in (2.1), and this completes the proof. □

To obtain the a priori estimate, it is important to ensure the positive definiteness of the following matrix.

$$A := \begin{pmatrix} 1-\gamma & 0 & -\beta\phi-\psi/2 \\ 0 & \gamma & -\beta\psi \\ -\beta\phi-\psi/2 & -\beta\psi & 2\beta \end{pmatrix} \tag{2.4}$$

for $(\phi, \psi) \in \mathbb{R}^2$ and $(\beta, \gamma) \in \Lambda$. Then Lemma 2.2 gives the following lemma.

Lemma 2.3 *There exists $(\beta_0, \gamma_0) \in \tilde{\Lambda}$ such that the matrix A is positive definite if and only if $(\phi, \psi) \in S^1$ holds.*

Proof To show the positive definiteness of A, we employ Sylvester's criterion. Namely, we will take β and γ such that $1-\gamma > 0$, $\gamma > 0$, and $|A| > 0$ to get the positive definiteness of A. Now, we have

$$\begin{aligned} |A| &= 2\beta\gamma(1-\gamma) - (1-\gamma)\beta^2\psi^2 - \gamma\left(\beta\phi + \frac{\psi}{2}\right)^2 \\ &= 2\beta\gamma(1-\gamma)\left\{1 - \frac{\beta}{2\gamma}\psi^2 - \frac{\beta}{2(1-\gamma)}\left(\phi + \frac{1}{2\beta}\psi\right)^2\right\}. \end{aligned}$$

Since Lemma 2.2 with $r = 1$, we find that there exist $(\beta_0, \gamma_0) \in \tilde{\Lambda}$ such that $|A| > 0$ if and only if $(\phi, \psi) \in S^1$, and complete the proof. □

Remark 2.4 Lemma 2.3 means that there exists a positive constant c_τ such that

$$(Aa, a) \geq c_\tau |a|^2 \tag{2.5}$$

for any $a \in \mathbb{R}^3$. The constant c_τ is given by

$$c_\tau = \begin{cases} \dfrac{\psi(1-(\phi+\psi)\psi)}{6(\phi+\psi)}, & \psi \neq 0, \\ \dfrac{1}{3(\phi^2+1)}, & \psi = 0. \end{cases} \tag{2.6}$$

Substituting

$$\phi = \sqrt{\frac{\tau}{\nu}}|V(\rho_*)|, \qquad \psi = \sqrt{\frac{\tau}{\nu}}|\rho_* V'(\rho_*)| \tag{2.7}$$

into (2.6), we obtain (1.12). The estimate (2.5) with (2.7) will be used in the derivation of the a priori estimate in Section 3.

To derive the explicit representation (2.6), we introduce a new lemma. For $r > 0$, we set

$$\partial S^r := \{(\phi, \psi) \in \mathbb{R}^2; (\phi + \psi)\psi = r, \phi \geq 0, \psi \geq 0\},$$
$$\partial S^r_\beta := \left\{(\phi, \psi) \in \mathbb{R}^2; \frac{\beta}{2(1-\beta)}\left(\phi + \frac{1}{2\beta}\psi\right)^2 + \frac{1}{2}\psi^2 = r, \phi \geq 0, \psi \geq 0\right\}.$$

Then these sets also satisfy the following lemma.

Lemma 2.5 *For any $r > 0$, the following condition is satisfied.*

$$\partial S^r \subset \bigcup_{0<\beta\leq 1/2} \partial S^r_\beta.$$

Proof We show that, for any $(\phi, \psi) \in \partial S^r$, there exists $\beta_3 \in (0, 1/2]$ such that $(\phi, \psi) \in \partial S^r_{\beta_3}$. Since $(\phi, \psi) \in \partial S^r$, we have $0 < \psi \leq \sqrt{r}$ and $\phi = r/\psi - \psi$. Thus we compute

$$\frac{\beta}{2(1-\beta)}\left(\phi + \frac{1}{2\beta}\psi\right)^2 + \frac{1}{2}\psi^2 - r = \frac{1}{2(1-\beta)\psi^2}H_{\beta,\beta}(\psi^2),$$

where $H_{\beta,\beta}$ is defined in the proof of Lemma 2.2. Similarly as before, we take $\beta_3 = \psi^2/(2r)$ and get $H_{\beta_3,\beta_3}(\psi^2) = 0$. Consequently, we arrive at

$$\frac{\beta_3}{2(1-\beta_3)}\left(\phi + \frac{1}{2\beta_3}\psi\right)^2 + \frac{1}{2}\psi^2 = r,$$

and this completes the proof. □

We derive the representation (2.6). We calculate

$$\begin{aligned}(Aa, a) &= (1-\gamma)a_1^2 + \gamma a_2^2 + 2\beta a_3^2 - (2\beta\phi + \psi)a_1 a_3 - 2\beta\psi a_2 a_3 \\ &= \varepsilon_1 a_1^2 + \varepsilon_2 a_2^2 + \frac{Q}{(1-\gamma-\varepsilon_1)(\gamma-\varepsilon_2)}a_3^2 \\ &\quad + (1-\gamma-\varepsilon_1)\left(a_1 - \frac{\beta\phi + \psi/2}{1-\gamma-\varepsilon_1}a_3\right)^2 + (\gamma-\varepsilon_2)\left(a_2^2 - \frac{\beta\psi}{\gamma-\varepsilon_2}a_3\right)^2\end{aligned}$$

for $\varepsilon_1, \varepsilon_2 > 0$, where $a = (a_1, a_2, a_3)^\top \in \mathbb{R}^3$ and

$$\begin{aligned} Q &:= 2\beta(1-\gamma-\varepsilon_1)(\gamma-\varepsilon_2) - (\beta\phi+\psi/2)^2(\gamma-\varepsilon_2) - \beta^2\psi^2(1-\gamma-\varepsilon_1) \\ &= 2\beta\gamma(1-\gamma)(1-r) + 2\beta\gamma(1-\gamma)\left\{r - \frac{\beta}{2\gamma}\psi^2 - \frac{\beta}{2(1-\gamma)}\left(\phi+\frac{1}{2\beta}\psi\right)^2\right\} \\ &\quad - 2\beta\gamma\varepsilon_1 - 2\beta(1-\gamma)\varepsilon_2 + 2\beta\varepsilon_1\varepsilon_2 + \beta^2\psi^2\varepsilon_1 + (\beta\phi+\psi/2)^2\varepsilon_2 \end{aligned}$$

for $0 < r < 1$. Now, taking $\varepsilon_1 = (1-\gamma)(1-r)/3$ and $\varepsilon_2 = \gamma(1-r)/3$, we obtain

$$Q \geq \frac{2}{3}\beta\gamma(1-\gamma)(1-r) + 2\beta\gamma(1-\gamma)\mathcal{Q},$$

where

$$\mathcal{Q} := r - \frac{\beta}{2\gamma}\psi^2 - \frac{\beta}{2(1-\gamma)}\left(\phi+\frac{1}{2\beta}\psi\right)^2.$$

Namely, this gives

$$(Aa, a) \geq \frac{1}{3}(1-\gamma)(1-r)a_1^2 + \frac{1}{3}\gamma(1-r)a_2^2 + \frac{6\beta(1-r)}{(2+r)^2}a_3^2 + \frac{18\beta}{(2+r)^2}a_3^2\mathcal{Q}. \quad (2.8)$$

Now, we assume $\psi \neq 0$ and define $r_0 := (\phi+\psi)\psi$. Then, because of $(\phi, \psi) \in S^1$, we have $0 < r_0 < 1$. Employing Lemma 2.5 with $r = r_0$, we find that there exists $\beta_3 \in (0, 1/2]$ such that

$$\frac{\beta_3}{2(1-\beta_3)}\left(\phi+\frac{1}{2\beta_3}\psi\right)^2 + \frac{1}{2}\psi^2 = r_0 \quad (2.9)$$

for $(\phi, \psi) \in \partial S^{r_0}$. More precisely, we can choose $\beta_3 = \psi^2/(2r_0) \leq 1/2$ to get (2.9). For the details, see the proof of Lemma 2.5. Namely, substituting $\beta = \gamma = \beta_3$ and $r = r_0$ into (2.8), we have $\mathcal{Q} = 0$ and

$$\begin{aligned} (Aa, a) &\geq \frac{1}{3}(1-\beta_3)(1-r_0)a_1^2 + \frac{1}{3}\beta_3(1-r_0)a_2^2 + \frac{2}{3}\beta_3(1-r_0)a_3^2 \\ &\geq \frac{\beta_3}{3}(1-r_0)\left\{\left(\frac{1}{\beta_3}-1\right)a_1^2 + a_2^2 + a_3^2\right\} \\ &\geq \frac{\psi^2}{6r_0}(1-r_0)|a|^2. \end{aligned} \quad (2.10)$$

On the other hand, if $\psi = 0$, we have

$$\begin{aligned} (Aa, a) &= (1-\gamma)a_1^2 + \gamma a_2^2 + 2\beta a_3^2 - 2\beta\phi a_1 a_3 \\ &= (1-\gamma-\beta\phi^2)a_1^2 + \gamma a_2^2 + \beta a_3^2 + \beta(a_3-\phi a_1)^2. \end{aligned}$$

Thus, taking $\beta = 1/(3\phi^2)$ and $\gamma = 1/3$, we obtain

$$(Aa, a) \geq \frac{1}{3}a_1^2 + \frac{1}{3}a_2^2 + \frac{1}{3\phi^2}a_3^2 \geq \frac{1}{3(\phi^2+1)}|a|^2. \tag{2.11}$$

Consequently, since (2.10) and (2.11), we conclude that the constant c_τ is described as (2.6).

In the rest of this section, we shall recall some basic properties of solutions to the inhomogeneous heat equation $\partial_t w - \nu \partial_x^2 w = g$ with the initial data $w(t', x) = \tilde{w}_{t'}(x)$. Since Duhamel's principle, the solution of this Cauchy problem is described as $w(t) = w_L(t) + w_N(t)$ with

$$w_L(t) := S(t-t')\tilde{w}_{t'}, \qquad w_N(t) := \int_{t'}^{t} S(t-s)g(s)ds,$$

where $S(t)$ denotes the heat semigroup defined by

$$S(t)f = \mathcal{F}^{-1}\left[e^{-\nu t|\xi|^2}\mathcal{F}[f](\xi)\right] = \frac{1}{\sqrt{4\pi\nu t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\nu t}} f(y)dy,$$

and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are usual Fourier transform and its Fourier inverse transform on $\mathbb{R}$ defined by

$$\mathcal{F}[f](\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)dx, \qquad \mathcal{F}^{-1}[f](x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi} f(\xi)d\xi.$$

The heat semigroup $S(t)$ has useful properties, which are corrected in the following lemma.

Lemma 2.6 *The heat semigroup $S(t)$ satisfies the following properties.*

(i) *Let $k \in \mathbb{N} \cup \{0\}$ and $t' \geq 0$. Suppose that $\tilde{w}_{t'} \in H^k$. Then $w_L \in C([t', \infty); H^k)$, $\partial_x w_L \in L^2(t', \infty; H^k)$ with the estimates*

$$\|\partial_x^\ell w_L(t)\|_{L^2} \leq \|\partial_x^\ell \tilde{w}_{t'}\|_{L^2}, \qquad \int_{t'}^{\infty} \|\partial_x^{\ell+1} w_L(s)\|_{L^2}^2 ds \leq \frac{1}{2\nu}\|\partial_x^\ell \tilde{w}_{t'}\|_{L^2}^2$$

for $t \geq t'$ and $\ell = 0, \cdots, k$.

(ii) *Let $t'' > t' \geq 0$. Suppose that $g \in C([t', t'']; L^2)$. Then $w_N \in C([t', t'']; H^1) \cap L^2(t', t''; H^2)$ with the estimates*

$$\sup_{t' \leq t \leq t''} \|w_N(t)\|_{H^1} \leq \sqrt{t''-t'}\left(\sqrt{t''-t'} + \sqrt{\frac{2}{\nu}}\right) \sup_{t' \leq t \leq t''} \|g(t)\|_{L^2},$$

$$\int_{t'}^{t''} \|\partial_x^2 w_N(t)\|_{L^2}^2 dt \leq \frac{t''-t'}{\nu^2} \sup_{t' \leq t \leq t''} \|g(t)\|_{L^2}^2.$$

3 A priori estimate

In this section, we derive a priori estimates for (1.5), (1.11) to construct the global solution in time. To mention the propositions, we use the notation that $N(t) := \sup_{-\tau \le s \le t} \|u(s)\|_{L^\infty}$, and the function $\omega(t)$ is defined by

$$\omega(t) := \begin{cases} 1 & \text{for} \quad t > \tau, \\ 0 & \text{for} \quad 0 \le t \le \tau. \end{cases}$$

Then our a priori estimates are described as follows.

Proposition 3.1 *Assume* (1.13). *Let $T > 0$ and suppose that $u \in C([-\tau, T]; H^1)$ is a solution to* (1.5), (1.11), *which satisfies $\partial_t u \in L^2(0, T; L^2)$ and $\partial_x u \in L^2(0, T; H^1)$ with the condition*

$$\sqrt{\frac{\tau}{\nu}} W(N(T))N(T) \le c_\tau, \tag{3.1}$$

where c_τ is defined by (1.12), *then the solution satisfies the estimate that*

$$\begin{aligned} &\|u(t)\|_{H^1}^2 + \frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds \\ &+ \frac{c_\tau}{2}\left(\nu \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds + \omega(t) \int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\ &\le C_1 I_1^2 \end{aligned} \tag{3.2}$$

for $t \in [0, T]$, where I_1 and C_1 are defined by (1.15).

Proposition 3.2 *Suppose the same assumption as in Proposition 3.1. Furthermore, assume that V' is uniformly bounded. Then the solution satisfies the estimate* (3.2) *for $t \in [0, T]$, where I_1 and C_1 are defined by* (1.17).

To derive the energy estimate, we introduce a new function that

$$z(t, \theta, x) := u(t + \theta, x), \qquad t > 0, \quad \theta \in [-\tau, 0], \quad x \in \mathbb{R}.$$

Then, we find that $z(t, 0, x) = u(t, x)$, $z(t, -\tau, x) = u_\tau(t, x)$, $z(0, \theta, x) = u_0(\theta, x)$, and

$$\partial_t z - \partial_\theta z = 0, \qquad t > 0, \quad \theta \in [-\tau, 0], \quad x \in \mathbb{R}. \tag{3.3}$$

Applying the energy method to (1.5) and (3.3), we construct the energy estimate. To derive Propositions 3.1 and 3.2, we prepare three lemmas. The first one describes the lower-order estimate of solutions.

Lemma 3.3 *Suppose the same assumption as in Proposition 3.1. Then the solution u satisfies*

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + (\beta_0+\gamma_0)\nu\int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\
&+ (\beta_0+\gamma_0)\varepsilon_0\nu\int_\tau^t\int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&+ \frac{c_\tau}{2}\left(\nu\int_0^t\|\partial_x u(s)\|_{L^2}^2 ds + \nu\int_0^t\|\partial_x u_\tau(s)\|_{L^2}^2 ds + \omega(t)\int_\tau^t\int_{s-\tau}^s\|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\
&\le 41 I_0^2
\end{aligned} \tag{3.4}$$

for $t\in[0,T]$, where β_0, γ_0, and ε_0 are certain positive constants that depends on $\sqrt{\tau/\nu}$, ρ_*, and V.

Proof To obtain (3.4), we construct the a priori estimates for $t\in[0,\tau]$ and $t\in[\tau,T]$, separately. Since (3.3), we have $\partial_x\partial_t z-\partial_x\partial_\theta z=0$. Then, multiplying this equation by $e^{\varepsilon\theta}\partial_x z$ and integrating the resultant equation with respect to θ over $[-\tau,0]$, we obtain

$$\partial_t\left(\int_{-\tau}^0 e^{\varepsilon\theta}(\partial_x z)^2 d\theta\right) - (\partial_x u)^2 + e^{-\varepsilon\tau}(\partial_x u_\tau)^2 + \varepsilon\int_{-\tau}^0 e^{\varepsilon\theta}(\partial_x z)^2 d\theta = 0, \tag{3.5}$$

where ε is a positive parameter determined later. On the other hand, multiplying (1.5) by u, we get

$$\begin{aligned}
&\frac12\partial_t(u^2) + \partial_x\left(V(\rho_*+u_\tau)(\rho_*+u)u - \nu u\partial_x u - \int^{\rho_*+u} V(\eta)\eta d\eta\right) \\
&+ \nu(\partial_x u)^2 + \left(V(\rho_*+u) - V(\rho_*+u_\tau)\right)(\rho_*+u)\partial_x(\rho_*+u) = 0.
\end{aligned}$$

Then, using the relation that

$$\begin{aligned}
V(\rho_*+u(t)) - V(\rho_*+u_\tau(t)) &= \int_{t-\tau}^t V'(\rho_*+u(s))\partial_t u(s) ds \\
&= V'(\rho_*)\int_{t-\tau}^t \partial_t u(s) ds + \int_{t-\tau}^t (V'(\rho_*+u(s)) - V'(\rho_*))\partial_t u(s) ds,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac12\partial_t(u^2) + \partial_x\left(V(\rho_*+u_\tau)(\rho_*+u)u - \nu u\partial_x u - \int^{\rho_*+u} V(\eta)\eta d\eta\right) + \nu(\partial_x u)^2 \\
&+ \rho_* V'(\rho_*)\partial_x u\int_{t-\tau}^t \partial_t u(s) ds + \rho_*\partial_x u\int_{t-\tau}^t (V'(\rho_*+u(s)) - V'(\rho_*))\partial_t u(s) ds \\
&+ u\partial_x u\int_{t-\tau}^t V'(\rho_*+u(s))\partial_t u(s) ds = 0.
\end{aligned} \tag{3.6}$$

Furthermore, multiplying (1.5) by $\partial_t u$, we have

$$\frac{\nu}{2}\partial_t(\partial_x u)^2 - \nu\partial_x(\partial_t u\partial_x u) + (\partial_t u)^2 + \partial_t u\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right) = 0. \tag{3.7}$$

Then, integrating this equation with respect to t over $[t-\tau, t]$ and employing

$$\begin{aligned}\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right) &= \partial_x u V(\rho_* + u_\tau) + (\rho_* + u)\partial_x u_\tau V'(\rho_* + u_\tau)\\ &= V(\rho_*)\partial_x u + \rho_* V'(\rho_*)\partial_x u_\tau\\ &\quad + \partial_x u(V(\rho_* + u_\tau) - V(\rho_*))\\ &\quad + \rho_*\partial_x u_\tau(V'(\rho_* + u_\tau) - V'(\rho_*))\\ &\quad + u\partial_x u_\tau V'(\rho_* + u_\tau),\end{aligned}$$

we obtain

$$\begin{aligned}&\nu(\partial_x u)^2 - \nu(\partial_x u_\tau)^2 - 2\nu\partial_x\left(\int_{t-\tau}^t \partial_t u\partial_x u ds\right)\\ &\quad + 2\int_{t-\tau}^t (\partial_t u)^2 ds + 2V(\rho_*)\int_{t-\tau}^t \partial_t u\partial_x u ds\\ &\quad + 2\rho_* V'(\rho_*)\int_{t-\tau}^t \partial_t u\partial_x u_\tau ds + 2\int_{t-\tau}^t \partial_t u\partial_x u(V(\rho_* + u_\tau) - V(\rho_*))ds\\ &\quad + 2\rho_*\int_{t-\tau}^t \partial_t u\partial_x u_\tau(V'(\rho_* + u_\tau) - V'(\rho_*))ds\\ &\quad + 2\int_{t-\tau}^t u\partial_t u\partial_x u_\tau V'(\rho_* + u_\tau)ds = 0.\end{aligned} \tag{3.8}$$

for $t \geq \tau$. Thus, calculating (3.6) + (3.5) $\times\,\alpha$ + (3.8) $\times\,\beta$, we obtain

$$\partial_t E + \partial_x F + D + \alpha\varepsilon\int_{-\tau}^0 e^{\varepsilon\theta}(\partial_x z)^2 d\theta + R = 0, \tag{3.9}$$

where α and β are positive parameters determined later, and

$$\begin{aligned}E &:= \frac{1}{2}u^2 + \alpha\int_{-\tau}^0 e^{\varepsilon\theta}(\partial_x z)^2 d\theta,\\ F &:= V(\rho_* + u_\tau)(\rho_* + u)u - \nu u\partial_x u - \int^{\rho_* + u} V(\eta)\eta d\eta - 2\beta\nu\int_{t-\tau}^t \partial_t u\partial_x u ds,\\ D &:= ((1+\beta)\nu - \alpha)(\partial_x u)^2 + (\alpha e^{-\varepsilon\tau} - \beta\nu)(\partial_x u_\tau)^2 + 2\beta\int_{t-\tau}^t (\partial_t u)^2 ds\\ &\quad + \rho_* V'(\rho_*)\partial_x u\int_{t-\tau}^t \partial_t u ds\\ &\quad + 2\beta V(\rho_*)\int_{t-\tau}^t \partial_t u\partial_x u ds + 2\beta\rho_* V'(\rho_*)\int_{t-\tau}^t \partial_t u\partial_x u_\tau ds,\end{aligned}$$

$$
\begin{aligned}
R \ &:= \rho_* \partial_x u \int_{t-\tau}^{t} \partial_t u (V'(\rho_* + u) - V'(\rho_*)) ds + u \partial_x u \int_{t-\tau}^{t} \partial_t u V'(\rho_* + u) ds \\
&+ 2\beta \int_{t-\tau}^{t} \partial_t u \partial_x u (V(\rho_* + u_\tau) - V(\rho_*)) ds \\
&+ 2\beta\rho_* \int_{t-\tau}^{t} \partial_t u \partial_x u_\tau (V'(\rho_* + u_\tau) - V'(\rho_*)) ds \\
&+ 2\beta \int_{t-\tau}^{t} u \partial_t u \partial_x u_\tau V'(\rho_* + u_\tau) ds.
\end{aligned}
$$

It is important to estimate the dissipation term D. Using the Hölder inequality and Lemma 2.1, we estimate

$$
\begin{aligned}
\int_{\mathbb{R}} D dx \ &\geq ((1+\beta)\nu - \alpha)\|\partial_x u\|_{L^2}^2 + (\alpha e^{-\varepsilon\tau} - \beta\nu)\|\partial_x u_\tau\|_{L^2}^2 + 2\beta \int_{t-\tau}^{t} \|\partial_t u\|_{L^2}^2 ds \\
&- |\rho_* V'(\rho_*)| \sqrt{\tau} \|\partial_x u\|_{L^2} \left(\int_{t-\tau}^{t} \|\partial_t u\|_{L^2}^2 ds \right)^{1/2} \\
&- 2\beta |V(\rho_*)| \int_{t-\tau}^{t} \|\partial_t u\|_{L^2} \|\partial_x u\|_{L^2} ds \\
&- 2\beta |\rho_* V'(\rho_*)| \int_{t-\tau}^{t} \|\partial_t u\|_{L^2} \|\partial_x u_\tau\|_{L^2} ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\tau^t \int_{\mathbb{R}} D dx ds \\
&\geq \left(1 + \beta - \frac{\alpha}{\nu}\right) \nu \int_\tau^t \|\partial_x u\|_{L^2}^2 ds + \left(\frac{\alpha}{\nu} e^{-\varepsilon\tau} - \beta\right) \nu \int_\tau^t \|\partial_x u_\tau\|_{L^2}^2 ds \\
&\quad + 2\beta \int_\tau^t \int_{s-\tau}^{s} \|\partial_t u\|_{L^2}^2 d\sigma ds \\
&\quad - \sqrt{\frac{\tau}{\nu}} |\rho_* V'(\rho_*)| \left(\nu \int_\tau^t \|\partial_x u\|_{L^2}^2 ds \right)^{1/2} \left(\int_\tau^t \int_{s-\tau}^{s} \|\partial_t u\|_{L^2}^2 d\sigma ds \right)^{1/2} \\
&\quad - 2\beta |V(\rho_*)| \left(\int_\tau^t \int_{s-\tau}^{s} \|\partial_t u\|_{L^2}^2 d\sigma ds \right)^{1/2} \left(\int_\tau^t \int_{s-\tau}^{s} \|\partial_x u\|_{L^2}^2 d\sigma ds \right)^{1/2} \\
&\quad - 2\beta |\rho_* V'(\rho_*)| \left(\int_\tau^t \int_{s-\tau}^{s} \|\partial_t u\|_{L^2}^2 d\sigma ds \right)^{1/2} \left(\int_\tau^t \int_{s-\tau}^{s} \|\partial_x u_\tau\|_{L^2}^2 d\sigma ds \right)^{1/2} \\
&\geq (Aa, a) - \alpha(1 - e^{-\varepsilon\tau}) \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds \\
&\quad - \left(1 + \beta - \frac{\alpha}{\nu}\right) \nu \int_0^\tau \|\partial_x u\|_{L^2}^2 ds - \left(\frac{\alpha}{\nu} e^{-\varepsilon\tau} - \beta\right) \nu \int_0^\tau \|\partial_x u_\tau\|_{L^2}^2 ds,
\end{aligned}
$$

where $a = (a_1, a_2, a_3)^\top$ with

$$
a_1 = \left(\nu \int_0^t \|\partial_x u\|_{L^2}^2 ds \right)^{1/2}, \ a_2 = \left(\nu \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds \right)^{1/2},
$$

$$a_3 = \left(\int_\tau^t \int_{s-\tau}^s \|\partial_t u\|_{L^2}^2 d\sigma ds \right)^{1/2},$$

the matrix A is defined in (2.4) with $\gamma = \alpha/\nu - \beta$, and ϕ and ψ are defined in (2.7). Then, since Lemma 2.3, there exists a positive constant c_τ such that $(Aa, a) \geq c_\tau |a|^2$ under (1.13). Consequently, we obtain

$$\begin{aligned}\int_\tau^t \int_{\mathbb{R}} D dx ds \geq c_\tau &\left(\nu \int_0^t \|\partial_x u\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds + \int_\tau^t \int_{s-\tau}^s \|\partial_t u\|_{L^2}^2 d\sigma ds \right) \\ &- (1 - e^{-\varepsilon\tau})(\beta_0 + \gamma_0)\nu \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds \\ &- (1 - \gamma_0)\nu \int_0^\tau \|\partial_x u\|_{L^2}^2 ds - \gamma_0 \nu \int_0^\tau \|\partial_x u_\tau\|_{L^2}^2 ds\end{aligned}$$

for $t \geq \tau$. Here, β_0 and γ_0 are taken in Lemma 2.3. Furthermore, taking $\varepsilon = \varepsilon_0$ such that $(1 - e^{-\varepsilon_0 \tau})(\beta_0 + \gamma_0) \leq c_\tau/2$, we arrive at

$$\begin{aligned}\int_\tau^t \int_{\mathbb{R}} D dx ds \geq \frac{c_\tau}{2} &\left(\nu \int_0^t \|\partial_x u\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds \right. \\ &\left. + \int_\tau^t \int_{s-\tau}^s \|\partial_t u\|_{L^2}^2 d\sigma ds \right) \\ &- (1 - \gamma_0)\nu \int_0^\tau \|\partial_x u\|_{L^2}^2 ds - \gamma_0 \nu \int_0^\tau \|\partial_x u_\tau\|_{L^2}^2 ds\end{aligned} \tag{3.10}$$

for $t \geq \tau$.

We drive the estimate for the remainder term R. Using Taylor's theorem, we have

$$V^{(j)}(\rho_* + v) - V^{(j)}(\rho_*) = V^{(j+1)}(\rho_* + \theta_j v) v$$

for some $\theta_j \in (0, 1)$, where $j = 0, 1$ and $v = u$ or $v = u_\tau$. Then we estimate

$$\begin{aligned}|R| \leq &\sqrt{\tau} V_2(N(T)) |\rho_*| |\partial_x u| \left(\int_{t-\tau}^t u^2 (\partial_t u)^2 ds \right)^{1/2} \\ &+ \sqrt{\tau} V_1(N(T)) |u| |\partial_x u| \left(\int_{t-\tau}^t (\partial_t u)^2 ds \right)^{1/2} \\ &+ 2\beta V_1(N(T)) \left(\int_{t-\tau}^t (\partial_x u)^2 ds \right)^{1/2} \left(\int_{t-\tau}^t u_\tau^2 (\partial_t u)^2 ds \right)^{1/2} \\ &+ 2\beta |\rho_*| V_2(N(T)) \left(\int_{t-\tau}^t (\partial_x u_\tau)^2 ds \right)^{1/2} \left(\int_{t-\tau}^t u_\tau^2 (\partial_t u)^2 ds \right)^{1/2} \\ &+ 2\beta V_1(N(T)) \left(\int_{t-\tau}^t (\partial_x u_\tau)^2 ds \right)^{1/2} \left(\int_{t-\tau}^t u^2 (\partial_t u)^2 ds \right)^{1/2},\end{aligned}$$

and hence

$$\int_{\mathbb{R}} |R| dx \leq \sqrt{\tau}(V_1 + |\rho_*|V_2) N(T) \|\partial_x u\|_{L^2} \left(\int_{t-\tau}^{t} \|\partial_t u\|_{L^2}^2 ds \right)^{1/2}$$
$$+ 2\beta V_1 N(T) \left(\int_{t-\tau}^{t} \|\partial_x u\|_{L^2}^2 ds \right)^{1/2} \left(\int_{t-\tau}^{t} \|\partial_t u\|_{L^2}^2 ds \right)^{1/2}$$
$$+ 2\beta(V_1 + |\rho_*|V_2) N(T) \left(\int_{t-\tau}^{t} \|\partial_x u_\tau\|_{L^2}^2 ds \right)^{1/2} \left(\int_{t-\tau}^{t} \|\partial_t u\|_{L^2}^2 ds \right)^{1/2}.$$

Thus, employing Lemma 2.1, we obtain

$$\begin{aligned} \int_\tau^t \int_{\mathbb{R}} |R| dx ds \ &\leq ((1+2\beta)V_1 + |\rho_*|V_2) \sqrt{\frac{\tau}{\nu}} N(T) a_1 a_3 \\ &+ 2\beta(V_1 + |\rho_*|V_2) \sqrt{\frac{\tau}{\nu}} N(T) a_2 a_3 \end{aligned} \tag{3.11}$$

for $t \geq \tau$. Combing (3.10) and (3.11), we get

$$\begin{aligned} &\int_\tau^t \int_{\mathbb{R}} D dx ds - \int_\tau^t \int_{\mathbb{R}} |R| dx ds \\ &\geq (Ba, a) - (1-\gamma_0)\nu \int_0^\tau \|\partial_x u\|_{L^2}^2 ds - \gamma_0 \nu \int_0^\tau \|\partial_x u_\tau\|_{L^2}^2 ds, \end{aligned}$$

where

$$B := \begin{pmatrix} c_\tau/2 & 0 & b_1\sqrt{\tau/\nu}N(T) \\ 0 & c_\tau/2 & b_2\sqrt{\tau/\nu}N(T) \\ b_1\sqrt{\tau/\nu}N(T) & b_2\sqrt{\tau/\nu}N(T) & c_\tau/2 \end{pmatrix}$$

with $b_1 := -((1+2\beta_0)V_1 + |\rho_*|V_2)/2$ and $b_2 := -\beta_0(V_1 + |\rho_*|V_2)$. The simple computation gives

$$\begin{aligned} (Ba, a) &= \frac{c_\tau}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{\frac{\tau}{\nu}} N(T)(a_1 b_1 + a_2 b_2) a_3 \\ &= \frac{\chi}{2}(a_1^2 + a_2^2) + \frac{c_\tau - \chi}{2} \left(a_1 + \sqrt{\frac{\tau}{\nu}} \frac{2N(T)}{c_\tau - \chi} b_1 a_3 \right)^2 \\ &\quad + \frac{c_\tau - \chi}{2} \left(a_2 + \sqrt{\frac{\tau}{\nu}} \frac{2N(T)}{c_\tau - \chi} b_2 a_3 \right)^2 \\ &\quad + \frac{2}{c_\tau - \chi} \left\{ \frac{c_\tau(c_\tau - \chi)}{4} - \frac{\tau}{\nu} N(T)^2 (b_1^2 + b_2^2) \right\} a_3^2 \end{aligned}$$

for $0 < \chi < c_\tau$. Then, letting $\chi = c_\tau/2$, this gives

$$(Ba, a) \geq \frac{c_\tau}{4}|a|^2 + \frac{4}{c_\tau} \left\{ \frac{c_\tau^2}{16} - \frac{\tau}{\nu} N(T)^2 (b_1^2 + b_2^2) \right\} a_3^2.$$

Since Lemma 2.2, we have $0 < \beta_0 < 1/2$, and this gives

$$\begin{aligned} b_1^2 + b_2^2 &= \frac{1}{4}\left\{(1+2\beta_0)V_1(N(T)) + |\rho_*|V_2(N(T))\right\}^2 \\ &\quad + \beta_0^2\left\{V_1(N(T)) + |\rho_*|V_2(N(T))\right\}^2 \\ &< \frac{1}{2}\left\{2V_1(N(T)) + |\rho_*|V_2(N(T))\right\}^2. \end{aligned}$$

Therefore, using the condition (3.1), we obtain $(Ba, a) \geq c_\tau |a|^2/4$, and hence this gives

$$\begin{aligned} &\int_\tau^t \int_{\mathbb{R}} D dx ds - \int_\tau^t \int_{\mathbb{R}} |R| dx ds \\ &\quad \geq \frac{c_\tau}{4}\left(\nu \int_0^t \|\partial_x u\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds + \int_\tau^t \int_{s-\tau}^s \|\partial_t u\|_{L^2}^2 d\sigma ds\right) \\ &\quad - (1-\gamma_0)\nu \int_0^\tau \|\partial_x u\|_{L^2}^2 ds - \gamma_0 \nu \int_0^\tau \|\partial_x u_\tau\|_{L^2}^2 ds \end{aligned} \tag{3.12}$$

for $t \geq \tau$. Consequently, integrating (3.9) and using (3.12) we arrive at

$$\begin{aligned} &\frac{1}{2}\|u(t)\|_{L^2}^2 + (\beta_0+\gamma_0)\nu \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\ &\quad + (\beta_0+\gamma_0)\varepsilon_0 \nu \int_\tau^t \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\ &\quad + \frac{c_\tau}{4}\left(\nu \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds \right. \\ &\quad \left. + \int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\ &\leq \frac{1}{2}\|u_0(\tau)\|_{L^2}^2 + (\beta_0+\gamma_0)\nu \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(\tau,\theta)\|_{L^2}^2 d\theta \\ &\quad + (1-\gamma_0)\nu \int_0^\tau \|\partial_x u(s)\|_{L^2}^2 ds + \gamma_0 \nu \int_0^\tau \|\partial_x u_\tau(s)\|_{L^2}^2 ds \end{aligned} \tag{3.13}$$

for $t \geq \tau$.

On the other hand, multiplying (1.5) by u, we have

$$\begin{aligned} &\frac{1}{2}\partial_t(u^2) + \partial_x\left(-\nu u \partial_x u + u(\rho_*+u)V(\rho_*+u_\tau) - \frac{1}{2}V(\rho_*)(\rho_*+u)^2\right) \\ &+ \nu(\partial_x u)^2 - (\rho_*+u)\partial_x u\left(V(\rho_*+u_\tau) - V(\rho_*)\right) = 0. \end{aligned}$$

Then, combining this equation and (3.5), we obtain

$$
\begin{aligned}
&\partial_t \left(u^2 + \nu \int_{-\tau}^{0} e^{\varepsilon\theta} (\partial_x z)^2 d\theta \right) \\
&+2\partial_x \left(-\nu u \partial_x u + u(\rho_* + u) V(\rho_* + u_\tau) - \frac{1}{2} V(\rho_*)(\rho_* + u)^2 \right) \\
&+\nu(\partial_x u)^2 + \nu e^{-\varepsilon\tau} (\partial_x u_\tau)^2 + \varepsilon\nu \int_{-\tau}^{0} e^{\varepsilon\theta} (\partial_x z)^2 d\theta \\
&-2(\rho_* + u)\partial_x u \left(V(\rho_* + u_\tau) - V(\rho_*) \right) = 0.
\end{aligned}
$$

Therefore, integrating the resultant equation concerning t and x, and employing the fact that

$$
\begin{aligned}
&(\rho_* + u) \left(V(\rho_* + u_\tau) - V(\rho_*) \right) \\
&= \rho_* \left(u_\tau V'(\rho_*) + \frac{1}{2} u_\tau^2 V''(\rho_* + \theta_2 u_\tau) \right) + u u_\tau V'(\rho_* + \theta_1 u_\tau)
\end{aligned}
$$

for some $\theta_1, \theta_2 \in (0, 1)$, we arrive at

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2 + \nu \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\
&\quad + \frac{\nu}{4} \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu e^{-\varepsilon\tau} \int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds \\
&\quad + \varepsilon\nu \int_0^t \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\le \|u_0(0)\|_{L^2}^2 + \nu \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x u_0(\theta)\|_{L^2}^2 d\theta \\
&\quad + \frac{4\rho_*^2 V'(\rho_*)^2}{\nu} \int_0^t \|u_\tau\|_{L^2}^2 ds + \frac{\rho_*^2 V_2^2}{\nu} \int_0^t \|u_\tau\|_{L^\infty}^2 \|u_\tau\|_{L^2}^2 ds \\
&\quad + \frac{4V_1^2}{\nu} \int_0^t \|u\|_{L^\infty}^2 \|u_\tau\|_{L^2}^2 ds \\
&\le \|u_0(0)\|_{L^2}^2 + \nu \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x u_0(\theta)\|_{L^2}^2 d\theta \\
&\quad + \frac{\tau}{\nu} \left\{ 4\rho_*^2 V'(\rho_*)^2 + (4V_1^2 + \rho_*^2 V_2^2) N(t)^2 \right\} \sup_{-\tau \le \theta \le 0} \|u_0(\theta)\|_{L^2}^2 \\
&\le \|u_0(0)\|_{L^2}^2 + \nu \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x u_0(\theta)\|_{L^2}^2 d\theta + \left(4 + \frac{c_\tau^2}{8} \right) \sup_{-\tau \le \theta \le 0} \|u_0(\theta)\|_{L^2}^2
\end{aligned}
$$

for $0 \le t \le \tau$. Here, we used that $4V_1(M)^2 + \rho_*^2 V_2(M)^2 \le W(M)^2/8$. Consequently, this estimate and (3.13) gives

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2 + 2(\beta_0+\gamma_0)\nu \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\
&\quad + 2(\beta_0+\gamma_0)\varepsilon_0\nu \int_\tau^t \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\quad + \frac{c_\tau}{2}\left(\nu\int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu\int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds + \int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\
&\leq 8\|u_0(0)\|_{L^2}^2 + 8\nu\int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x u_0(\theta)\|_{L^2}^2 d\theta + (32+c_\tau^2)\sup_{-\tau\leq\theta\leq 0}\|u_0(\theta)\|_{L^2}^2
\end{aligned}
$$

for $t \geq \tau$. Here, we also chose ε_0 such that $e^{-\varepsilon_0\tau} \geq 1/4$. Furthermore, these estimates give

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2 + (\beta_0+\gamma_0)\nu \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\
&\quad + (\beta_0+\gamma_0)\varepsilon_0\nu \int_\tau^t \int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\quad + \frac{c_\tau}{2}\left(\nu\int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu\int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds + \omega(t)\int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\
&\leq 8\|u_0(0)\|_{L^2}^2 + 8\nu\int_{-\tau}^0 e^{\varepsilon_0\theta}\|\partial_x u_0(\theta)\|_{L^2}^2 d\theta + (32+c_\tau^2)\sup_{-\tau\leq\theta\leq 0}\|u_0(\theta)\|_{L^2}^2 \\
&\leq 41 I_0^2
\end{aligned}
$$

for $t \geq 0$. Here, we used $c_\tau \leq 1/3$, which is given by (1.12). Thus, we complete the proof. ◻

Our next purpose is to show the higher-order estimate of solutions.

Lemma 3.4 *Suppose the same assumption as in Proposition 3.1. Then the solution u satisfies*

$$
\begin{aligned}
&\|\partial_x u(t)\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \nu\int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds \\
&\leq \|\partial_x u_0(0)\|_{L^2}^2 + \frac{84}{\nu}\left\{\frac{c_\tau}{\tau} + \frac{4}{\nu c_\tau}(V(\rho_*)^2 + 2\rho_*^2 V'(\rho_*)^2)\right\} I_0^2
\end{aligned}
\tag{3.14}
$$

for $t \in [0, T]$.

Proof Similarly as before, multiplying (1.5) by $-\partial_x^2 u$, we have

$$
\frac{1}{2}\partial_t(\partial_x u)^2 + \nu(\partial_x^2 u)^2 - \partial_x(\partial_t u\partial_x u) - \partial_x^2 u\partial_x\left((\rho_*+u)V(\rho_*+u_\tau)\right) = 0.
$$

Combining (3.7) and this equation, we get

$$\partial_t(\partial_x u)^2 + \frac{1}{\nu}(\partial_t u)^2 + \nu(\partial_x^2 u)^2 - 2\partial_x(\partial_t u \partial_x u) - \frac{1}{\nu}\left(\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right)\right)^2 = 0.$$

Then, integrating this with respect to x over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\|\partial_x u\|_{L^2}^2 + \frac{1}{\nu}\|\partial_t u\|_{L^2}^2 + \nu\|\partial_x^2 u\|_{L^2}^2 - \frac{1}{\nu}\|\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right)\|_{L^2}^2 = 0, \quad (3.15)$$

and this gives

$$\begin{aligned} &\|\partial_x u(t)\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \nu\int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds \\ &= \|\partial_x u_0(0)\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right)(s)\|_{L^2}^2 ds. \end{aligned} \quad (3.16)$$

The remainder term is decomposed as

$$\begin{aligned} &\int_0^t \|\partial_x\left((\rho_* + u)V(\rho_* + u_\tau)\right)\|_{L^2}^2 ds \\ &\le 2\int_0^t \|\partial_x u V(\rho_* + u_\tau)\|_{L^2}^2 ds + 2\int_0^t \|(\rho_* + u)\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2}^2 ds, \end{aligned} \quad (3.17)$$

and each terms are estimated that

$$\int_0^t \|\partial_x u V(\rho_* + u_\tau)\|_{L^2}^2 ds \le 2\left(V(\rho_*)^2 + V_1^2 N(T)^2\right)\int_0^t \|\partial_x u\|_{L^2}^2 ds,$$

and

$$\begin{aligned} &\int_0^t \|(\rho_* + u)\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2}^2 ds \\ &\le 2\int_0^t \|\rho_* \partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2}^2 ds + 2\int_0^t \|u\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2}^2 ds \\ &\le 2\left(2\rho_*^2 V'(\rho_*)^2 + 2\rho_*^2 V_2^2 N(T)^2 + V_1^2 N(T)^2\right)\int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds. \end{aligned}$$

Thus, using (3.1) and Lemma 3.3, the remainder term is estimated as

$$\begin{aligned}
&\frac{1}{\nu}\int_0^t \|\partial_x\left((\rho_*+u)V(\rho_*+u_\tau)\right)\|_{L^2}^2 ds\\
&\le \frac{4}{\nu}\left(V(\rho_*)^2+V_1^2N(T)^2\right)\int_0^t \|\partial_x u\|_{L^2}^2 ds\\
&\quad+\frac{4}{\nu}\left(2\rho_*^2V'(\rho_*)^2+2\rho_*^2V_2^2N(T)^2+V_1^2N(T)^2\right)\int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds\\
&\le \left(\frac{4}{\nu}V(\rho_*)^2+\frac{c_\tau^2}{8\tau}\right)\int_0^t \|\partial_x u\|_{L^2}^2 ds+\left(\frac{8}{\nu}\rho_*^2V'(\rho_*)^2+\frac{c_\tau^2}{\tau}\right)\int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds\\
&\le \frac{84}{\nu}\left\{\frac{c_\tau}{\tau}+\frac{4}{\nu c_\tau}(V(\rho_*)^2+2\rho_*^2V'(\rho_*)^2)\right\}I_0^2,
\end{aligned} \tag{3.18}$$

and this leads to the desired estimate. □

Lemma 3.5 *Suppose that the same assumption as in Proposition 3.2. Then the solution u satisfies*

$$\begin{aligned}
&\|\partial_x u(t)\|_{L^2}^2+\frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds+\frac{\nu}{2}\int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds\\
&\le \|\partial_x u_0(0)\|_{L^2}^2+\frac{\tilde{C}_1}{\nu^2 c_\tau}\left(1+\frac{I_0^4}{\nu^2}\right)I_0^2+\frac{c_\tau^2}{8}\sup_{-\tau\le\theta\le 0}\|\partial_x u_0(\theta)\|_{L^2}^2
\end{aligned} \tag{3.19}$$

for $t \in [0, T]$, where $\tilde{C}_1$ is defined in Theorem 1.4.

Proof We start the proof from (3.16) and (3.17). The first term in (3.17) is estimated that

$$\begin{aligned}
\int_0^t \|\partial_x u V(\rho_*+u_\tau)\|_{L^2}^2 ds &\le 2V(\rho_*)^2\int_0^t \|\partial_x u\|_{L^2}^2 ds+2V_*^2\int_0^t \|u_\tau\partial_x u\|_{L^2}^2 ds\\
&\le 2V(\rho_*)^2\int_0^t \|\partial_x u\|_{L^2}^2 ds+2V_*^2\sup_{0\le s\le t}\|u_\tau\|_{L^2}^2\int_0^t \|\partial_x u\|_{L^\infty}^2 ds\\
&\le 2V(\rho_*)^2\int_0^t \|\partial_x u\|_{L^2}^2 ds+84V_*^2I_0^2\int_0^t \|\partial_x u\|_{L^2}\|\partial_x^2 u\|_{L^2} ds\\
&\le \frac{\nu^2}{8}\int_0^t \|\partial_x^2 u\|_{L^2}^2 ds+2\left(V(\rho_*)^2+\frac{84^2V_*^4I_0^4}{\nu^2}\right)\int_0^t \|\partial_x u\|_{L^2}^2 ds.
\end{aligned} \tag{3.20}$$

We also estimate the second term in (3.17) that

$$\begin{aligned}
&\int_0^t \|(\rho_*+u)\partial_x u_\tau V'(\rho_*+u_\tau)\|_{L^2}^2 ds\\
&\le 2\int_0^t \|\rho_*\partial_x u_\tau V'(\rho_*+u_\tau)\|_{L^2}^2 ds+2\int_0^t \|u\partial_x u_\tau V'(\rho_*+u_\tau)\|_{L^2}^2 ds\\
&\le 2\rho_*^2V_*^2\int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds+2V_*^2\int_0^t \|u\partial_x u_\tau\|_{L^2}^2 ds.
\end{aligned}$$

Here, we have

$$
\begin{aligned}
V_*^2 \int_\tau^t \|u\partial_x u_\tau\|_{L^2}^2 ds &\le V_*^2 \sup_{\tau\le s\le t} \|u\|_{L^2}^2 \int_\tau^t \|\partial_x u_\tau\|_{L^\infty}^2 ds \\
&\le 42 V_*^2 I_0^2 \int_0^{t-\tau} \|\partial_x u\|_{L^2} \|\partial_x^2 u\|_{L^2} ds \\
&\le \frac{\nu^2}{16} \int_0^t \|\partial_x^2 u\|_{L^2}^2 ds + \frac{84^2 V_*^4 I_0^4}{\nu^2} \int_0^t \|\partial_x u\|_{L^2}^2 ds
\end{aligned}
$$

and

$$
V_*^2 \int_0^\tau \|u\partial_x u_\tau\|_{L^2}^2 ds \le V_*^2 \int_0^\tau \|u\|_{L^\infty}^2 \|\partial_x u_\tau\|_{L^2}^2 ds \le \tau V_*^2 N(\tau)^2 \sup_{-\tau\le s\le 0} \|\partial_x u\|_{L^2}^2.
$$

Thus, these give

$$
\begin{aligned}
&\int_0^t \|(\rho_* + u)\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2}^2 ds \\
&\le 2\rho_*^2 V_*^2 \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds + \frac{\nu^2}{8} \int_0^t \|\partial_x^2 u\|_{L^2}^2 ds \\
&\quad + 2\frac{84^2 V_*^4 I_0^4}{\nu^2} \int_0^t \|\partial_x u\|_{L^2}^2 ds + 2\tau V_*^2 N(\tau)^2 \sup_{-\tau\le s\le 0} \|\partial_x u\|_{L^2}^2.
\end{aligned} \tag{3.21}
$$

Substituting (3.20) and (3.21) into (3.17), and employing (3.1) and Lemma 3.3, we obtain

$$
\begin{aligned}
&\frac{1}{\nu} \int_0^t \|\partial_x \left((\rho_* + u) V(\rho_* + u_\tau)\right)(s)\|_{L^2}^2 ds \\
&\le \frac{\nu}{2} \int_0^t \|\partial_x^2 u\|_{L^2}^2 ds + \frac{2}{\nu} \left(2V(\rho_*)^2 + \frac{168^2 V_*^4 I_0^4}{\nu^2} \right) \int_0^t \|\partial_x u\|_{L^2}^2 ds \\
&\quad + \frac{4\rho_*^2 V_*^2}{\nu} \int_0^t \|\partial_x u_\tau\|_{L^2}^2 ds + \frac{4\tau V_*^2}{\nu} N(\tau)^2 \sup_{-\tau\le s\le 0} \|\partial_x u\|_{L^2}^2 \\
&\le \frac{\nu}{2} \int_0^t \|\partial_x^2 u\|_{L^2}^2 ds + \frac{168}{\nu^2 c_\tau} \left(2V(\rho_*)^2 + 2\rho_*^2 V_*^2 + \frac{168^2 V_*^4 I_0^4}{\nu^2} \right) I_0^2 \\
&\quad + \frac{c_\tau^2}{8} \sup_{-\tau\le s\le 0} \|\partial_x u_0\|_{L^2}^2.
\end{aligned} \tag{3.22}
$$

Finally, combining (3.16) and (3.22), we arrive at (3.19) and complete the proof. □

Proof of Proposition 3.1 Combining (3.4) and (3.14), we obtain

$$\begin{aligned}
&\|u(t)\|_{H^1}^2 + \frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds \\
&\quad + \frac{c_\tau}{2}\left(\nu \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds + \omega(t)\int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\
&\le 41 I_0^2 + \|\partial_x u_0(0)\|_{L^2}^2 + \frac{84}{\nu}\left\{\frac{c_\tau}{\tau} + \frac{4}{\nu c_\tau}(V(\rho_*)^2 + 2\rho_*^2 V'(\rho_*)^2)\right\} I_0^2,
\end{aligned}$$

and this estimate with $c_\tau \le 1/3$ give the desired estimate (3.2). □

Proof of Proposition 3.2 Instead of (3.14), we employ (3.19). Namely, the estimates (3.4) and (3.19) leads to

$$\begin{aligned}
&\|u(t)\|_{H^1}^2 + \frac{1}{\nu}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \frac{\nu}{2} \int_0^t \|\partial_x^2 u(s)\|_{L^2}^2 ds \\
&\quad + \frac{c_\tau}{2}\left(\nu \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds + \nu \int_0^t \|\partial_x u_\tau(s)\|_{L^2}^2 ds + \omega(t)\int_\tau^t \int_{s-\tau}^s \|\partial_t u(\sigma)\|_{L^2}^2 d\sigma ds\right) \\
&\le 41 I_0^2 + \|\partial_x u_0(0)\|_{L^2}^2 + \frac{\tilde{C}_1}{\nu^2 c_\tau}\left(1 + \frac{I_0^4}{\nu^2}\right) I_0^2 + \frac{c_\tau^2}{8}\sup_{-\tau\le\theta\le 0}\|\partial_x u_0(\theta)\|_{L^2}^2,
\end{aligned}$$

and this gives the desired estimate. □

4 Global existence

We shall prove the global-in-time existence of solutions in this section. To this end, we first derive the local-in-time solution for the initial history problem. For any fixed $t' \ge 0$, we consider (1.5) for $t > t'$ and $x \in \mathbb{R}$, and the corresponding initial history

$$u(\theta, x) = \tilde{u}_{t'}(\theta, x), \tag{4.1}$$

for $t' - \tau \le \theta \le t'$ and $x \in \mathbb{R}$. Here, $\tilde{u}_{t'} = \tilde{u}_{t'}(\theta, x)$ is given function. Then, the local-in-time existence theorem is derived as follows.

Lemma 4.1 *Let ε and M be positive numbers, and suppose that an initial history $\tilde{u}_{t'} \in C([t'-\tau, t']; H^1)$ satisfies*

$$\sup_{t'-\tau\le\theta\le t'} \|\tilde{u}_{t'}(\theta)\|_{H^1} \le M. \tag{4.2}$$

Then there exists a positive constant $t_0 = t_0(\varepsilon, M)$, which does not depend on t', such that the problem (1.5), (4.1) has a unique solution $u \in C([t'-\tau, t'+t_0]; H^1) \cap L^2(t', t'+t_0; H^2)$ which satisfies $\partial_t u \in L^2(t', t'+t_0; L^2)$ and

$$\sup_{t'-\tau\le s\le t'+t_0} \|u(s)\|_{H^1} \le (1+\varepsilon)M. \tag{4.3}$$

Remark that the small positive constant t_0 appeared in Lemma 4.1 depends not only on ε and M but also on τ, ν, ρ_* and V.

Proof of Lemma 4.1 In order to prove Lemma 4.1, we consider the integral equation corresponding to (1.5), (4.1) that

$$u(t) = S(t-t')\tilde{u}_{t'}(t') - \int_{t'}^{t} S(t-s)\partial_x \left((\rho_* + u)V(\rho_* + u_\tau)\right)(s)ds. \tag{4.4}$$

By using the contraction mapping theorem, we shall prove the existence of the solution to (4.4) in $C([t', t'+t_0]; H^1) \cap L^2(t', t'+t_0; H^2)$. Suppose that t_0 is a positive number with $t_0 \le \tau$. Then we introduce the mapping Φ defined by

$$\Phi[u](t) := S(t-t')\tilde{u}_{t'}(t') - \int_{t'}^{t} S(t-s)\partial_x \left((\rho_* + u)V(\rho_* + u_\tau)\right)(s)ds$$

for $t \in [t', t'+t_0]$ in the underlying space B_{t_0} defined by

$$B_{t_0} := \left\{ u \in C([t', t'+t_0]; H^1) \cap L^2(t', t'+t_0; H^2)\,;\ \sup_{t' \le s \le t'+t_0} \|u(s)\|_{H^1} \le (1+\varepsilon)M \right\}.$$

Here, we remark that $u_\tau \in C([t', t'+\tau]; H^1)$ is described as the initial history. This means

$$\sup_{t' \le s \le t'+\tau} \|u_\tau(s)\|_{H^1} \le M,$$

which comes from (4.2).

Firstly, we show that there exists t_0 such that the mapping Φ satisfies $\Phi : B_{t_0} \to B_{t_0}$. To this end, we set u_L and u_N as follows

$$u_L(t) := S(t-t')\tilde{u}_{t'}(t'), \qquad u_N(t) := \int_{t'}^{t} S(t-s)\partial_x \left((\rho_* + u)V(\rho_* + u_\tau)\right)(s)ds.$$

Since (i) in Lemma 2.6 and (4.2), we find that $u_L \in C([t', t'+t_0]; H^1) \cap L^2(t', t'+t_0; H^2)$ with

$$\sup_{t' \le s \le t'+t_0} \|u_L(s)\|_{H^1} \le \|\tilde{u}_{t'}(t')\|_{H^1} \le M. \tag{4.5}$$

On the other hand, because of $u, u_\tau \in C([t', t'+t_0]; H^1)$ and $V \in C^1$, we find that

$$\partial_x((\rho_* + u)V(\rho_* + u_\tau)) \in C([t', t'+t_0]; L^2)$$

for $u \in B_{t_0}$. Thus, we can apply (ii) in Lemma 2.6 to the Duhamel term u_N and obtain that $u_N \in C([t', t'+t_0]; H^1) \cap L^2(t', t'+t_0; H^2)$ and

$$\sup_{t' \le s \le t'+t_0} \|u_N(s)\|_{H^1} \le \sqrt{t_0}\left(\sqrt{t_0} + \sqrt{\frac{2}{\nu}}\right) \sup_{t' \le s \le t'+t_0} \|\partial_x((\rho_* + u)V(\rho_* + u_\tau))(s)\|_{L^2}. \tag{4.6}$$

The right hand side in (4.6) is estimated as

$$
\begin{aligned}
&\|\partial_x((\rho_* + u)V(\rho_* + u_\tau))\|_{L^2} \\
&\leq \|\partial_x u V(\rho_* + u_\tau)\|_{L^2} + \|(\rho_* + u)\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2} \\
&\leq |V(\rho_*)|\|\partial_x u\|_{L^2} + V_1(M)\|u_\tau\|_{L^\infty}\|\partial_x u\|_{L^2} + V_1(M)(|\rho_*| + \|u\|_{L^\infty})\|\partial_x u_\tau\|_{L^2} \\
&\leq \{|V(\rho_*)| + V_1(M)(|\rho_*| + 2(1+\varepsilon)M)\}(1+\varepsilon)M
\end{aligned}
$$

for $u \in B_{t_0}$. Therefore, substituting this estimate into (4.6), we get

$$
\begin{aligned}
&\sup_{t' \leq s \leq t'+t_0} \|u_N(s)\|_{H^1} \\
&\leq \sqrt{t_0}\left(\sqrt{t_0} + \sqrt{\frac{2}{\nu}}\right)\{|V(\rho_*)| + V_1(M)(|\rho_*| + 2(1+\varepsilon)M)\}(1+\varepsilon)M.
\end{aligned}
$$

This estimate tells us that we should take t_0 such that

$$
\left(\sqrt{\tau} + \sqrt{\frac{2}{\nu}}\right)\{|V(\rho_*)| + V_1(M)(|\rho_*| + 2(1+\varepsilon)M)\}(1+\varepsilon)\sqrt{t_0} \leq \varepsilon
$$

to get

$$
\sup_{t' \leq s \leq t'+t_0} \|u_N(s)\|_{H^1} \leq \varepsilon M. \tag{4.7}
$$

Namely, taking t_0 such that

$$
t_0 \leq \frac{\varepsilon^2}{(1+\varepsilon)^2}\left(\sqrt{\tau} + \sqrt{\frac{2}{\nu}}\right)^{-2}\{|V(\rho_*)| + V_1(M)(|\rho_*| + 2(1+\varepsilon)M)\}^{-2}, \tag{4.8}
$$

we obtain (4.7) for $u \in B_{t_0}$. Hence, combining (4.5) and (4.7), we arrive at

$$
\sup_{t' \leq s \leq t'+t_0} \|\Phi[u](s)\|_{H^1} \leq (1+\varepsilon)M
$$

for $u \in B_{t_0}$. This means that Φ is the mapping on B_{t_0}.

Secondly, we shall show that Φ is the contraction mapping on B_{t_0} for small t_0. For this purpose, we consider

$$
\Phi[u](t) - \Phi[v](t) = -\int_{t'}^{t} S(t-s)\partial_x((u-v)V(\rho_* + u_\tau))(s)ds
$$

for $u, v \in B_{t_0}$. Here, we remark that $u_\tau = v_\tau$ in B_{t_0}. Similarly to before, we employ (ii) in Lemma 2.6, we have

$$\begin{aligned}
&\sup_{t' \le s \le t'+t_0} \|(\Phi[u] - \Phi[v])(s)\|_{H^1} \\
&\le \sqrt{t_0}\left(\sqrt{t_0} + \sqrt{\frac{2}{\nu}}\right) \sup_{t' \le s \le t'+t_0} \|\partial_x((u-v)V(\rho_* + u_\tau))(s)\|_{L^2}.
\end{aligned}$$

Furthermore, we compute

$$\begin{aligned}
&\|\partial_x((u-v)V(\rho_* + u_\tau))\|_{L^2} \\
&\le \|\partial_x(u-v)V(\rho_* + u_\tau)\|_{L^2} + \|(u-v)\partial_x u_\tau V'(\rho_* + u_\tau)\|_{L^2} \\
&\le |V(\rho_*)|\|\partial_x(u-v)\|_{L^2} + V_1(M)\|u_\tau\|_{L^\infty}\|\partial_x(u-v)\|_{L^2} \\
&\quad + V_1(M)\|u-v\|_{L^\infty}\|\partial_x u_\tau\|_{L^2} \\
&\le (|V(\rho_*)| + 2MV_1(M))\|u-v\|_{H^1}
\end{aligned}$$

for $u, v \in B_{t_0}$. Thus, combining the above two estimates, we get

$$\begin{aligned}
&\sup_{t' \le s \le t'+t_0} \|(\Phi[u] - \Phi[v])(s)\|_{H^1} \\
&\le \sqrt{t_0}\left(\sqrt{t_0} + \sqrt{\frac{2}{\nu}}\right)(|V(\rho_*)| + 2MV_1(M)) \sup_{t' \le s \le t'+t_0} \|(u-v)(s)\|_{H^1}.
\end{aligned}$$

Namely, taking t_0 such that (4.8), this yields

$$\sup_{t' \le s \le t'+t_0} \|(\Phi[u] - \Phi[v])(s)\|_{H^1} \le \frac{\varepsilon}{1+\varepsilon} \sup_{t' \le s \le t'+t_0} \|(u-v)(s)\|_{H^1},$$

which means the mapping Φ is the contraction mapping on B_{t_0}.

Consequently, taking

$$t_0 = \min\left\{\tau, \frac{\varepsilon^2}{(1+\varepsilon)^2}\left(\sqrt{\tau}+\sqrt{\frac{2}{\nu}}\right)^{-2}\{|V(\rho_*)| + V_1(M)(|\rho_*|+2(1+\varepsilon)M)\}^{-2}\right\},$$

the mapping Φ becomes the contraction mapping on B_{t_0}. Thus, because of the contraction mapping theorem, there exists a unique solution $u \in B_{t_0}$ to (1.5), (1.11), which means (4.3) is satisfied. Furthermore, the equation (1.5) leads to $\partial_t u \in L^2(t', t'+t_0; L^2)$, and this completes the proof. □

In the rest of this section, we shall prove Theorems 1.2 and 1.4 by using Propositions 3.1 and 3.2, and Lemma 4.1.

Proof of Theorems 1.2 and 1.4 Because both proofs are the same arguments, we only prove Theorem 1.2. We suppose that the initial data satisfies (1.14) and define $\delta_0 > 0$ such that

$$\sqrt{\frac{\tau}{\nu}} W(\delta_0)\delta_0 = c_\tau, \tag{4.9}$$

Then, because $W(\delta)\delta$ is strictly monotonically increasing, these give $\sqrt{C_1} I_1 < \delta_0$, and there exists ε_0 such that $(1+\varepsilon_0)\sqrt{C_1} I_1 = \delta_0$. Employing Lemma 4.1 with $t' = 0$, $\varepsilon = \varepsilon_0$ and $M = \delta_0/(1+\varepsilon_0)$, we obtain the solution $u \in C([-\tau, t_0]; H^1)$ with

$$\sup_{-\tau \le s \le t_0} \|u(s)\|_{H^1} \le \delta_0.$$

Therefore, because of (4.9) and $N(t_0) \le \delta_0$, this solution satisfy (3.1) with $T = t_0$ and Proposition 3.1 gives

$$\sup_{-\tau \le s \le t_0} \|u(s)\|_{H^1} \le \sqrt{C_1} I_1 = \frac{\delta_0}{1+\varepsilon_0}.$$

Furthermore, using this estimate, we can apply Lemma 4.1 with $t' = t_0$, $\varepsilon = \varepsilon_0$ and $M = \delta_0/(1+\varepsilon_0)$ again to obtain the solution $u \in C([t_0 - \tau, 2t_0]; H^1)$ with

$$\sup_{t_0 - \tau \le s \le 2t_0} \|u(s)\|_{H^1} \le \delta_0.$$

Combining this and the fact in the previous step, we get the solution $u \in C([-\tau, 2t_0]; H^1)$ with

$$\sup_{-\tau \le s \le 2t_0} \|u(s)\|_{H^1} \le \delta_0.$$

Therefore, we can utilize Proposition 3.1 with $T = 2t_0$ and get

$$\sup_{-\tau \le s \le 2t_0} \|u(s)\|_{H^1} \le \frac{\delta_0}{1+\varepsilon_0}.$$

Eventually, repeating this argument inductionary, we arrive at the global-in-time solution $u \in C([-\tau, \infty); H^1)$ to (1.5), (1.11). Furthermore, this solution satisfy (3.2) which gives (1.16).

Finally we shall show $\|u(t)\|_{L^\infty} \to 0$ as $t \to \infty$. Because of (1.16), we find $\|\partial_x u(\cdot)\|_{L^2}^2 \in L^1(0, \infty)$. Furthermore, since (3.15) and (3.18), we have

$$\begin{aligned}
\int_0^\infty \left| \frac{d}{dt} \|\partial_x u(t)\|_{L^2}^2 \right| dt &\le \frac{1}{\nu} \int_0^\infty \|\partial_t u(t)\|_{L^2}^2 dt + \nu \int_0^\infty \|\partial_x^2 u(t)\|_{L^2}^2 dt \\
&\quad + \frac{1}{\nu} \int_0^\infty \|\partial_x ((\rho_* + u)V(\rho_* + u_\tau))(t)\|_{L^2}^2 dt \\
&< \infty.
\end{aligned}$$

Therefore, we also find that $\|\partial_x u(\cdot)\|_{L^2}^2 \in W^{1,1}(0,\infty)$, and this gives $\|\partial_x u(t)\|_{L^2}^2 \to 0$ as $t \to \infty$. Consequently, we obtain $\|u(t)\|_{L^\infty} \le \sqrt{2}\|u(t)\|_{L^2}^{1/2}\|\partial_x u(t)\|_{L^2}^{1/2} \to 0$ as $t \to \infty$, and complete the proof. □

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Declarations

Conflict of Interest The author declared that he has no conflict of interest to this work.

References

1. Cole, J.D.: On a quasilinear parabolic equation occurring in aerodynamics. Q. Appl. Math. **9**, 225–236 (1951)
2. Garavello, M., Han, K., Piccoli, B.: Models for vehicular traffic on Networks, AIMS on Applied Mathematics Vol.9, American Institute of Mathematical Sciences, (2016)
3. Hale, J.K., Lunel, S.M.V.: Introduction to functional-differential equations, Applied Mathematical Sciences, 99. Springer-Verlag, New York (1993)
4. Hattori, Y., Nishihara, K.: A note on the stability of rarefaction wave of the Burgers equation. Jpn. J. Ind. Appl. Math. **8**, 85–96 (1991)
5. Herron, I., McCalla, C., Mickens, R.: Traveling wave solutions of Burgers' equation with time delay. Appl. Math. Lett. **107**, 106496 (2020)
6. Hopf, E.: The partial differential equation $u_t + uu_x = \mu u_{xx}$, Communications of. Pure Appl. Math. **3**, 201–230 (1950)
7. Jordan, P.M.: A note on Burgers' equation with time delay: Instability via finite-time blow-up. Phys. Lett. A **372**, 6363–6367 (2008)
8. Kubo, T., Ueda, Y.: Existence theorem for global in time solutions to Burgers equation with a time delay. J. Differential Equations **333**, 184–230 (2022)
9. T. Kubo and Y. Ueda, *Large time behavior of solutions to Burgers equation with a time delay*, to be published in Communications in Mathematical Analysis and Applications (2025)
10. Liu, W.: Asymptotic behavior of solutions of time-delayed Burgers' equation. Discrete and Continuous Dynamical Systems Series B **2**(1), 47–56 (2002)
11. Nishihara, K.: A note on the stability of traveling wave solutions of Burgers' equation, Japan. J. Appl. Math. **2**, 27–35 (1985)
12. Smaoui, N., Mekkaoui, M.: The generalized Burgers equation with and without a time delay. J. Appl. Math. Stoch. Anal. **1**, 73–96 (2004)

13. Tang, Y.: Exponential stability of nonlocal time-delayed Burgers equation, Perspectives in Mathematical Sciences, Interdisciplinary Mathematical. Sciences **9**, 265–274 (2009)
14. Tang, Y., Wang, M.: A remark on exponential stability of time-delayed Burgers equation. Discrete and Continuous Dynamical Systems Series B **12**(1), 219–225 (2009)
15. Tanthanuch, J.: Symmetry analysis of the nonhomogeneous inviscid Burgers equation with delay. Commun Nonlinear Sci Numer Simulat **17**, 4978–4987 (2012)
16. Ueda, Y.: Linear stability of the non-zero equilibrium state for the viscous Burgers equation with time delay. Jpn. J. Ind. Appl. Math. **42**, 2057–2072 (2025)
17. Wazwaz, A.M.: Multiple-front solutions for the Burgers equation and the coupled Burgers equations, Applied Mathematics and Computation, 190. No. **2**, 1198–1206 (2007)