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(Citation)

神戸大学大学院海事科学研究科紀要, 22:1-26

(Issue Date)

2026

(Resource Type)

departmental bulletin paper

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100505627>

(URL)

<https://hdl.handle.net/20.500.14094/0100505627>



Formal asymptotic expansions of solutions of the fourth-order phase field equations

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Received 2026/06/24

Abstract

We study formal asymptotic expansions of solutions of the fourth-order phase field equation (PFE for short). Loreti–March [18] obtained the formal asymptotic expansion up to the order ε^2 of solutions of the system of PDE's equivalent to the PFE. Fei–Liu [12] rigorously prove the convergence of the approximate solutions of the equivalent system to the PFE. Their approximate solutions are given by the asymptotic expansion and they present a procedure to determine the coefficient of the order ε^k ($k \in \mathbb{N} \cup \{0\}$). In this article, we derive the formal asymptotic expansion of solutions of the PFE up to the order ε^4 . It is more precise than that by [18] and the coefficient of ε^k is obtained in a different way from [12]. In addition the formal asymptotic expansion of the radial solutions of the PFE is discussed.

1 Introduction

In this article we study formal asymptotic expansions of solutions of the following fourth-order phase field equation (PFE for short):

$$(1.1) \quad u_t^\varepsilon = -\Delta \left(\Delta u^\varepsilon - \frac{1}{\varepsilon^2} W'(u^\varepsilon) \right) + \frac{1}{\varepsilon^2} W''(u^\varepsilon) \left(\Delta u^\varepsilon - \frac{1}{\varepsilon^2} W'(u^\varepsilon) \right) \quad \text{in } (0, +\infty) \times \mathbb{R}^N,$$

where $N \geq 2$ and $W(s) = \frac{1}{4}(1 - s^2)^2$.

The above equation is the gradient flow of the following functional:

$$(1.2) \quad \mathcal{W}_\varepsilon(u) := \begin{cases} \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} \left(\varepsilon \Delta u - \frac{W'(u)}{\varepsilon} \right)^2 dx & \text{if } u \in L^1(\mathbb{R}^N) \cap W^{2,2}(\mathbb{R}^N), \\ +\infty & \text{otherwise in } L^1(\mathbb{R}^N), \end{cases}$$

This functional is introduced by Bellettini–Paolini [2] to study the De Giorgi conjecture simpler (see also Bellettini–Freguglia–Picenni [1]). It is also understood as an approximation of the Willmore energy defined by

$$(1.3) \quad \mathcal{W}(\Gamma) := \frac{1}{2} \int_\Gamma |H|^2 d\mathcal{H}^{N-1}.$$

Here Γ is the boundary of an open set in $\mathbb{R}^N$, H is the sum of all the principal curvatures (in other words, $(N-1)$ -times mean curvature) of Γ with respect to the inner unit normal of Γ and $\mathcal{H}^{N-1}$ denotes the $(N-1)$ -dimensional Hausdorff measure. Indeed, the term $\varepsilon \Delta u - \frac{W'(u)}{\varepsilon}$ is related to the mean curvature of the level set $\{x \in \mathbb{R}^N \mid u(x) = 0\} (= \{u(\cdot) = 0\} \text{ for short})$ for small $\varepsilon > 0$ in the phase-field theory.

In the study of the fourth-order partial differential equations (PDE's for short), we sometimes consider the equivalent system of the second order PDE's. For example, we treat the following system instead of (1.1):

$$(1.4) \quad \begin{cases} \varepsilon^2 u_t^\varepsilon = \Delta \mu^\varepsilon - \frac{1}{\varepsilon^2} W''(u^\varepsilon) \mu^\varepsilon, \\ \mu^\varepsilon = -\varepsilon^2 \Delta u^\varepsilon + W'(u^\varepsilon). \end{cases}$$

In [18] Loreti and March used the method of matched asymptotic expansion and obtained the formal asymptotic expansions of a pair $(u^\varepsilon, \mu^\varepsilon)$ of solutions of (1.4) (plus a lower order term) for small $\varepsilon > 0$ in the case of $N = 2, 3$. As a result, they showed that the zero level set $\Gamma^\varepsilon(t) := \{u^\varepsilon(t, \cdot) = 0\}$ formally moves by (1.7) below. Bretin–Masnou–Oudet [5] considered the asymptotic expansions of solutions of several modified models for (1.1) and (1.2) and derived the motion laws of the zero level sets of solutions of modified models. They also gave some numerical results (including the case (1.1)). Fei–Liu [12] rigorously proved the convergence of the approximate solutions of (1.4) in $L^\infty(0, T; L^2(\Omega))$ in the case of $N = 2, 3$ if the smooth Willmore flow exists. They construct the approximate solutions based on the asymptotic expansion and present a procedure to determine the coefficient of the order ε^k ($k \in \mathbb{N} \cup \{0\}$).

The main purposes of this article are to directly derive the formal asymptotic expansions up to the order ε^4 of solutions of (1.1) and to confirm that the expansions of a pair $(u^\varepsilon, \mu^\varepsilon)$ of solutions of (1.4) by [18] is also derived from ours. The first motivation of this study is to compare the formal asymptotic expansions of solutions of (1.1) with that of solutions of the initial value problem of the fourth-order heat equation:

$$(1.5) \quad \begin{cases} u_t + (-\Delta)^2 u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \chi_{C_0}(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

where C_0 is a given compact set of $\mathbb{R}^N$, χ_{C_0} denotes the characteristic function for C_0 . This problem is used to define a threshold-type algorithm for the fourth-order geometric motions. See Ishii–Kohsaka–Miyake–Sakakibara [16] for details. In [16] the authors obtained the formal asymptotic expansion up to the order $t^{3/4}$ of solutions of (1.5) near ∂C_0 . We need a higher order expansion of solutions of (1.1) than those of [18] and [5] to compare the asymptotic expansion of solutions of (1.1) with that of (1.5). The second motivation is that the author would like to know the formal asymptotics of the signed distance function to $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$, which appears in the stretched variable of the expansion near $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$. The author thinks that the formal asymptotic expansion for the signed distance function to $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$ suggests the optimal error estimate between $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$ and the Willmore flow. Indeed, in the case of the Allen–Cahn equation:

$$(1.6) \quad u_t - \Delta u + \frac{1}{\varepsilon^2} W'(u) = 0 \quad (0, +\infty) \times \mathbb{R}^N,$$

Ishii [15] obtained the optimal rate of convergence by using the asymptotic expansion of the solution of (1.6) and that of the signed distance function to $\{u(t, \cdot) = 0\}$. Based on these facts, we consider the formal asymptotic expansion of solutions of (1.1) up to the order ε^4 and that of the signed distance function to $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$ up to the order ε^2 . As for this, in [12] Fei and Liu gives a procedure to define the coefficients of the order ε^k ($k \in \mathbb{N}$) in the asymptotic expansions for the solution $(u^\varepsilon, \mu^\varepsilon)$ of (1.4) and the signed distance function to $\Gamma^\varepsilon(t)$. However, the coefficients of ε^k ($k \geq 3$) of u^ε are not explicitly given. In this article we give the coefficients of ε^k for the solution u^ε of (1.1) (somehow) explicitly although we do it up to the order ε^4 . This is one of the differences between their expansion and ours.

This article is organized in the following way. In section 2, we state our assumptions and the method of matched asymptotic expansion developed by Cahn–Elliott–Novick–Cohen [9] and Pego [19]. Then we also introduce the matching conditions between the outer expansion and the inner one. Section 3 is devoted to the formal asymptotic expansion of solutions of (1.1). We give the outer expansion in subsection 3.1 and the inner one in subsection 3.2. Another difference between the result in [12] and ours is an application of Bellettini–Paolini [3, Lemma 3.1, 3.2] to choose the coefficients U_i 's in

subsection 3.2 below. In considering the inner expansion, we obtain the following equation:

$$(1.7) \quad V = -\Delta_{\Gamma(t)}H - H|A|^2 + \frac{1}{2}H^3 \quad \text{on } \Gamma(t), \quad t > 0.$$

Here $\Delta_{\Gamma(t)}$ is the Laplace - Beltrami operator on $\Gamma(t)$, V (resp, $|A|^2$) is the velocity (resp., the sum of the squares of all the principal curvatures) of $\Gamma(t)$ with respect to the inner unit normal of $\Gamma(t)$. This is the gradient flow equation of (1.3) and it will be seen that the level set $\{u^\varepsilon(t, \cdot) = 0\}$ of the solution of (1.1) formally approximates the motion of hypersurface by (1.7). In subsection 3.3 we treat the case of the radially symmetric solution of (1.1). We then see that the expansion in the case $N = 3$ is different from the case $N \neq 3$. This seems to reflect that all the spheres are stationary solutions of (1.7). In section 4 we compare between the formal asymptotic expansion of solutions of (1.1) and those of (1.5).

2 Preliminaries

In this section we state our assumptions and the method of matched asymptotic expansion to the solutions of (1.1).

Before stating our assumptions, we prepare some notations. Fix $\varepsilon > 0$. Let $u^\varepsilon = u^\varepsilon(t, x) : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a classical solution of (1.1). For $t \geq 0$, set $\Gamma^\varepsilon(t) := \{u^\varepsilon(t, \cdot) = 0\}$, $\Omega_+^\varepsilon(t) := \{u^\varepsilon(t, \cdot) > 0\}$ and $\Omega_-^\varepsilon(t) := \{u^\varepsilon(t, \cdot) < 0\}$. Denote by $d^\varepsilon = d^\varepsilon(t, x) : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ the signed distance function to $\Gamma^\varepsilon(t)$ such that $d^\varepsilon(t, \cdot) > 0$ in $\Omega_+^\varepsilon(t)$ for all $t \geq 0$. Let $\{\Gamma(t)\}_{t \geq 0}$ be a family of compact hypersurfaces of $\mathbb{R}^N$ and for each $t \geq 0$, let $D(t)$ be the bounded domain with $\partial D(t) = \Gamma(t)$ and denote by $d_0 = d_0(t, x) : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ the signed distance function to $\Gamma(t)$ such that $d_0(t, x) > 0$ in $D(t)$. We denote by, respectively, Du , D^2u , Δu etc. the gradient, the Hessian and the Laplacian of u etc. with respect to the variable x . We also denote by, respectively, u_t , u_z the derivatives with respect to the variable t and z . Let $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{L^2}$ be, respectively, the inner product in $\mathbb{R}^N$ and that in $L^2(\mathbb{R})$. Set $\|f\|_{L^2} := \sqrt{\langle f, f \rangle_{L^2}}$ for $f \in L^2(\mathbb{R})$.

We make the following assumptions.

- (A1) For all $\varepsilon > 0$, $(u^\varepsilon(t, x), Du^\varepsilon(t, x), D^2u^\varepsilon(t, x)) \rightarrow \exists (u^\varepsilon(0, x), Du^\varepsilon(0, x), D^2u^\varepsilon(0, x))$ as $t \rightarrow 0$. In addition, for any small $\delta > 0$, there is an $\varepsilon_0 > 0$ such that $u^\varepsilon(0, x) \simeq 1$ (resp., $u^\varepsilon(0, x) \simeq -1$) in $\Omega_+^\varepsilon(0) \setminus \{|d^\varepsilon(0, \cdot)| < \delta\}$ (resp., in $\Omega_-^\varepsilon(0) \setminus \{|d^\varepsilon(0, \cdot)| < \delta\}$) for all $\varepsilon \in (0, \varepsilon_0)$.
- (A2) $\sup_{0 < \varepsilon < 1} \mathcal{W}_\varepsilon(u^\varepsilon(0, \cdot)) < +\infty$.
- (A3) For each $\varepsilon > 0$, $\Gamma^\varepsilon(t) = \partial\Omega_+^\varepsilon(t) = \partial\Omega_-^\varepsilon(t)$ for all $t \in [0, T)$ and $\Gamma^\varepsilon(0) = \Gamma(0)$.
- (A4) For each $t \geq 0$, $(\Gamma^\varepsilon(t), \Omega_+^\varepsilon(t)) \rightarrow (\Gamma(t), D(t))$ as $\varepsilon \rightarrow 0$. Moreover, $\Gamma^\varepsilon(t)$ and $\Gamma(t)$ are smooth hypersurfaces for all $t \geq 0$ and they move smoothly with respect to $t \geq 0$.
- (A5) d^ε (resp, d_0) is smooth in a neighborhood of $\bigcup_{t \geq 0} \{t\} \times \Gamma^\varepsilon(t)$ (resp., $\bigcup_{t \geq 0} \{t\} \times \Gamma(t)$) and d^ε satisfies

$$(2.1) \quad d^\varepsilon = d_0 + \varepsilon \bar{d}, \quad \bar{d} := d_1 + \varepsilon d_2 + \varepsilon^2 d_3 + O(\varepsilon^3),$$

where d_l are smooth and bounded functions satisfying $d_l(0, x) = 0$ ($l = 1, 2, 3$).

Remark 2.1. (1) The assumption (A2) will be used to determine U_0 below. Besides we are able to expect that $\mathcal{W}_\varepsilon(u^\varepsilon)$ is an approximation of (1.3), which we will see in Remark 3.3 below.

(2) (A4) implies that $\Omega_+^\varepsilon(t)$ is a bounded domain in $\mathbb{R}^N$ for small $\varepsilon > 0$ and all $t \geq 0$.

(3) It follows from (A5) and the fact $|Dd^\varepsilon|^2 = |Dd|^2 = 1$ that

$$\langle Dd_0, Dd_1 \rangle = 0, \quad |Dd_1|^2 + 2\langle Dd_0, Dd_2 \rangle = 0, \quad \langle Dd_0, Dd_3 \rangle + \langle Dd_1, Dd_2 \rangle = 0.$$

(4) As for the well-posedness of (1.4), Colli and Laurençot studied in [10] the well-posedness of (1.4) with a volume constraint fixing the average of u and in [11] that with both volume and area constraints. In both of the papers they considered (1.4) in a bounded domain.

(5) In (A5), we assume that d^ε is expanded up to the order ε^3 . We need the ε^3 -term in this expansion to derive the equation for d_2 . We will see in Step 5 of subsection 3.2 below that the equation for U_5 includes both of d_2 and d_3 .

Following the method of matched asymptotic expansion developed by Cahn–Elliott–Novick–Cohen [9], Pego [19] etc., we look for the outer- and the inner- expansion of the solution of (1.1). As for the outer expansion away from $\Gamma^\varepsilon(t)$, set

$$(2.2) \quad u^\varepsilon(t, x) = u_0(t, x) + \varepsilon u_1(t, x) + \varepsilon^2 u_2(t, x) + \varepsilon^3 u_3(t, x) + \varepsilon^4 u_4(t, x) + \varepsilon^5 u_5(t, x) + O(\varepsilon^6).$$

On the other hand, since it is known that the width of the transition layer is of order ε (cf. Pego [19]), we introduce a stretched variable $z = d^\varepsilon(t, x)/\varepsilon$. We seek the inner expansion of u^ε near $\Gamma^\varepsilon(t)$:

$$(2.3) \quad u^\varepsilon(t, x) = U^\varepsilon(t, x, z) = U_0(t, x, z) + \varepsilon U_1(t, x, z) + \varepsilon^2 U_2(t, x, z) + \varepsilon^3 U_3(t, x, z) + \varepsilon^4 U_4(t, x, z) + \varepsilon^5 U_5(t, x, z) + O(\varepsilon^6),$$

The expansions (2.2) and (2.3) are related by matching conditions (cf. Caginalp–Fife [8, Appendix]):

$$(2.4) \quad (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)(t, x + \varepsilon z \mathbf{n}) = (U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots)(t, x, z),$$

for x near $\Gamma^\varepsilon(t)$ and εz between $O(\varepsilon)$ and $o(1)$. Here $\mathbf{n} = \mathbf{n}(t, x) (:= Dd^\varepsilon(t, x))$ denotes the inner unit normal to $\Gamma^\varepsilon(t)$ at $x \in \Gamma^\varepsilon(t)$. We use the following notation:

$$u_{i,\pm}(t, x) := \lim_{s \rightarrow \pm 0} u_i(t, x + s \mathbf{n}) \quad \text{for } i = 0, 1, 2, \dots, 5, \quad t > 0 \text{ and } x \in \Gamma^\varepsilon(t).$$

The left-hand side of (2.4) can be expanded in powers of ε as $\varepsilon z \rightarrow 0$ and $z \rightarrow \pm\infty$ as follows:

the left-hand side of (2.4)

$$\begin{aligned} &= u_0(t, x) + \varepsilon(u_1 + z \partial_{\mathbf{n}} u_0)(t, x) + \varepsilon^2 \left(u_2 + z \partial_{\mathbf{n}} u_1 + \frac{1}{2} z^2 \partial_{\mathbf{n}}^2 u_0 \right)(t, x) \\ &\quad + \varepsilon^3 \sum_{k=0}^3 \frac{1}{k!} z^k \partial_{\mathbf{n}}^k u_{3-k}(t, x) + \varepsilon^4 \sum_{k=0}^4 \frac{1}{k!} z^k \partial_{\mathbf{n}}^k u_{4-k}(t, x) + \cdots, \end{aligned}$$

where $\partial_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$. By comparing the both sides of (2.4), the following conditions appear (cf. [19]).

$$\begin{aligned} u_{0,\pm}(t, x) &= \lim_{z \rightarrow \pm\infty} U_0(t, x, z), \quad \lim_{z \rightarrow \pm\infty} (u_{1,\pm} + z \partial_{\mathbf{n}} u_{0,\pm})(t, x) = \lim_{z \rightarrow \pm\infty} U_1(t, x, z), \\ \lim_{z \rightarrow \pm\infty} \left(u_{2,\pm} + z \partial_{\mathbf{n}} u_{1,\pm} + \frac{1}{2} z^2 \partial_{\mathbf{n}}^2 u_{0,\pm} \right)(t, x) &= \lim_{z \rightarrow \pm\infty} U_2(t, x, z), \\ \lim_{z \rightarrow \pm\infty} \sum_{k=0}^3 \frac{1}{k!} z^k \partial_{\mathbf{n}}^k u_{3-k,\pm}(t, x) &= \lim_{z \rightarrow \pm\infty} U_3(t, x, z), \\ \lim_{z \rightarrow \pm\infty} \sum_{k=0}^4 \frac{1}{k!} z^k \partial_{\mathbf{n}}^k u_{4-k,\pm}(t, x) &= \lim_{z \rightarrow \pm\infty} U_4(t, x, z), \dots \end{aligned}$$

for $t > 0$ and $x \in \Gamma^\varepsilon(t)$.

3 Formal asymptotic expansion

This section is devoted to the formal asymptotic expansion of solutions of the PFE (1.1). We give the outer expansion in subsection 3.1 and the inner one in subsection 3.2.

3.1 Outer expansion

Assume that the solution u^ε of (1.1) is expanded by (2.2) away from $\Gamma^\varepsilon(t)$. In this subsection we derive the following:

$$(3.1) \quad u^\varepsilon(t, \cdot) = \begin{cases} 1 + O(\varepsilon^6) & \text{in } \Omega_{\varepsilon,+}(t) \setminus \{|d^\varepsilon(t, \cdot)| < \delta\}, \\ -1 + O(\varepsilon^6) & \text{in } \Omega_{\varepsilon,-}(t) \setminus \{|d^\varepsilon(t, \cdot)| < \delta\}, \end{cases}$$

for all $t \geq 0$ and $\varepsilon > 0$.

We substitute the expansion given by (2.2) the into (1.1). Then we get

$$\begin{aligned} \sum_{k=0}^5 \varepsilon^k u_{k,t} = & - \sum_{k=0}^5 \varepsilon^k \Delta^2 u_k + \frac{1}{\varepsilon^2} \Delta \left\{ \sum_{l=1}^4 \frac{W^{(l)}(u_0)}{(l-1)!} \left(\sum_{k=1}^5 \varepsilon^k u_k \right)^{l-1} \right\} \\ & + \frac{1}{\varepsilon^2} \left\{ \sum_{l=2}^4 \frac{W^{(l)}(u_0)}{(l-2)!} \left(\sum_{k=1}^5 \varepsilon^k u_k \right)^{l-2} \right\} \sum_{k=0}^5 \varepsilon^k \Delta u_k \\ & - \frac{1}{\varepsilon^4} \left\{ \sum_{l=2}^4 \frac{W^{(l)}(u_0)}{(l-2)!} \left(\sum_{k=1}^5 \varepsilon^k u_k \right)^{l-2} \right\} \left\{ \sum_{l=1}^4 \frac{W^{(l)}(u_0)}{(l-1)!} \left(\sum_{k=1}^5 \varepsilon^k u_k \right)^{l-1} \right\} + O(\varepsilon^6). \end{aligned}$$

Since it is seen that the ε^{-4} -term of (1.1) turns to $W''(u)W'(u) = 0$, we have $W''(u_0)W'(u_0) = 0$. Hence we get

$$(3u_0^2 - 1)(u_0^3 - u_0) = 0.$$

By (A1), we have $u_0 = \pm 1$ in $\Omega_{\varepsilon,\pm}(t) \setminus \{|d^\varepsilon(t, \cdot)| < \delta\}$. The ε^{-3} -term is given by $\{W''(u_0)\}^2 u_1 + W'(u_0)W'''(u_0)u_1 = 4u_1 = 0$. Thus $u_1 = 0$. Similarly, from the ε^k -terms ($k = -2, -1, 0, 1$) it follows that $u_2 = u_3 = u_4 = u_5 = 0$. Therefore we obtain (3.1) for all $t \geq 0$ and $\varepsilon > 0$.

3.2 Inner expansion

Let u^ε be a smooth solution of (1.1) represented by (2.3). The purpose of this subsection is to derive the following expansion near $\Gamma^\varepsilon(t)$:

$$(3.2) \quad u(t, x) = U_0(z) + \varepsilon^2(2|A|^2 - H^2)\tilde{U}_2(z) + \varepsilon^3(4|B|^3 - H^3)\tilde{U}_3(z) \\ + \varepsilon^4 U_4(t, x, z) + O(\varepsilon^5),$$

$$(3.3) \quad z = \frac{d^\varepsilon(t, x)}{\varepsilon} = \frac{d_0(t, x) + \varepsilon^2 d_2(t, x) + O(\varepsilon^3)}{\varepsilon}$$

Here $U_0(z) = \tanh \frac{z}{\sqrt{2}}$, $\tilde{U}_2$ is the solution of (3.22), satisfying (3.4) - (3.6), $\tilde{U}_3$ is the solution of (3.25) satisfying (3.4) - (3.6), U_4 and d_2 are given in Step 4 and 5 below, respectively. The ODE's (3.22) and (3.25) are derived in Step 2, 3, respectively. The notations H , $|A|^2$, $|B|^3$ are given after (3.13) below.

Before obtaining (3.2), we give some preliminary observations. Set $U = U^\varepsilon$ and $d = d^\varepsilon$ for simplicity. We add the following assumption.

(A6) For each $z \in \mathbb{R}$,

$$\langle DU(t, x, z), \mathbf{n}(t, x) \rangle = 0 \quad \text{for all } t \geq 0 \text{ and } x \in \Gamma^\varepsilon(t).$$

Recall that $\mathbf{n}(= Dd^\varepsilon)$ is the inner unit normal to $\Gamma^\varepsilon(t)$. This assumption means that $U(t, x + s\mathbf{n}, z) \simeq U(t, x, z)$ for all $t \geq 0$, $x \in \Gamma_\varepsilon(t)$ and $|s| \ll 1$. In other words, the behavior of U with respect to the normal direction is not affected by the variable x , but by the variable $z = d^\varepsilon/\varepsilon$. See Pego [19, p.266]

for this assumption. As for the matching condition between the outer solution and the inner one, it follows from the result in the previous subsection that

$$U(t, x, z) = \begin{cases} 1 + O(\varepsilon^6) & \text{for } z \gg 1, \ 0 < \varepsilon z \ll 1, \\ -1 + O(\varepsilon^6) & \text{for } z \ll -1, \ -1 \ll \varepsilon z < 0. \end{cases}$$

Combining the matching conditions mentioned in section 2, the outer expansion with this, we can derive the following:

$$(3.4) \quad \lim_{z \rightarrow \pm\infty} U_0(t, x, z) = \pm 1, \quad \lim_{z \rightarrow \pm\infty} U_i(t, x, z) = 0 \quad \text{for } i = 1, 2, 3, 4.$$

Since u^ε is a classical solution of (1.1) and we directly treat (1.1), we impose some matching conditions for the derivatives between the outer solution and the inner one:

$$(3.5) \quad \lim_{z \rightarrow \pm\infty} U_{i,zz}(t, x, z) = 0 \quad \text{for } i = 0, 1, 2, 3, 4.$$

In addition, we also need the following conditions:

$$(3.6) \quad U_i(t, x, 0) = 0 \quad \text{for } i = 0, 1, 2, 3, 4,$$

because the fact $z = d^\varepsilon(t, x)/\varepsilon = 0$ implies that $U(t, x, 0) = 0$.

In order to determine the asymptotic expansion of U , we need some preliminary calculations. As for the inner solution U , we see that

$$u_t = U_t + U_z \frac{d_t}{\varepsilon}, \quad Du = DU + U_z \frac{Dd}{\varepsilon}, \quad \Delta u = \Delta U + \frac{1}{\varepsilon} U_z \Delta d + \frac{1}{\varepsilon^2} U_{zz},$$

where we have used (A6). Moreover we calculate that

$$\begin{aligned} \Delta^2 u &= \Delta^2 U + \frac{1}{\varepsilon} \left\{ 2\langle D(\Delta U_z), Dd \rangle + 2\Delta U_z \Delta d + 2\langle D(\Delta d), DU_z \rangle + U_z \Delta^2 d \right\} \\ &\quad + \frac{1}{\varepsilon^2} \{ 2\Delta U_{zz} + 2U_{zz} \langle D(\Delta d), Dd \rangle + U_{zz} (\Delta d)^2 \} + \frac{2}{\varepsilon^3} U_{zzz} \Delta d + \frac{1}{\varepsilon^4} U_{zzzz}. \end{aligned}$$

Here we have used (A6) again.

As for the nonlinear terms, it is observed that

$$\Delta W'(u) = W'''(U) \left(|DU|^2 + \frac{1}{\varepsilon^2} U_z^2 \right) + W''(U) \left(\Delta U + \frac{1}{\varepsilon} U_z \Delta d + \frac{1}{\varepsilon^2} U_{zz} \right).$$

Hence we conclude that

$$\begin{aligned} (3.7) \quad U_t + \frac{1}{\varepsilon} U_z d_t &= -\Delta^2 U - \frac{1}{\varepsilon} \left\{ 2\langle D(\Delta U_z), Dd \rangle + 2\Delta U_z \Delta d + 2\langle DU_z, D(\Delta d) \rangle + U_z \Delta^2 d \right\} \\ &\quad - \frac{1}{\varepsilon^2} \{ 2\Delta U_{zz} + 2U_{zz} \langle D(\Delta d), Dd \rangle + U_{zz} (\Delta d)^2 \} \\ &\quad + \frac{1}{\varepsilon^2} [\{ W''(U) \Delta U + W'''(U) |DU|^2 \} + W''(U) \Delta U] \\ &\quad - \frac{2}{\varepsilon^3} (\mathcal{A}_0 U)_z \Delta d - \frac{1}{\varepsilon^4} \{ (\mathcal{A}_0 U)_{zz} - W''(U) \mathcal{A}_0 U \} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

where $\mathcal{A}_0 u := u_{zz} - W'(u)$. Denote by I_0 the left-hand side of this equation.

For later use, we expand Δd_0 and its derivatives. Recall that d_0 is the signed distance function to $\Gamma(t)$ such that $d_0(t, \cdot) > 0$ in $D(t)$. Let $\kappa_j = \kappa_j(t, x) = \kappa_j(t, \pi(t, x))$ ($j = 1, 2, \dots, N-1$) is the principal curvature of $\Gamma(t)$ with respect to the inner unit normal ($= Dd_0$) and $\pi(t, \cdot)$ is defined by

$$(3.8) \quad |d_0(t, x)| = |x - \pi(t, x)|, \quad \pi(t, x) \in \Gamma(t) \quad \text{for } x \text{ near } \Gamma(t).$$

Then it is seen by the fact $d_0 = \varepsilon z - \varepsilon \bar{d}$ that near $\Gamma(t)$

$$(3.9) \quad \Delta d_0 = -\{H + \varepsilon|A|^2(z - \bar{d}) + \varepsilon^2|B|^3(z - \bar{d})^2 + \varepsilon^3|C|^4(z - \bar{d})^3 + \varepsilon^4|D|^5(z - \bar{d})^4\} \\ + O(|\varepsilon(z - \bar{d})|^5),$$

$$(3.10) \quad D(\Delta d_0) = -[\{DH + \varepsilon(z - \bar{d})D(|A|^2) + \varepsilon^2(z - \bar{d})^2D(|B|^3) + \varepsilon^3(z - \bar{d})^3D(|C|^4) \\ + \varepsilon^4(z - \bar{d})^4D(|D|^5)\} + \{|A|^2 + 2\varepsilon(z - \bar{d})|B|^3 + 3\varepsilon^2(z - \bar{d})^2|C|^4 \\ + 4\varepsilon^3(z - \bar{d})^3|D|^5\}Dd_0] + O(|\varepsilon(z - \bar{d})|^5),$$

$$(3.11) \quad \langle D(\Delta d_0), Dd_0 \rangle = -\{|A|^2 + 2\varepsilon|B|^3(z - \bar{d}) + 3\varepsilon^2|C|^4(z - \bar{d})^2 + 4\varepsilon^3|D|^5(z - \bar{d})^3\} \\ + O(|\varepsilon(z - \bar{d})|^4),$$

$$(3.12) \quad D(\langle D(\Delta d_0), Dd_0 \rangle) = -\{D(|A|^2) + 2\varepsilon(z - \bar{d})D(|B|^3) + 3\varepsilon^2(z - \bar{d})^2D(|C|^4) \\ + 4\varepsilon^3(z - \bar{d})^3D(|D|^5)\} + O(|\varepsilon(z - \bar{d})|^3)$$

$$(3.13) \quad \langle D(\langle D(\Delta d_0), Dd_0 \rangle), Dd_0 \rangle = -\{2|B|^3 + 6\varepsilon|C|^4(z - \bar{d}) + 12\varepsilon^2|D|^5(z - \bar{d})^2\} \\ + O(|\varepsilon(z - \bar{d})|^3).$$

Here

$$H := \sum_{i=1}^{N-1} \kappa_i(t, x), \quad |A|^2 := \sum_{i=1}^{N-1} (\kappa_i(t, x))^2, \quad |B|^3 := \sum_{i=1}^{N-1} (\kappa_i(t, x))^3, \\ |C|^4 := \sum_{i=1}^{N-1} (\kappa_i(t, x))^4, \quad |D|^5 := \sum_{i=1}^{N-1} (\kappa_i(t, x))^5.$$

(cf. Gilbarg–Trudinger [13, Section 14.6]). Moreover we have used $\langle Df, Dd_0 \rangle = 0$ ($f = \kappa_j, H, |A|^2, |B|^3, |C|^4, |D|^5, j = 1, 2, \dots, N-1$) near $\Gamma(t)$ because $\kappa_j(t, \pi(t, x))$ is constant in the normal direction of $\Gamma(t)$

Furthermore, we calculate that

$$(3.14) \quad \Delta^2 d_0 = -[(\Delta H - H|A|^2 + 2|B|^3) + \{\Delta|A|^2 - (|A|^2)^2 - 2H|B|^3 + 6|C|^4\}d_0 \\ + (\Delta|B|^3 - 3|A|^2|B|^3 - 3H|C|^4 + 12|D|^5)d_0^2] + O(|d_0|^3) \\ =: -(\Delta H - H|A|^2 + 2|B|^3) - \varepsilon(z - \bar{d})P_1 - \varepsilon^2(z - \bar{d})^2P_2 + O(|\varepsilon(z - \bar{d})|^3).$$

It is observed from the above calculations that

$$(3.15) \quad \Delta_{\Gamma(t)} H = -\operatorname{div}(D(\Delta d_0) - \langle D(\Delta d_0), Dd_0 \rangle Dd_0) = \Delta H \quad \text{on } \Gamma(t).$$

Remark 3.1. The formula $\Delta_{\Gamma(t)} H = \Delta H$ on $\Gamma(t)$ is natural. Indeed, recalling that $H(t, x) = H(t, \pi(t, x))$ near $\Gamma(t)$ and that $\pi = \pi(t, x)$ is defined by (3.8), we see that $\langle DH, Dd_0 \rangle = 0$ near $\Gamma(t)$ and hence $\Delta_{\Gamma(t)} H = \operatorname{div} DH$.

To select U_i ($i = 0, 1, 2, 3, 4$), we divide our considerations into some steps.

Step 1. Choice of U_0 .

We may assume $U_0 = U_0(z)$ from the first condition of (3.4). We apply (A1), (A2), (3.4) and the fact that the width of transition layer is of order ε to get

$$\mathcal{W}_\varepsilon(u^\varepsilon(0, \cdot)) = \frac{1}{2\varepsilon} \int_{\{|u^\varepsilon(0, \cdot)| < 1\}} \left[\varepsilon \Delta \left\{ U_0 \left(\frac{d_0(0, x)}{\varepsilon} \right) + \varepsilon U_1 \left(0, x, \frac{d_0(0, x)}{\varepsilon} \right) + \dots \right\} \right. \\ \left. - \frac{1}{\varepsilon} W' \left(U_0 \left(\frac{d_0(0, x)}{\varepsilon} \right) + \varepsilon U_1 \left(0, x, \frac{d_0(0, x)}{\varepsilon} \right) + \dots \right) \right]^2 dx$$

$$\begin{aligned}
&= \frac{1}{2\varepsilon} \int_{\Gamma(0)} \int_{-O(\varepsilon)}^{O(\varepsilon)} \left[\left\{ U_{0,z} \left(\frac{d_0(0,x)}{\varepsilon} \right) \Delta d + \frac{1}{\varepsilon} U_{0,zz} \left(\frac{d_0(0,x)}{\varepsilon} \right) \right\} \right. \\
&\quad \left. - \frac{1}{\varepsilon} W' \left(U_0 \left(\frac{d_0(0,x)}{\varepsilon} \right) \right) - W'' \left(U_0 \left(\frac{d_0(0,x)}{\varepsilon} \right) \right) \left(U_1 \left(0, x, \frac{d_0(0,x)}{\varepsilon} \right) + \dots \right) \right. \\
&\quad \left. + O(1) \right]^2 dx + o_\varepsilon(1) = *.
\end{aligned}$$

Here $\lim_{\varepsilon \rightarrow 0} o_\varepsilon(1) = 0$. Set $x = y + rDd(0, x)$ and $r = d(0, x)$, where $y \in \Gamma(0)$ satisfies $|d(0, x)| = |x - y|$. Then we see that

$$\begin{aligned}
* &= \frac{1}{2\varepsilon} \int_{\Gamma(0)} \int_{-O(\varepsilon)}^{O(\varepsilon)} \left[\frac{1}{\varepsilon} \left\{ U_{0,zz} \left(\frac{r}{\varepsilon} \right) - W' \left(U_0 \left(\frac{r}{\varepsilon} \right) \right) \right\} + U_{0,z} \left(\frac{r}{\varepsilon} \right) H + O(1) \right]^2 dr d\mathcal{H}^{N-1} \\
&\quad + o_\varepsilon(1) \\
&= \frac{1}{2} \int_{\Gamma(0)} \int_{\mathbb{R}} \left[\frac{1}{\varepsilon} \{ U_{0,zz}(s) - W'(U_0(s)) \} + U_{0,z}(s)H + O(1) \right]^2 ds d\mathcal{H}^{N-1} + o_\varepsilon(1).
\end{aligned}$$

Here we have used the change of variable $s = r/\varepsilon$. By (A2), U_0 should satisfy

$$(3.16) \quad \mathcal{A}_0 U_0 = 0 \quad \text{in } \mathbb{R}.$$

It is well known that $U_0(z) = \tanh \frac{z}{\sqrt{2}}$ is a unique and monotone increasing solution of this ODE satisfying (3.4) and (3.6) with $i = 0$. Hence we choose $U_0(z) = \tanh \frac{z}{\sqrt{2}}$. Note that U_0 also satisfies (3.5).

Remark 3.2. $U_0, U_{0,zz}$ are odd in $\mathbb{R}$ and $U_{0,z}, U_{0,zzz}$ are even in $\mathbb{R}$.

We look at I_6 since the ε^{-3} -term appears only from I_6 . Indeed, applying Taylor's theorem to I_6 and (3.16), we see that

$$\begin{aligned}
I_6 &= -\frac{1}{\varepsilon^3} \mathcal{A}_1(\mathcal{A}_1 U_1) - \frac{1}{\varepsilon^2} \left[\sum_{k=2}^5 \varepsilon^{k-2} \mathcal{A}_1(\mathcal{A}_1 U_k) \right. \\
&\quad \left. + \mathcal{A}_1 \left\{ \frac{1}{2} W'''(U_0) \left(\sum_{k=1}^5 \varepsilon^{k-1} U_k \right)^2 + \frac{\varepsilon}{6} W^{(4)}(U_0) \left(\sum_{k=1}^5 \varepsilon^{k-1} U_k \right)^3 \right\} \right. \\
&\quad \left. + \left\{ W'''(U_0) \sum_{k=1}^5 \varepsilon^{k-1} U_k + \frac{\varepsilon}{2} W^{(4)}(U_0) \left(\sum_{k=1}^5 \varepsilon^{k-1} U_k \right)^2 \right\} \right. \\
&\quad \left. \cdot \left\{ \sum_{k=1}^5 \varepsilon^{k-1} \mathcal{A}_1 U_k - \frac{\varepsilon}{2} W''' \left(\sum_{k=1}^5 \varepsilon^k U_k \right)^2 - \frac{\varepsilon^2}{6} W^{(4)}(U_0) \left(\sum_{k=1}^5 \varepsilon^k U_k \right)^3 \right\} \right],
\end{aligned}$$

where $\mathcal{A}_1 u := u_{zz} - W''(U_0)u$.

Here and in the sequel, we often solve the following ODE:

$$(3.17) \quad \mathcal{A}_1 u = f \quad \text{in } \mathbb{R},$$

where $f \in L^2(\mathbb{R})$ satisfies

$$(3.18) \quad |f| \leq C(1 + |z|^m) U_{0,z} \quad \text{in } \mathbb{R} \quad \text{for some } C > 0 \text{ and } m \in \mathbb{N} \cup \{0\}.$$

The following lemma plays an important role in solving (3.32). Note that $U_{0,z}$ satisfies $\mathcal{A}_1 U_{0,z} = 0$ in $\mathbb{R}$, $|U_{0,zz}| \leq \sqrt{2} U_{0,z}$ in $\mathbb{R}$ and that $U_{0,z}$ and $U_{0,zz}$ exponentially decay as $z \rightarrow \pm\infty$.

Lemma 3.1. (cf. Bellettini–Paolini [3, Lemma 3.1, 3.2]) *Let $f \in L^2(\mathbb{R})$ satisfy (3.18). If $\langle f, U_{0,z} \rangle_{L^2} = 0$, then there exists a solution $u \in H^2(\mathbb{R})$ of (3.32). Such a solution u is unique up to an addition of $cU_{0,z}$ ($c \in \mathbb{R}$). Moreover, if $f \in H^1(\mathbb{R})$ and f_z satisfies (3.18), the solution u satisfies $u \in H^3(\mathbb{R})$ and*

$$(3.19) \quad |u| + |u_z| \leq C_1(1 + |z|^{m+1})U_{0,z} \quad \text{in } \mathbb{R} \quad \text{for some } C_1 > 0.$$

Step 2. Choice of U_1 .

From the above calculations to I_6 , we get the equation for U_1 :

$$(3.20) \quad \mathcal{A}_1(\mathcal{A}_1 U_1) = 0 \quad \text{in } \mathbb{R}.$$

We may consider $U_1(t, x, z) = g(t, x)\tilde{U}_1(z)$ because $\mathcal{A}_1$ is a linear operator and does not depend on the variables t, x . Thus we look for a solution $\tilde{U}_1$ of (3.20) satisfying (3.4) - (3.6) with $i = 1$. Setting $\tilde{V}_1 = \mathcal{A}_1 \tilde{U}_1$ and applying Lemma 3.1, we have $\tilde{V}_1 = cU_{0,z}$ ($c \in \mathbb{R}$). Hence $\tilde{U}_1$ satisfies $\mathcal{A}_1 \tilde{U}_1 = cU_{0,z}$ in $\mathbb{R}$. We use the orthogonality condition of Lemma 3.1 to obtain $c = 0$ and thus $\tilde{U}_1 \equiv 0$ is a unique solution of $\mathcal{A}_1 u = 0$ in $\mathbb{R}$ satisfying (3.6) with $i = 1$. Therefore we conclude $U_1 \equiv 0$ and it is a solution of (3.20) satisfying (3.4) - (3.6) with $i = 1$.

Note by the above choice of U_1 that

$$\begin{aligned} I_0 &= \frac{1}{\varepsilon} U_{0,z} \sum_{l=0}^2 \varepsilon^l d_{l,t} + \varepsilon U_{2,z} d_{0,t} + O(\varepsilon^2), \quad I_1 = O(\varepsilon^2), \\ I_2 &= -\frac{1}{\varepsilon} U_{0,z} \sum_{l=0}^2 \varepsilon^l \Delta^2 d_l - \varepsilon (2\langle D(\Delta U_{2,z}), \Delta d_0 \rangle + 2\Delta U_{2,z} \Delta d_0 \\ &\quad + 2\langle D U_{2,z}, D(\Delta d_0) \rangle + U_{2,z} \Delta^2 d_0) + O(\varepsilon^2), \\ I_3 &= -\frac{1}{\varepsilon^2} U_{0,zz} \sum_{0 \leq l+m \leq 3} \varepsilon^{l+m} (2\langle D(\Delta d_l), D d_m \rangle + \Delta d_l \Delta d_m) \\ &\quad - U_{2,zz} \sum_{0 \leq l+m \leq 1} \varepsilon^{l+m} (2\langle D(\Delta d_l), D d_m \rangle + \Delta d_l \Delta d_m) \\ &\quad - \varepsilon U_{3,zz} (2\langle D(\Delta d_0), D d_0 \rangle + (\Delta d_0)^2) - 2 \sum_{k=2,3} \varepsilon^{k-2} \Delta U_{k,zz} + O(\varepsilon^2), \\ I_4 &= 2W''(U_0)(\Delta U_2 + \varepsilon \Delta U_3) + O(\varepsilon^2), \\ I_5 &= -\frac{2}{\varepsilon} \sum_{k=2}^4 \varepsilon^{k-2} (\mathcal{A}_1 U_k)_z \Delta d_0 - 2 \sum_{k=2}^3 \varepsilon^{k-2} (\mathcal{A}_1 U_k)_z \Delta d_1 - 2\varepsilon (\mathcal{A}_1 U_2)_z \Delta d_2, \\ &\quad - \varepsilon W'''(U_0) U_2^2 \Delta d_0 + O(\varepsilon^2), \\ I_6 &= -\frac{1}{\varepsilon^2} \sum_{k=2}^5 \varepsilon^{k-2} \mathcal{A}_1(\mathcal{A}_1 U_k) - \frac{1}{2} \mathcal{A}_1(W'''(U_0)(U_2^2 + 2\varepsilon U_2 U_3)) \\ &\quad - W'''(U_0) \{U_2 \mathcal{A}_1 U_2 + \varepsilon(U_2 \mathcal{A}_1 U_3 + U_3 \mathcal{A}_1 U_2)\} + O(\varepsilon^2). \end{aligned}$$

Step 3. Choice of U_2 .

We observe by (3.9), (3.11) and the calculations in the previous step that the ε^{-2} -terms appear from I_3 and I_6 :

$$\begin{aligned} I_3 &= \frac{1}{\varepsilon^2} (2|A|^2 - H^2) U_{0,zz} + \frac{2}{\varepsilon} (2|B|^3 - H|A|^2)(z - \bar{d}) U_{0,zz} + \dots, \\ I_6 &= -\frac{1}{\varepsilon^2} \mathcal{A}_1(\mathcal{A}_1 U_2) - \frac{1}{\varepsilon} \sum_{k=3}^5 \varepsilon^{k-3} \mathcal{A}_1(\mathcal{A}_1 U_k) - \dots. \end{aligned}$$

Hence we have the equation for U_2 :

$$(3.21) \quad \mathcal{A}_1(\mathcal{A}_1 U_2) = (2|A|^2 - H^2)U_{0,zz} \quad \text{in } \mathbb{R}.$$

In the following of this step, we have the solution of this equation under the conditions (3.4) - (3.6) with $i = 2$. Since we may consider $U_2 = (2|A|^2 - H^2)\tilde{U}_2$ ($\tilde{U}_2 = \tilde{U}_2(z)$) by the same reason as in Step 2, we solve the following equation:

$$(3.22) \quad \mathcal{A}_1(\mathcal{A}_1 \tilde{U}_2) = U_{0,zz} \quad \text{in } \mathbb{R}.$$

Set $\tilde{V}_2 := \mathcal{A}_1 \tilde{U}_2$. Then $\tilde{V}_2$ satisfies $\mathcal{A}_1 \tilde{V}_2 = U_{0,zz}$ in $\mathbb{R}$. As $\langle U_{0,zz}, U_{0,z} \rangle_{L^2} = 0$ and $|U_{0,zz}| \leq \sqrt{2}U_{0,z}$, we use Lemma 3.1 to show that $\tilde{V}_2 = \left(\frac{z}{2} + c_0\right)U_{0,z}$ ($c_0 \in \mathbb{R}$) is the solution of this ODE.

Next we solve

$$(3.23) \quad \mathcal{A}_1 \tilde{U}_2 = \left(\frac{z}{2} + c_0\right)U_{0,z} \quad \text{in } \mathbb{R}.$$

We have $c_0 = 0$ from the orthogonality condition of Lemma 3.1. Thus we obtain a unique solution $\tilde{U}_2$ of (3.23) satisfying (3.6). In addition it follows from (3.19) that

$$(3.24) \quad |\tilde{U}_2| \leq C_1(1 + |z|^2)U_{0,z} \quad \text{in } \mathbb{R} \text{ for some } C_1 > 0.$$

Hence (3.4) - (3.5) with $i = 2$ are satisfied by (3.23) with $c_0 = 0$ and (3.24). Therefore we obtain the desired solution $U_2 = (2|A|^2 - H^2)\tilde{U}_2$ of (3.21) satisfying (3.4) - (3.6) with $i = 2$.

Remark 3.3. Now we have $U = U_0 + \varepsilon^2 U_2 + O(\varepsilon^3)$. Then the energy $\mathcal{W}_\varepsilon(u^\varepsilon(0, \cdot))$ is formally calculated in the following:

$$\begin{aligned} \mathcal{W}_\varepsilon(u(0, \cdot)) &= \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} \left[\varepsilon \Delta \left\{ U_0 \left(\frac{d_0(0, x)}{\varepsilon} \right) + \varepsilon^2 U_2 \left(0, x, \frac{d_0(0, x)}{\varepsilon} \right) + \dots \right\} \right. \\ &\quad \left. - \frac{1}{\varepsilon} W' \left(U_0 \left(\frac{d_0(0, x)}{\varepsilon} \right) + \varepsilon^2 U_2 \left(0, x, \frac{d_0(0, x)}{\varepsilon} \right) + \dots \right) \right]^2 dx \\ &= \frac{1}{2\varepsilon} \int_{\Gamma(0)} \int_{-O(\varepsilon)}^{O(\varepsilon)} \left\{ U_{0,z} \left(\frac{d_0(0, x)}{\varepsilon} \right) \Delta d_0(0, x) \right\}^2 dx + O(\varepsilon) = *. \end{aligned}$$

Set $x = y + rDd_0(x)$ and $r = d_0(x)$, where $y \in \Gamma(0)$ satisfies $|d_0(x)| = |x - y|$. Then we see by (3.9) that

$$\begin{aligned} * &= \frac{1}{2\varepsilon} \int_{\Gamma(0)} \int_{-O(\varepsilon)}^{O(\varepsilon)} \left\{ U_{0,z} \left(\frac{r}{\varepsilon} \right) H(0, x) \right\}^2 dr d\mathcal{H}^{N-1} + O(\varepsilon) \\ &= \frac{1}{2} \int_{\Gamma(0)} \int_{\mathbb{R}} \{U_{0,z}(z)H(0, x)\}^2 dz d\mathcal{H}^{N-1} + O(\varepsilon) \\ &= \frac{\|U_{0,z}\|_{L^2(\mathbb{R})}^2}{2} \int_{\Gamma(0)} (H(0, x))^2 d\mathcal{H}^{N-1} + O(\varepsilon). \end{aligned}$$

This means that $\mathcal{W}_\varepsilon$ approximates the $(\|U_{0,z}\|_{L^2(\mathbb{R})}^2\text{-times})$ Willmore energy (1.3) (at least formally).

Since we have chosen U_2 , we can update I_3 and I_6 as follows:

$$\begin{aligned} I_3 &= \frac{2}{\varepsilon} (2|B|^3 - H|A|^2)(z - \bar{d})U_{0,zz} \\ &\quad - \frac{1}{\varepsilon} U_{0,zz} \sum_{1 \leq l+m \leq 3} \varepsilon^{l+m-1} (2\langle D(\Delta d_l), Dd_m \rangle + \Delta d_l \Delta d_m) \\ &\quad + (6|C|^4 - (|A|^2)^2 - 2H|B|^3)(z - \bar{d})^2 U_{0,zz} \end{aligned}$$

$$\begin{aligned}
& -U_{2,zz} \sum_{0 \leq l+m \leq 1} \varepsilon^{l+m} (2\langle D(\Delta d_l), Dd_m \rangle + \Delta d_l \Delta d_m) \\
& + 2\varepsilon(4|D|^5 - |A|^2|B|^3 - H|C|^4)(z - \bar{d})^3 U_{0,zz} \\
& - \varepsilon U_{3,zz} (2\langle D(\Delta d_0), Dd_0 \rangle + (\Delta d_0)^2) - 2 \sum_{k=2,3} \varepsilon^{k-2} \Delta U_{k,zz} + O(\varepsilon^2), \\
I_6 = & -\frac{1}{\varepsilon} \sum_{k=3}^5 \varepsilon^{k-3} \mathcal{A}_1(\mathcal{A}_1 U_k) - \frac{1}{2} \mathcal{A}_1(W''''(U_0)(U_2^2 + 2\varepsilon U_2 U_3)) \\
& - W''''(U_0)\{U_2 \mathcal{A}_1 U_2 + \varepsilon(U_2 \mathcal{A}_1 U_3 + U_3 \mathcal{A}_1 U_2)\} + O(\varepsilon^2).
\end{aligned}$$

The following lemma is needed for later use.

Lemma 3.2. *Let $\tilde{U}_2$ be a solution of (3.23). Then $\tilde{U}_2$ and $\tilde{U}_{2,zz}$ are odd and $\tilde{U}_{2,z}$ and $\tilde{U}_{2,zzz}$ are even in $\mathbb{R}$.*

Proof. Recall that $\tilde{U}_2$ satisfies

$$(3.25) \quad \mathcal{A}_1 \tilde{U}_2 = \frac{z}{2} U_{0,z} \quad \text{in } \mathbb{R}, \quad \lim_{z \rightarrow \pm\infty} \tilde{U}_2(z) = 0, \quad \tilde{U}_2(0) = 0.$$

Since $W''(U_0(-z)) = W''(U_0(z))$ and $U_{0,z}$ is even, $\tilde{U}_2$ satisfies

$$\tilde{U}_{2,zz}(-z) - W''(U_0(z))\tilde{U}_2(-z) = -\frac{z}{2} U_{0,z}(z).$$

Set $V(z) := -\tilde{U}_2(-z)$. Then V satisfies (3.25). Thus $V(z) = \tilde{U}_2(z)$ by Lemma 3.1. As $\tilde{U}_2$ is smooth, the assertion of Lemma 3.2 holds. $\square$

To have explicit form of $\tilde{U}_2$, we set $\tilde{U}_2(z) := v_2(z)U_{0,z}$ and solve the following ODE of v_2 :

$$(3.26) \quad v_{2,zz} - 2\sqrt{2}U_0 v_{2,z} = \frac{z}{2} \quad \text{in } \mathbb{R}, \quad v_2(-z) = -v_2(z), \quad |v_2(z)| \leq C(1 + |z|^2) \quad \text{for } z \in \mathbb{R}.$$

Here the second and the third condition follows from (3.24) and Lemma 3.2. We may explicitly solve (3.26). However we omit the detail since the actual calculations seems to be complicated.

Step 4. Choices of d_0 and U_3 .

The ε^{-1} -terms appear from I_0 , I_2 , I_3 , I_5 and I_6 . As for I_2 and I_3 we have

$$\begin{aligned}
I_2 = & \frac{1}{\varepsilon} U_{0,z} (\Delta H - H|A|^2 + 2|B|^3) - \{\Delta^2 d_1 - P_1(z - \bar{d})\} U_{0,z} - \cdots, \\
I_3 = & \frac{2}{\varepsilon} (2|B|^3 - H|A|^2)(z - d_1 - \varepsilon d_2) U_{0,zz} \\
& + \frac{2}{\varepsilon} (\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1) U_{0,zz} + \cdots.
\end{aligned}$$

Here we have used $\bar{d} = d_1 + \varepsilon d_2 + \varepsilon^2 d_3 + O(\varepsilon^3)$. Since $\mathcal{A}_1 U_2 = \frac{1}{2}(2|A|^2 - H^2)z U_{0,z}$, we have

$$(\mathcal{A}_1 U_2)_z = \frac{1}{2}(2|A|^2 - H^2)(U_{0,z} + z U_{0,zz}).$$

Using (3.9) and this equation, we observe that

$$I_5 = \frac{1}{\varepsilon} \{H + \varepsilon(z - \bar{d})|A|^2 + \varepsilon^2(z - \bar{d})^2|B|^3\} (2|A|^2 - H^2)(U_{0,z} + z U_{0,zz}) + \cdots.$$

It is seen by (3.21) that

$$I_6 = -\frac{1}{\varepsilon} \mathcal{A}_1(\mathcal{A}_1 U_3) - \sum_{k=4}^5 \varepsilon^{k-4} \mathcal{A}_1(\mathcal{A}_1 U_k) - \cdots.$$

Note that $I_4 = O(1)$. Consequently we obtain the equation for U_3 from the ε^{-1} -term:

$$(3.27) \quad \mathcal{A}_1(\mathcal{A}_1 U_3) = (-d_t + \Delta H + H|A|^2 - H^3 + 2|B|^3)U_{0,z} + (4|B|^3 - H^3)zU_{0,zz} \\ + 2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}U_{0,zz}.$$

In addition the conditions (3.4) - (3.6) with $i = 3$ are imposed. Set $U_3 := U_{3,1} + U_{3,2}$ and let $U_{3,1}$, $U_{3,2}$ satisfy, respectively,

$$(3.28) \quad \mathcal{A}_1(\mathcal{A}_1 U_{3,1}) = (-d_{0,t} + \Delta H + H|A|^2 - H^3 + 2|B|^3)U_{0,z} + (4|B|^3 - H^3)zU_{0,zz},$$

$$(3.29) \quad \mathcal{A}_1(\mathcal{A}_1 U_{3,2}) = 2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}U_{0,zz}.$$

Based on Lemma 3.1, the following is the solvability condition to find $\mathcal{A}_1 U_{3,1}$:

$$\langle \text{RHS of (3.28)}, U_{0,z} \rangle_{L^2} = 0.$$

It is seen by $\langle zU_{0,zz}, U_{0,z} \rangle_{L^2} = -\frac{1}{2}\|U_{0,z}\|_{L^2}^2$ that

$$\langle \text{RHS of (3.28)}, U_{0,z} \rangle = \left(-d_{0,t} + \Delta H + H|A|^2 - \frac{1}{2}H^3 \right) \|U_{0,z}\|_{L^2}^2.$$

Therefore using (3.15) and the fact $V = -d_t$, we get

$$(3.30) \quad \langle \text{RHS of (3.28)}, U_{0,z} \rangle_{L^2} = 0 \iff V = -\Delta_{\Gamma(t)}H - H|A|^2 + \frac{1}{2}H^3.$$

Hence $\Gamma(t)$ moves by (1.7) and (3.28) is reduced to

$$(3.31) \quad \mathcal{A}_1(\mathcal{A}_1 U_{3,1}) = (4|B|^3 - H^3) \left(\frac{1}{2}U_{0,z} + zU_{0,zz} \right).$$

Similarly to Step 2, we may assume that $U_{3,1} = (4|B|^3 - H^3)\tilde{U}_{3,1}$ and we solve the following equation:

$$(3.32) \quad \mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{3,1}) = \frac{1}{2}U_{0,z} + zU_{0,zz} \quad \text{in } \mathbb{R}.$$

Set $\tilde{V}_{3,1} = \mathcal{A}_1 \tilde{U}_{3,1}$ and find a solution $\tilde{V}_{3,1}$ of the ODE $\mathcal{A}_1 \tilde{V}_{3,1} = \frac{1}{2}U_{0,z} + zU_{0,zz}$ in $\mathbb{R}$. Lemma 3.1 yields that $\tilde{V}_{3,1} = \left(\frac{z^2}{4} + c_1 \right) U_{0,z}$ ($c_1 \in \mathbb{R}$) is the solution of this ODE.

Next we solve

$$(3.33) \quad \mathcal{A}_1 \tilde{U}_{3,1} = \tilde{V}_{3,1} = \left(\frac{z^2}{4} + c_1 \right) U_{0,z}.$$

Note that the right-hand side of this equation satisfies (3.18) with $m = 2$. It is seen by the orthogonality condition of Lemma 3.1 that

$$\left\langle \left(\frac{z^2}{4} + c_1 \right) U_{0,z}, U_{0,z} \right\rangle_{L^2} = 0 \iff c_1 = -\frac{\|zU_{0,z}\|_{L^2(\mathbb{R})}^2}{4\|U_{0,z}\|_{L^2(\mathbb{R})}^2}.$$

Thus by Lemma 3.1, we obtain a unique solution $\tilde{U}_{3,1}$ of (3.33) with this c_1 satisfying (3.6) with $i = 3$. The (3.4) - (3.5) with $i = 3$ follow from (3.19) and (3.33). Therefore we conclude that $U_{3,1} = (4|B|^3 - H^3)\tilde{U}_{3,1}$ is a solution of (3.28) satisfying (3.4) - (3.6). Note that $\tilde{U}_{3,1}$ is even in $\mathbb{R}$ by the same argument as the proof of Lemma 3.2.

In the case of $U_{3,2}$, we get

$$U_{3,2} = 2\{\langle DH, Dd_1 \rangle + \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}\tilde{U}_2$$

by a similar way to the case of U_2 . Here $\tilde{U}_2(= \tilde{U}_2(z))$ is the solution of (3.25) satisfying (3.4) - (3.6). In fact, $U_{3,2}$ (hence U_3) will be determined in the next step since $U_{3,2}$ includes d_1 .

We are able to update $I_0 - I_6$ as follows:

$$\begin{aligned}
I_0 &= U_{0,z}d_{1,t} + \varepsilon(U_{0,z}d_{2,t} + U_{2,z}d_{0,t}) + O(\varepsilon^2), \\
I_2 &= -\{\Delta^2 d_1 - (z - d_1 - \varepsilon d_2)P_1\}U_{0,z} - \varepsilon\{\Delta^2 d_2 - (z - \bar{d})^2 P_2\}U_{0,z} \\
&\quad - \varepsilon\{2\langle D(\Delta U_{2,z}), D(\Delta d_0) \rangle + 2\Delta U_{2,z}\Delta d_0 + 2\langle DU_{2,z}, D(\Delta d_0) \rangle + U_{2,z}\Delta^2 d_0\} + O(\varepsilon^2), \\
I_3 &= -2d_2(2|B|^3 - H|A|^2)U_{0,zz} + 2(z - d_1 - \varepsilon d_2)U_{0,zz}\{\langle D(|A|^2), Dd_1 \rangle + |A|^2\Delta d_1\} \\
&\quad + 2\varepsilon(z - \bar{d})^2 U_{0,zz}\{\langle D(|B|^3), Dd_1 \rangle + |B|^3\Delta d_1\} \\
&\quad + (6|C|^4 - (|A|^2)^2 - 2H|B|^3)(z - \bar{d})^2 U_{0,zz} \\
&\quad - U_{0,zz}\{2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_1), Dd_1 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 + (\Delta d_1)^2\} \\
&\quad - \varepsilon U_{0,zz} \sum_{l+m=3} (2\langle D(\Delta d_l), Dd_m \rangle + \Delta d_l \Delta d_m) \\
&\quad - (U_{2,zz} + \varepsilon U_{3,zz})(2|A|^2 - H^2) - 2\varepsilon(z - \bar{d})U_{2,zz}(2|B|^3 - H|A|^2) \\
&\quad - \varepsilon U_{2,zz} \sum_{l+m=1} (2\langle D(\Delta d_l), Dd_m \rangle + \Delta d_l \Delta d_m) + \varepsilon(8|D|^5 - 2|A|^2|B|^3 - 2H|C|^4)z^3 U_{0,zz} \\
&\quad - 2\Delta U_{2,zz} - 2\varepsilon\Delta U_{3,zz} + O(\varepsilon^2), \\
I_4 &= 2W''(U_0)(\Delta U_2 + \varepsilon\Delta U_3) + O(\varepsilon^2) \\
I_5 &= |A|^2(2|A|^2 - H^2)(z - d_1 - \varepsilon d_2)(U_{0,z} + zU_{0,zz}) - 2(\mathcal{A}_1 U_3)_z(H + \varepsilon(z - \bar{d})|A|^2) \\
&\quad - 2(\mathcal{A}_1 U_2)_z\Delta d_1 + \varepsilon(z - \bar{d})^2|B|^3(2|A|^2 - H^2)(U_{0,z} + zU_{0,zz}) \\
&\quad - 2\varepsilon \sum_{l=0}^2 (\mathcal{A}_1 U_{4-l})_z\Delta d_l - \varepsilon W'''(U_0)U_2^2\Delta d_0 + O(\varepsilon^2), \\
I_6 &= -\mathcal{A}_1(\mathcal{A}_1 U_4) - \varepsilon\mathcal{A}_1(\mathcal{A}_1 U_5) - \frac{1}{2}\mathcal{A}_1(W'''(U_0)U_2^2) \\
&\quad - W'''(U_0)\{U_2\mathcal{A}_1 U_2 + \varepsilon(U_2\mathcal{A}_1 U_3) + U_3\mathcal{A}_1 U_2\} + O(\varepsilon^2).
\end{aligned}$$

Step 5. Choices of d_1 and U_4 .

We note by (3.25) that

$$\begin{aligned}
-2\Delta U_{2,zz} + 2W''(U_0)\Delta U_2 &= -2\Delta(2|A|^2 - H^2)\{\tilde{U}_{2,zz} - W''(U_0)\tilde{U}_2\} \\
&= -\Delta(2|A|^2 - H^2)zU_{0,z}.
\end{aligned}$$

Since U_3 satisfies

$$\begin{aligned}
\mathcal{A}_1 U_3 &= (4|B|^3 - H^3)\left(\frac{z^2}{4} + c_1\right)U_{0,z} \\
&\quad + 2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}zU_{0,z},
\end{aligned}$$

we get

$$\begin{aligned}
(\mathcal{A}_1 U_3)_z &= (4|B|^3 - H^3)\left\{\frac{z}{2}U_{0,z} + \left(\frac{z^2}{4} + c_1\right)U_{0,zz}\right\} \\
&\quad + 2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}(U_{0,z} + zU_{0,zz}).
\end{aligned}$$

Thus from (3.25) I_5 turns to

$$\begin{aligned}
I_5 &= |A|^2(2|A|^2 - H^2)(z - d_1)(U_{0,z} + zU_{0,zz}) + (-2|A|^2 + H^2)(U_{0,z} + zU_{0,zz})\Delta d_1 \\
&\quad + \left[(4|B|^3 - H^3)\left\{\frac{z}{2}U_{0,z} + \left(\frac{z^2}{4} + c_1\right)U_{0,zz}\right\}\right.
\end{aligned}$$

$$\begin{aligned}
& -2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}(U_{0,z} + zU_{0,zz}) \Big] H \\
& + O(\varepsilon).
\end{aligned}$$

Besides notice that

$$W'''(U_0)U_2\mathcal{A}_1U_2 = \frac{1}{2}(2|A|^2 - H^2)^2W'''(U_0)z\tilde{U}_2U_{0,z}.$$

Consequently we have the equation for U_4 from the ε^0 -term:

$$\begin{aligned}
& \mathcal{A}_1(\mathcal{A}_1U_4) \\
& = -U_{0,z}d_{1,t} - \{\Delta^2d_1 - (z - d_1)P_1\}U_{0,z} - 2d_2(2|B|^3 - H|A|^2)U_{0,zz} \\
& \quad + 2(z - d_1)\{\langle D(|A|^2), Dd_1 \rangle + |A|^2\Delta d_1\}U_{0,zz} \\
& \quad + (z - d_1 - \varepsilon d_2)^2U_{0,zz}(6|C|^4 - (|A|^2)^2 - 2H|B|^3) \\
& \quad - U_{0,zz}\{2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_1), Dd_1 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 + (\Delta d_1)^2\} \\
& \quad + (2|A|^2 - H^2)U_{2,zz} - zU_{0,z}\Delta(2|A|^2 - H^2) \\
& \quad + (2|A|^2 - H^2)|A|^2(z - d_1)(U_{0,z} + zU_{0,zz}) + (-2|A|^2 + H^2)(U_{0,z} + zU_{0,zz})\Delta d_1 \\
& \quad + \left[(4|B|^3 - H^3) \left\{ \frac{z}{2}U_{0,z} + \left(\frac{z^2}{4} + c_1 \right) U_{0,zz} \right\} \right. \\
& \quad \left. - 2\{\langle DH, Dd_1 \rangle - \langle D(\Delta d_1), Dd_0 \rangle + H\Delta d_1 - d_1(2|B|^3 - H|A|^2)\}(U_{0,z} + zU_{0,zz}) \right] H \\
& \quad - \frac{1}{2}(2|A|^2 - H^2)^2 \left\{ \mathcal{A}_1(W'''(U_0)\tilde{U}_2^2) + W'''(U_0)z\tilde{U}_2U_{0,z} \right\}.
\end{aligned}$$

Recalling Lemma 3.2 and Remark 3.2, we observe that

$$(3.34) \quad \langle \tilde{U}_{2,zz}, U_{0,z} \rangle_{L^2} = 0, \quad \langle zU_{0,z}U_{0,z} \rangle_{L^2} = 0, \quad \langle zU_{0,zz}U_{0,z} \rangle_{L^2} = -\frac{1}{2}\|U_{0,z}\|_{L^2}^2,$$

$$(3.35) \quad \langle z^2U_{0,zz}, U_{0,z} \rangle_{L^2} = 0, \quad \langle U_{0,zz}, U_{0,z} \rangle_{L^2} = 0,$$

$$(3.36) \quad \langle \mathcal{A}_1(W'''(U_0)\tilde{U}_2^2), U_{0,z} \rangle_{L^2} = 6\langle U_0\tilde{U}_2^2, \mathcal{A}_1U_{0,z} \rangle_{L^2} = 0,$$

$$(3.37) \quad \langle W'''(U_0)z\tilde{U}_2U_{0,z}, U_{0,z} \rangle_{L^2} = 6\langle zU_0\tilde{U}_2U_{0,z}, U_{0,z} \rangle_{L^2} = 0.$$

Thus the orthogonality condition in Lemma 3.2 yields the equation for d_1 :

$$\begin{aligned}
& -d_{1,t} - \Delta^2d_1 + H\langle D(\Delta d_1), Dd_0 \rangle - \frac{1}{2}H^2\Delta d_1 - \langle HDH + D(|A|^2), Dd_1 \rangle \\
& + \left\{ -P_1 - 3|C|^4 - \frac{1}{2}(|A|^2)^2 + H|B|^3 + H^2|A|^2 + 2|B|^3 - H|A|^2 \right\} d_1 = 0.
\end{aligned}$$

Since we impose $d_1(0, \cdot) = 0$ in $\mathbb{R}^N$, we can take $d_1 \equiv 0$. Thus the equation for U_4 turns to the following:

$$\begin{aligned}
\mathcal{A}_1(\mathcal{A}_1U_4) = & \left\{ P_1 + \|A\|^2 - \Delta(2|A|^2 - H^2) \right\} + \frac{1}{2}H(4|B|^3 - H^3) \Big\} zU_{0,z} \\
& - \{2(2|B|^3 - H|A|^2)d_2 + 2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 \\
& \quad - c_1H(4|B|^3 - H^3)\}U_{0,zz} \\
& + \left\{ 6|C|^4 + (|A|^2)^2 - H \left(|B|^3 - H|A|^2 + \frac{1}{4}H^3 \right) \right\} z^2U_{0,zz} \\
& + (2|A|^2 - H^2)^2 \left\{ \tilde{U}_{2,zz} - \frac{1}{2}\mathcal{A}_1(W'''(U_0)\tilde{U}_2^2) - \frac{1}{2}W'''(U_0)z\tilde{U}_2U_{0,z} \right\}.
\end{aligned}$$

Notice that $U_3 = U_{3,1}$ because $U_{3,2} = 0$ due to $d_1 = 0$.

We decompose $U_4 = U_{4,1} + U_{4,2} + U_{4,3} + U_{4,4}$ and assume that $U_{4,i}$ ($i = 1, 2, 3, 4$) satisfies

$$(3.38) \quad \mathcal{A}_1(\mathcal{A}_1 U_{4,1}) = \left\{ P_1 + \{|A|^2 - \Delta(2|A|^2 - H^2)\} + \frac{1}{2}H(4|B|^3 - H^3) \right\} z U_{0,z},$$

$$(3.39) \quad \mathcal{A}_1(\mathcal{A}_1 U_{4,2}) = -\{2(2|B|^3 - H|A|^2)d_2 + 2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 - c_1 H(4|B|^3 - H^3)\} U_{0,zz}$$

$$(3.40) \quad \mathcal{A}_1(\mathcal{A}_1 U_{4,3}) = \left\{ 6|C|^4 + (|A|^2)^2 - H \left(|B|^3 - H|A|^2 + \frac{1}{4}H^3 \right) \right\} z^2 U_{0,zz},$$

$$(3.41) \quad \mathcal{A}_1(\mathcal{A}_1 U_{4,4}) = (2|A|^2 - H^2)^2 \left\{ \tilde{U}_{2,zz} - \frac{1}{2}\mathcal{A}_1(W'''(U_0)\tilde{U}_2^2) - \frac{1}{2}W'''(U_0)z\tilde{U}_2 U_{0,z} \right\}.$$

First we choose $U_{4,1}$. For this purpose, we set

$$U_{4,1} = \left[P_1 + \{|A|^2 - \Delta(2|A|^2 - H^2)\} + \frac{1}{2}H(4|B|^3 - H^3) \right] \tilde{U}_{4,1}$$

and hence $\tilde{U}_{4,1}$ satisfies $\mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{4,1}) = z U_{0,z}$ in $\mathbb{R}$. It follows from (3.25) that $\mathcal{A}_1 \tilde{U}_{4,1} = 2\tilde{U}_2$ in $\mathbb{R}$. Since $\tilde{U}_2$ is odd (cf. Lemma 3.2) and satisfies (3.23), applying Lemma 3.1 we are able to choose a unique solution of the equation for $\tilde{U}_{4,1}$ satisfying (3.4) - (3.6). Thus $U_{4,1}$ is determined. Next it follows from (3.22) that

$$U_{4,2} = -\{2(2|B|^3 - H|A|^2)d_2 + 2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 + c_1 H(4|B|^3 - H^3)\} \tilde{U}_2,$$

where $\tilde{U}_2$ is a solution of (3.22) satisfying (3.4) - (3.6).

As for $U_{4,3}$, set

$$U_{4,3} = \left\{ 6|C|^4 + (|A|^2)^2 - H \left(|B|^3 - H|A|^2 + \frac{1}{4}H^3 \right) \right\} \tilde{U}_{4,3}.$$

Then $\tilde{U}_{4,3}$ satisfies $\mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{4,3}) = z^2 U_{0,zz}$ in $\mathbb{R}$. Looking at the right-hand side of this equation, we define

$$\tilde{V}_{4,3} = \tilde{V}_{4,3}(z) := \left(\frac{z^3}{6} + c_2 \right) U_{0,z} \quad (d_0 \in \mathbb{R}).$$

Then we see that $\tilde{V}_{4,3}$ satisfies $\mathcal{A}_1 \tilde{V}_{4,3} = z U_{0,z} + z^2 U_{0,zz}$ in $\mathbb{R}$. As $\tilde{U}_2$ satisfies (3.25), we observe that $\bar{V}_{4,3} := \tilde{V}_{4,3} - 2\tilde{U}_2$ is a solution of

$$\mathcal{A}_1 \bar{V}_{4,3} = z^2 U_0 \quad \text{in } \mathbb{R}.$$

Next we solve the following equation:

$$\mathcal{A}_1 \tilde{U}_{4,3} = \left(\frac{z^3}{6} + c_2 \right) U_{0,z} - 2\tilde{U}_2 \quad \text{in } \mathbb{R}.$$

We use the orthogonality condition of Lemma 3.1 to have $c_2 = 0$ and hence there is a unique solution $\tilde{U}_{4,3}$ of the above equation with $c_2 = 0$ and it satisfies (3.4) - (3.6). Thus we have determined $U_{4,3}$.

Finally setting $U_{4,4} := (2|A|^2 - H^2)^2 \tilde{U}_{4,4}$, we choose $\tilde{U}_{4,4}$ as a solution of

$$(3.42) \quad \mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{4,4}) = \tilde{U}_{2,zz} - \frac{1}{2}\mathcal{A}_1(W'''(U_0)\tilde{U}_2^2) - \frac{1}{2}W'''(U_0)z\tilde{U}_2 U_{0,z} =: f_0 \quad \text{in } \mathbb{R}.$$

Since f_0 satisfies $\langle f_0, U_{0,z} \rangle_{L^2} = 0$ by (3.45) - (3.37), there is a solution $\tilde{V}_{4,4}$ of the equation $\mathcal{A}_1 \tilde{V}_{4,4} = f_0$ in $\mathbb{R}$ by Lemma 3.1. Moreover, the $\tilde{V}_{4,4} + c_3 U_{0,z}$ ($c_3 \in \mathbb{R}$) is also a solution of the same equation. By the fact $|f_0| \leq C_4(1 + |z|^4)U_{0,z}$ for $z \in \mathbb{R}$ and some $C_4 > 0$ and Lemma 3.1 we have

$$|\tilde{V}_{4,4}| \leq C_5(1 + |z|^5)U_{0,z} \quad \text{for } z \in \mathbb{R} \text{ and some } C_5 > 0.$$

As f_0 is odd in $\mathbb{R}$, we can show the oddness of $\tilde{V}_{4,4}$ by the same way as the proof of Lemma 3.2.

Next we solve $\mathcal{A}_1 U_{4,4} = \tilde{V}_{4,4} + c_3 U_{0,z}$ in $\mathbb{R}$, that is (3.42). We easily see by the orthogonality condition of Lemma 3.1 that $\langle \tilde{V}_{4,4} + c_3 U_{0,z}, U_{0,z} \rangle_{L^2} = c_3 \|U_{0,z}\|_{L^2}^2 = 0$, which implies $c_3 = 0$. Therefore we have a unique solution $\tilde{U}_{4,4}$ of (3.42) satisfying (3.4) - (3.6).

Consequently we obtain

$$(3.43) \quad \begin{aligned} U_4 = & \left\{ P_1 + (|A|^2 - \Delta)(2|A|^2 - H^2) + \frac{1}{2}H(4|B|^3 - H^3) \right\} \tilde{U}_{4,1} \\ & - \{2(2|B|^3 - H|A|^2)d_2 + 2\langle DH, Dd_2 \rangle + 2\langle D(\Delta d_2), Dd_0 \rangle + 2H\Delta d_2 \\ & - c_1 H(4|B|^3 - H^3)\} \tilde{U}_2 \\ & + \left\{ 6|C|^4 + (|A|^2)^2 - H \left(|B|^3 - H|A|^2 + \frac{1}{4}H^3 \right) \right\} \tilde{U}_{4,3} \\ & + (2|A|^2 - H^2)^2 \tilde{U}_{4,4}. \end{aligned}$$

We have the following lemma by the same way as in the proof of Lemma 3.2.

Lemma 3.3. *For each $i = 1, 3, 4$, $\tilde{U}_{4,i}$ is odd in $\mathbb{R}$.*

To prove this lemma, we use $\mathcal{A}_1(W'''(U_0)\tilde{U}_2^2)$ and $W'''(U_0)z\tilde{U}_2U_{0,z}$ are odd in $\mathbb{R}$.

From the above result, we can update I_i ($i = 0, 2, 3, 4, 5, 6$):

$$\begin{aligned} I_0 &= \varepsilon(U_{0,z}d_{2,t} + U_{2,z}d_{0,t}) + O(\varepsilon^2), \\ I_2 &= -\varepsilon(d_2P_1 + \Delta^2d_2 - z^2P_2)U_{0,z} - \varepsilon\{2\langle D(\Delta U_{2,z}), DH + |A|^2Dd_0 \rangle - 2H\Delta U_{2,z} \\ &\quad + 2\langle DU_{2,z}, DH + |A|^2Dd_0 \rangle - U_{2,z}(\Delta H - H|A|^2 + 2|B|^3)\} + O(\varepsilon^2), \\ I_3 &= -2\varepsilon d_2\{6|C|^4 - (|A|^2)^2 - 2H|B|^3\}zU_{0,zz} \\ &\quad + 2\varepsilon U_{0,zz}(\langle DH + |A|^2Dd_0, Dd_3 \rangle - \langle D(\Delta d_3), Dd_0 \rangle + H\Delta d_3) \\ &\quad - 2\varepsilon z(U_{2,zz} + U_{3,zz})(2|B|^3 - H|A|^2) \\ &\quad + 2\varepsilon(4|D|^5 - |A|^2|B|^3 - H|C|^4)z^3U_{0,zz} - 2\varepsilon\Delta U_{3,zz} + O(\varepsilon^2), \\ I_4 &= 2\varepsilon W''(U_0)\Delta U_3 + O(\varepsilon^2) \\ I_5 &= -\varepsilon d_2|A|^2(2|A|^2 - H^2)(U_{0,z} + zU_{0,zz}) - 2\varepsilon z(\mathcal{A}_1U_3)_z|A|^2 \\ &\quad - 2\varepsilon \sum_{l=0,2} (\mathcal{A}_1U_{4-l})_z\Delta d_l + \varepsilon W'''(U_0)U_2^2H + O(\varepsilon^2), \\ I_6 &= -\varepsilon\mathcal{A}_1(\mathcal{A}_1U_5) - \varepsilon W'''(U_0)(U_2\mathcal{A}_1U_3 + U_3\mathcal{A}_1U_2) + O(\varepsilon^2). \end{aligned}$$

Step 5. Choice of d_2 .

From the updated I_i 's, the ε -term turns to

$$\begin{aligned} \mathcal{A}_1(\mathcal{A}_1U_5) = & -\{d_{2,t} + \Delta^2d_2 + P_1d_2 + |A|^2(2|A|^2 - H^2)d_2\}U_{0,z} + P_2(z^2U_{0,z}) \\ & + 2U_{0,zz}(\langle DH + |A|^2Dd_0, Dd_3 \rangle - \langle D(\Delta d_3), Dd_0 \rangle + H\Delta d_3) \\ & - [2\{6|C|^4 - (|A|^2)^2 - 2H|B|^3\} + d_2|A|^2(2|A|^2 - H^2)]zU_{0,zz} \\ & + 2(4|D|^5 - |A|^2|B|^3 - H|C|^4)z^3U_{0,zz} \\ & - \{2\langle D\{(\Delta + 1)(2|A|^2 - H^2)\}, DH + |A|^2Dd_0 \rangle + 2H\Delta(2|A|^2 - H^2) \\ & \quad + (2|A|^2 - H^2)(\Delta H - H|A|^2 + 2|B|^3)\} \tilde{U}_{2,z} \\ & + 2z\{(2|A|^2 - H^2)\tilde{U}_{2,zz} + (4|B|^3 - H^3)\tilde{U}_{3,1,zz}\}(2|B|^3 - H|A|^2) \\ & - 2\Delta(4|B|^3 - H^3)\mathcal{A}_1\tilde{U}_{3,1} + 2|A|^2(4|B|^3 - H^3)z(\mathcal{A}_1\tilde{U}_{3,1})_z - 2H(\mathcal{A}_1U_4)_z \\ & - 2(2|A|^2 - H^2)\mathcal{A}_1\tilde{U}_2\Delta d_2 - H(2|A|^2 - H^2)^2W'''(U_0)\tilde{U}_2^2 \\ & - (2|A|^2 - H^2)(4|B|^3 - H^3)W'''(U_0)(\tilde{U}_2\mathcal{A}_1\tilde{U}_3 + \tilde{U}_3\mathcal{A}_1\tilde{U}_2). \end{aligned}$$

Applying the orthogonality condition in Lemma 3.1, we get the equation for d_2 :

$$-d_{2,t} - \Delta^2 d_2 - \left\{ \frac{1}{2} P_1 + |A|^2 (2|A|^2 - H^2) \right\} d_2 = F.$$

Here F depends on H , $|A|^2$, $|B|^3$, $|C|^4$, $|D|^5$ and their derivatives. We omit the detail because the actual calculations are very complicated.

We solve this equation in $(0, +\infty) \times \mathbb{R}^N$ under the initial condition $d_2(0, \cdot) = 0$ in $\mathbb{R}^N$. Then U_4 is determined and the equation for U_5 is obtained. We omit the details for U_5 because the actual arguments are harder than those in Step 4.

Consequently we obtain (3.2) and (3.3).

3.3 Radially symmetric case

This subsection is devoted to the formal asymptotic expansion of radial solutions of (1.1).

Assume that $u^\varepsilon(t, x) = u^\varepsilon(t, r)$ ($r = |x|$). Then (1.1) turns to the following:

$$(3.44) \quad u_t^\varepsilon = -\Delta_r \left(\Delta_r u^\varepsilon - \frac{1}{\varepsilon^2} W'(u^\varepsilon) \right) + \frac{1}{\varepsilon^2} W''(u^\varepsilon) \left(\Delta_r u^\varepsilon - \frac{1}{\varepsilon^2} W'(u^\varepsilon) \right) \\ \text{in } (0, +\infty) \times (0, +\infty), \\ \Delta_r := \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr}.$$

We also assume (A1) - (A5) mentioned in the beginning of section 2. In these assumptions, we set

$$\Gamma(t) = \{|x| = \phi_0(t)\}, \quad D(t) = \{|x| < \phi_0(t)\}, \\ \Gamma^\varepsilon(t) = \{|x| \leq \phi^\varepsilon(t)\}, \quad \Omega_+^\varepsilon(t) = \{|x| < \phi^\varepsilon(t)\}, \quad \Omega_-^\varepsilon(t) = \{|x| > \phi^\varepsilon(t)\}, \\ d^\varepsilon(t, r) = \sum_{l=0}^3 \varepsilon^3 \phi_l(t) - r + O(\varepsilon^4), \quad d_0(t, r) = \phi_0(t) - r,$$

where ϕ_l ($l = 0, 1, 2, 3$) is a smooth and bounded functions and $\phi_0(0) = R_0(> 0)$, $\phi_l(0) = 0$ ($l = 1, 2, 3$).

We get the outer expansion of the solution u^ε of (3.44) by the same argument as in subsection 2.1: For a given $\delta \in (0, 1)$,

$$u^\varepsilon(t, r) = \pm 1 + O(\varepsilon^6) \quad \text{in } \Omega_\pm^\varepsilon(t) \setminus \{|d^\varepsilon(t, \cdot)| \geq \delta\}$$

for all $t \geq 0$ and small $\varepsilon > 0$. We omit the detail.

As for the inner expansion, we suppose that u^ε is expanded as

$$(3.45) \quad u^\varepsilon(t, r) = U(t, z) = U_0(t, z) + \varepsilon U_1(t, z) + \varepsilon^2 U_2(t, z) + \varepsilon^3 U_3(t, z) \\ + \varepsilon^4 U_4(t, z) + \varepsilon^5 U_5(t, z) + O(\varepsilon^6),$$

$$(3.46) \quad z = \frac{d^\varepsilon(t, r)}{\varepsilon} = \frac{1}{\varepsilon} \left(\sum_{l=0}^3 \varepsilon^l \phi_l(t) - r \right) + O(\varepsilon^3).$$

We assume $N \neq 3$ for the moment and derive the following expansions:

$$(3.47) \quad U(t, r) = U_0(z) - \frac{\varepsilon^2 K_N}{\phi_0^2(t)} U_{0,zz}(z) - \frac{\varepsilon^3 (N+1) K_N}{\phi_0^3(t)} \tilde{U}_{3,1}(z) + \varepsilon^4 U_4(t, r) + O(\varepsilon^5),$$

$$(3.48) \quad z = \frac{\phi_0(t) + \varepsilon^2 \phi_2(t) - r}{\varepsilon} + O(\varepsilon^2),$$

where $\tilde{U}_{3,1}$, U_4 , ϕ_0 and ϕ_2 will be given in the following arguments.

We calculate that

$$\begin{aligned}
u_t &= U_t + \frac{1}{\varepsilon} U_z \sum_{l=0}^3 \varepsilon^l \phi_{l,t}, \quad \Delta_r u = \frac{1}{\varepsilon^2} U_{zz} - \frac{1}{\varepsilon} \frac{N-1}{r} U_z, \\
\Delta_r(\Delta_r u) &= \frac{1}{\varepsilon^4} U_{zzzz} - \frac{2}{\varepsilon^3} \frac{N-1}{r} U_{zzz} + \frac{1}{\varepsilon^2} \frac{(N-1)(N-3)}{r^2} U_{zz} + \frac{1}{\varepsilon} \frac{(N-1)(N-3)}{r^3} U_z, \\
\Delta_r W'(u) &= \frac{1}{\varepsilon^2} (W'''(U) U_z^2 + W''(U) U_{zz}) - \frac{1}{\varepsilon} \frac{N-1}{r} W''(U) U_z,
\end{aligned}$$

It follows from (3.44), (3.45) and the above calculations that

$$\begin{aligned}
(3.49) \quad U_t + \frac{1}{\varepsilon} U_z \sum_{l=0}^3 \varepsilon^l \phi_{l,t} &= -\frac{1}{\varepsilon} \frac{(N-1)(N-3)}{r^3} U_z - \frac{1}{\varepsilon^2} \frac{(N-1)(N-3)}{r^2} U_{zz} + \frac{2}{\varepsilon^3} \frac{N-1}{r} (\mathcal{A}_0 U)_z \\
&\quad - \frac{1}{\varepsilon^4} [(\mathcal{A}_0 U)_{zz} - W''(U) \mathcal{A}_0 U]. \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Denote by J_0 the left-hand side of this equation and set $K_N := (N-1)(N-3)$. Since it follows from (3.46) that

$$r = \phi_0 \left[1 - \left\{ -\frac{\varepsilon(\phi_1 - z)}{\phi_0} - \frac{\varepsilon^2 \phi_2}{\phi_0} - \frac{\varepsilon^3 \phi_3}{\phi_0} \right\} + O(\varepsilon^4) \right],$$

we see that

$$\begin{aligned}
\frac{1}{r} &= \frac{1}{\phi_0} \left[1 - \frac{\varepsilon(\phi_1 - z)}{\phi_0} - \frac{\varepsilon^2}{\phi_0} \left\{ \phi_2 - \frac{(\phi_1 - z)^2}{\phi_0} \right\} \right. \\
&\quad \left. - \frac{\varepsilon^3}{\phi_0} \left\{ \phi_3 - \frac{2(\phi_1 - z)\phi_2}{\phi_0} + \frac{(\phi_1 - z)^3}{\phi_0^2} \right\} + O(\varepsilon^4) \right], \\
\frac{1}{r^2} &= \frac{1}{\phi_0^2} \left[1 - \frac{2\varepsilon(\phi_1 - z)}{\phi_0} - \frac{\varepsilon^2}{\phi_0} \left\{ 2\phi_2 - 3\frac{(\phi_1 - z)^2}{\phi_0} \right\} \right. \\
&\quad \left. - \frac{2\varepsilon^3}{\phi_0} \left\{ \phi_3 - \frac{3(\phi_1 - z)\phi_2}{\phi_0} + \frac{2(\phi_1 - z)^3}{\phi_0^2} \right\} + O(\varepsilon^4) \right].
\end{aligned}$$

The arguments in subsection 2.2 also yield that

$$U_0(z) = \tanh \frac{z}{\sqrt{2}}, \quad U_1(z) \equiv 0, \quad U_2(t, z) = -\frac{K_N}{\phi_0^2(t)} \tilde{U}_2(z).$$

Thus we observe that

$$\begin{aligned}
J_0 &= \frac{1}{\varepsilon} U_{0,z} \sum_{l=0}^2 \varepsilon^l \phi'_l + \varepsilon U_{2,z} \phi'_0 + O(\varepsilon^2), \\
J_2 &= -\frac{K_N}{\varepsilon} \frac{1}{\phi_0^2} \left[-\frac{2(\phi_1 - z)}{\phi_0} - \frac{\varepsilon}{\phi_0} \left\{ 2\phi_2 - 3\frac{(\phi_1 - z)^2}{\phi_0} \right\} \right. \\
&\quad \left. - \frac{2\varepsilon^2}{\phi_0} \left\{ \phi_3 - \frac{3(\phi_1 - z)\phi_2}{\phi_0} + \frac{2(\phi_1 - z)^3}{\phi_0^2} \right\} \right] U_{0,zz} \\
&\quad - \frac{K_N}{\phi_0^2} \left\{ 1 - \frac{2\varepsilon(\phi_1 - z)}{\phi_0} \right\} U_{2,zz} - \frac{\varepsilon K_N}{\phi_0^2} U_{3,zz} + O(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
J_3 &= \frac{2(N-1)}{\varepsilon} \frac{1}{\phi_0} \left[1 - \frac{\varepsilon(\phi_1 - z)}{\phi_0} - \frac{\varepsilon^2}{\phi_0} \left\{ \phi_2 - \frac{(\phi_1 - z)^2}{\phi_0} \right\} \right] (\mathcal{A}_1 U_2)_z \\
&\quad + \frac{2(N-1)}{\phi_0} \left\{ 1 - \frac{\varepsilon(\phi_1 - z)}{\phi_0} \right\} (\mathcal{A}_1 U_3)_z + \frac{2\varepsilon(N-1)}{\phi_0} (\mathcal{A}_1 U_4)_z + O(\varepsilon^2), \\
J_4 &= -\frac{1}{\varepsilon} \sum_{k=3}^5 \varepsilon^{k-3} \mathcal{A}_1(\mathcal{A}_1 U_k) - \frac{1}{2} \mathcal{A}_1(W'''(U_0)(U_2^2 + 2\varepsilon U_2 U_3)) \\
&\quad - W'''(U_0)\{U_2 \mathcal{A}_1 U_2 + \varepsilon(U_2 \mathcal{A}_1 U_3 + U_3 \mathcal{A}_1 U_2)\} + O(\varepsilon^2).
\end{aligned}$$

In addition, since

$$\frac{1}{r^3} = \frac{1}{\phi_0^3} \left[1 - \frac{3\varepsilon(\phi_1 - z)}{\phi_0} - \frac{3\varepsilon^2}{\phi_0} \left\{ \phi_2 - \frac{2(\phi_1 - z)^2}{\phi_0} \right\} + O(\varepsilon^3) \right],$$

we get

$$\begin{aligned}
J_1 &= -\frac{1}{\varepsilon} \frac{K_N}{\phi_0^3} \left[1 - \frac{3\varepsilon(\phi_1 - z)}{\phi_0} - \frac{3\varepsilon^2}{\phi_0} \left\{ \phi_2 - \frac{2(\phi_1 - z)^2}{\phi_0} \right\} + O(\varepsilon^3) \right] U_{0,z} \\
&\quad - \frac{\varepsilon K_N}{\phi_0^3} U_{2,z} + O(\varepsilon^2).
\end{aligned}$$

Step 1. Choices of ϕ_0 and U_3

Comparing the both sides of (3.49) with the use of J_i 's, we have the ε^{-1} -term :

$$\mathcal{A}_1(\mathcal{A}_1 U_3) = -\left(\phi_0' + \frac{N K_N}{\phi_0^3} \right) U_{0,z} - \frac{(N+1)K_N}{\phi_0^3} z U_{0,zz} + \frac{2K_N}{\phi_0^3} \phi_1 U_{0,zz}.$$

Here we have used

$$(\mathcal{A}_1 U_2)_z = -\frac{K_N}{2\phi_0^2} (U_{0,z} + z U_{0,zz}).$$

Set $U_3 := U_{3,1} + U_{3,2}$, where $U_{3,1}$ and $U_{3,2}$, respectively, solve

$$(3.50) \quad \mathcal{A}_1(\mathcal{A}_1 U_{3,1}) = -\left(\phi_0' + \frac{N K_N}{\phi_0^3} \right) U_{0,z} - \frac{(N+1)K_N}{\phi_0^3} z U_{0,zz},$$

$$(3.51) \quad \mathcal{A}_1(\mathcal{A}_1 U_{3,2}) = \frac{2K_N}{\phi_0^3} \phi_1 U_{0,zz}.$$

To apply Lemma 3.1, we check the solvability condition. Then we obtain

$$-\left(\phi_0' + \frac{N K_N}{\phi_0^3} \right) \|U_{0,z}\|_{L^2}^2 - \frac{(N+1)K_N}{\phi_0^3} \langle z U_{0,zz}, U_{0,z} \rangle_{L^2} = 0.$$

By the fact $\langle z U_{0,zz}, U_{0,z} \rangle_{L^2} = -\frac{1}{2} \|U_{0,z}\|_{L^2}^2$, we have the equation for ϕ_0 :

$$(3.52) \quad \phi_0' = -\frac{(N-1)K_N}{2\phi_0^3}$$

Solving this equation under the initial condition $\phi_0(0) = R_0$, we get

$$\phi_0(t) = \{R_0^4 - 2(N-1)K_N t\}^{1/4}.$$

Then (3.50) turns to

$$\mathcal{A}_1(\mathcal{A}_1 U_{3,1}) = \frac{(N+1)K_N}{\phi_0^3} \left(\frac{1}{2} U_{0,z} + z U_{0,zz} \right).$$

Hence we obtain

$$U_{3,1} = \frac{(N+1)K_N}{\phi_0^3} \tilde{U}_{3,1},$$

where $\tilde{U}_{3,1}$ is the solution of (3.33) satisfying (3.4) - (3.6).

As for $U_{3,2}$, we use the solution of (3.23) satisfying (3.4) - (3.6) to have

$$U_{3,2} = \frac{2K_N}{\phi_0^3} \phi_1 \tilde{U}_2.$$

U_3 will be determined in the next step because $U_{3,2}$ includes ϕ_1 and it will be chosen in the next step.

Therefore J_i 's ($i = 0, 1, 2, 3, 4$) turn to the following:

$$\begin{aligned} J_0 &= \frac{1}{\varepsilon} U_{0,z} \sum_{l=1}^2 \varepsilon^l \phi'_l + \varepsilon U_{2,z} \phi'_0 + O(\varepsilon^2), \\ J_1 &= -\frac{K_N}{\phi_0^3} \left[-\frac{3(\phi_1 - z)}{\phi_0} - \frac{3\varepsilon}{\phi_0} \left\{ \phi_2 - \frac{2(\phi_1 - z)^2}{\phi_0} \right\} \right] U_{0,z} - \frac{\varepsilon K_N}{\phi_0^3} U_{2,z} + O(\varepsilon^2), \\ J_2 &= -\frac{K_N}{\phi_0^2} \left[-\frac{1}{\phi_0} \left\{ 2\phi_2 - 3\frac{(\phi_1 - z)^2}{\phi_0} \right\} - \frac{2\varepsilon}{\phi_0} \left\{ \phi_3 - \frac{3(\phi_1 - z)\phi_2}{\phi_0} + \frac{2(\phi_1 - z)^3}{\phi_0^2} \right\} \right] U_{0,zz} \\ &\quad - \frac{K_N}{\phi_0^2} \left\{ 1 - \frac{2\varepsilon(\phi_1 - z)}{\phi_0} \right\} U_{2,zz} - \frac{\varepsilon K_N}{\phi_0^2} U_{3,zz} + O(\varepsilon^2), \\ J_3 &= \frac{2(N-1)}{\phi_0} \left[-\frac{(\phi_1 - z)}{\phi_0} - \frac{\varepsilon}{\phi_0} \left\{ \phi_2 - \frac{(\phi_1 - z)^2}{\phi_0} \right\} \right] (\mathcal{A}_1 U_2)_z \\ &\quad + \frac{2(N-1)}{\phi_0} \left\{ 1 - \frac{\varepsilon(\phi_1 - z)}{\phi_0} \right\} (\mathcal{A}_1 U_3)_z + \frac{2\varepsilon(N-1)}{\phi_0} (\mathcal{A}_1 U_4)_z + O(\varepsilon^2), \\ J_4 &= -\sum_{k=4}^5 \varepsilon^{k-4} \mathcal{A}_1 (\mathcal{A}_1 U_k) - \frac{1}{2} \mathcal{A}_1 (W'''(U_0)(U_2^2 + 2\varepsilon U_2 U_3)) \\ &\quad - W'''(U_0) \{ U_2 \mathcal{A}_1 U_2 + \varepsilon (U_2 \mathcal{A}_1 U_3 + U_3 \mathcal{A}_1 U_2) \} + O(\varepsilon^2). \end{aligned}$$

Remark 3.4. The solution ϕ_0 of (3.52) is the radius of a circle or a sphere evolving by the Willmore flow equation (1.7).

Step 3. Choices of ϕ_1 and U_4

From the ε^0 -term, we have the following equation:

$$\begin{aligned} \mathcal{A}_1 (\mathcal{A}_1 U_4) &= U_{0,z} \phi'_1 + \frac{3K_N}{\phi_0^4} (\phi_1 - z) U_{0,z} + \frac{K_N}{\phi_0^3} \left\{ 2\phi_2 - 3\frac{(\phi_1 - z)^2}{\phi_0} \right\} U_{0,zz} \\ &\quad - \frac{K_N}{\phi_0^2} U_{2,zz} - \frac{2(N-1)}{\phi_0^2} (\phi_1 - z) (\mathcal{A}_1 U_2)_z \\ &\quad + \frac{2(N-1)}{\phi_0} (\mathcal{A}_1 U_3)_z - \frac{1}{2} \mathcal{A}_1 (W'''(U_0) U_2^2) - W'''(U_0) U_2 \mathcal{A}_1 U_2. \end{aligned}$$

We have the following equation by using (3.45) - (3.37), $\langle \tilde{U}_2, U_{0,z} \rangle_{L^2} = 0$ and the orthogonality condition of Lemma 3.1:

$$-\phi'_1 + \frac{(N+5)K_N}{2\phi_0^4} \phi_1 = 0.$$

We solve this equation under the condition $\phi_1(0) = 0$ to have $\phi_1 \equiv 0$. Thus we have

$$U_3 = \frac{(N+1)K_N}{\phi_0^3} \tilde{U}_{3,1}$$

and the equation for U_4 turns to the following:

$$\begin{aligned} \mathcal{A}_1(\mathcal{A}_1 U_4) = & -\frac{K_N}{\phi_0^3} \left\{ \frac{2N+1}{\phi_0} + (N+1)(N-1) \right\} z U_{0,z} + \frac{2K_N}{\phi_0^3} \{ \phi_2 - (N+1)(N-1)c_1 \} U_{0,zz} \\ & - \frac{K_N}{\phi_0^3} \left\{ \frac{2N+1}{\phi_0} + \frac{(N+1)(N-1)}{2} \right\} z^2 U_{0,zz} \\ & + \frac{K_N^2}{\phi_0^4} \left\{ \tilde{U}_{2,zz} - \frac{1}{2} \mathcal{A}_1(W'''(U_0) \tilde{U}_2^2) - \frac{1}{2} W'''(U_0) z \tilde{U}_2 U_{0,z} \right\} \end{aligned}$$

We decompose $U_4 := U_{4,1} + \dots + U_{4,4}$ and each $U_{4,i}$ is defined as follows:

$$\begin{aligned} U_{4,1} &:= -\frac{K_N}{\phi_0^3} \left\{ \frac{2N+1}{\phi_0} + (N+1)(N-1) \right\} \tilde{U}_{4,1}, \\ U_{4,2} &= \frac{2K_N}{\phi_0^3} \{ \phi_2 - (N+1)(N-1)c_1 \} \tilde{U}_2, \\ U_{4,3} &= -\frac{K_N}{\phi_0^3} \left\{ \frac{2N+1}{\phi_0} + \frac{(N+1)(N-1)}{2} \right\} \tilde{U}_{4,3}, \\ U_{4,4} &= \frac{K_N^2}{\phi_0^4} \tilde{U}_{4,4}, \end{aligned}$$

where $\tilde{U}_{4,i}$'s ($i = 1, 3, 4$) satisfy

$$\begin{aligned} \mathcal{A}_1 \tilde{U}_{4,1} &= \tilde{U}_2, \quad \mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{4,3}) = z^2 U_{0,zz}, \\ \mathcal{A}_1(\mathcal{A}_1 \tilde{U}_{4,4}) &= \tilde{U}_{2,zz} - \frac{1}{2} \mathcal{A}_1(W'''(U_0) \tilde{U}_2^2) - \frac{1}{2} W'''(U_0) z \tilde{U}_2 U_{0,z}. \end{aligned}$$

Since $\tilde{U}_{4,i}$'s are obtained in Step 4 of subsection 3.2, we can determine U_4 and J_i 's ($i = 0, 1, 2, 3, 4$) turn to

$$\begin{aligned} J_0 &= \varepsilon U_{0,z} \phi'_2 + \varepsilon U_{2,z} \phi'_0 + O(\varepsilon^2), \\ J_1 &= \frac{K_N}{\phi_0^3} \left[\frac{3\varepsilon}{\phi_0} \left\{ \phi_2 - \frac{2z^2}{\phi_0} \right\} + O(\varepsilon^2) \right] U_{0,z} - \frac{\varepsilon K_N}{\phi_0^3} U_{2,z} + O(\varepsilon^2), \\ J_2 &= \frac{2\varepsilon K_N}{\phi_0^3} \left(\phi_3 + \frac{3z\phi_2}{\phi_0} - \frac{2z^3}{\phi_0^2} \right) U_{0,zz} - \frac{2\varepsilon K_N}{\phi_0^3} z U_{2,zz} - \frac{\varepsilon K_N}{\phi_0^2} U_{3,zz} + O(\varepsilon^2), \\ J_3 &= -\frac{2\varepsilon(N-1)}{\phi_0^2} \left(\phi_2 - \frac{z^2}{\phi_0} \right) (\mathcal{A}_1 U_2)_z \\ &\quad + \frac{2\varepsilon(N-1)z}{\phi_0^2} (\mathcal{A}_1 U_3)_z + \frac{2\varepsilon(N-1)}{\phi_0} (\mathcal{A}_1 U_4)_z + O(\varepsilon^2), \\ J_4 &= -\varepsilon \mathcal{A}_1(\mathcal{A}_1 U_5) - \varepsilon \mathcal{A}_1(W''' U_2 U_3) - \varepsilon W'''(U_2 \mathcal{A}_1 U_3 + U_3 \mathcal{A}_1 U_2) + O(\varepsilon^2). \end{aligned}$$

Step 4. Choice of ϕ_2

Comparing the ε -terms of both sides of $J_0 = J_1 + \dots + J_4$, we have

$$\begin{aligned} \mathcal{A}_1(\mathcal{A}_1 U_5) = & -U_{0,z} \phi'_2 - \frac{(N-1)K_N^2}{2\phi_0^5} \tilde{U}_{2,z} + \frac{3K_N}{\phi_0^4} \left(\phi_2 - \frac{2z^2}{\phi_0} \right) U_{0,z} \\ & + \frac{K_N^2}{\phi_0^5} \tilde{U}_{2,z} + \frac{2K_N}{\phi_0^3} \left(\phi_3 + \frac{3z\phi_2}{\phi_0} - \frac{2z^3}{\phi_0^2} \right) U_{0,zz} \\ & + \frac{2K_N^2}{\phi_0^5} z \tilde{U}_{2,zz} - \frac{(N+1)K_N^2}{\phi_0^5} \tilde{U}_{3,1,zz} \end{aligned}$$

$$\begin{aligned}
& + \frac{2(N-1)K_N}{\phi_0^4} \left(\phi_2 - \frac{z^2}{\phi_0} \right) (\mathcal{A}_1 \tilde{U}_2)_z \\
& + \frac{2(N+1)(N-1)K_N}{\phi_0^5} z (\mathcal{A}_1 \tilde{U}_{3,1})_z + \frac{2(N-1)}{\phi_0} \sum_{k=1}^6 (\mathcal{A}_1 U_{4,k})_z \\
& + \frac{(N+1)K_N^2}{\phi_0^5} \{ \mathcal{A}_1 (W'''(U_0) \tilde{U}_2 \tilde{U}_{3,1}) + W'''(U_0) (\tilde{U}_2 \mathcal{A}_1 \tilde{U}_{3,1} + \tilde{U}_{3,1} \mathcal{A}_1 \tilde{U}_2) \}
\end{aligned}$$

As $N \neq 3$, the orthogonality condition of Lemma 3.1 yields the ODE for ϕ_2 :

$$\begin{aligned}
\phi_2' - \frac{(N-1)K_N}{2\phi_0^4} \phi_2 &= - \frac{L_1 K_N^3}{2\phi_0^5} + \frac{12L_2 K_N}{\phi_0^5} + \frac{L_4(N+1)K_N^2}{\phi_0^5} + \frac{L_2(N-1)K_N}{2\phi_0^5} \\
&\quad - \frac{(L_2 + 4c_1)(N+1)(N-1)K_N}{\phi_0^5} \\
&\quad + \frac{c_1(N+1)(N-1)^2 K_N}{\phi_0^4} \\
&\quad - \frac{2L_6(N-1)K_N}{\phi_0^4} \left\{ \frac{2N+1}{\phi_0} + \frac{(N+1)(N-1)}{2} \right\} + \frac{2L_7(N-1)K_N^2}{\phi_0^5} \\
&=: \frac{K_N M_1}{\phi_0^5} + \frac{K_N M_2}{\phi_0^4},
\end{aligned}$$

where

$$\begin{aligned}
L_1 &:= \frac{\langle \tilde{U}_{2,z}, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}, \quad L_2 := \frac{\|z U_{0,z}\|_{L^2}^2}{\|U_{0,z}\|_{L^2}^2} (= -4c_1), \quad L_3 := \frac{\langle z \tilde{U}_{2,zz}, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}, \\
L_4 &:= \frac{\langle \tilde{U}_{3,1,zz}, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}, \quad L_5 := \frac{\langle \tilde{U}_{3,1,zz}, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}, \quad L_6 := \frac{\langle (\mathcal{A}_1 \tilde{U}_{4,3})_z, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}, \\
L_7 &:= \frac{\langle (\mathcal{A}_1 \tilde{U}_{4,6})_z, U_{0,z} \rangle_{L^2}}{\|U_{0,z}\|_{L^2}^2}.
\end{aligned}$$

Multiplying both sides of the above equation by $\frac{\phi_0}{R_0}$, we see that

$$\left(\frac{\phi_0}{R_0} \phi_2 \right)' = \frac{K_N M_1}{R_0 \phi_0^4} + \frac{K_N M_2}{R_0 \phi_0^3}.$$

By solving this ODE under the condition $\phi_2(0) = 0$, we obtain

$$\phi_2(t) = \frac{2}{(N-1)} \left\{ -\frac{M_1}{\phi_0(t)} \log \frac{\phi_0(0)}{\phi_1(t)} + M_2 \left(\frac{\phi_0(0)}{\phi_0(t)} - 1 \right) \right\}.$$

Therefore we are able to determine U_4 , but we omit the detail. Consequently we obtain (3.47) and (3.48).

Step 5. The case $N = 3$

In the case $N = 3$, we have from the above arguments

$$\begin{aligned}
U(t, r) &= U_0(z) + O(\varepsilon^5), \quad z = \frac{R_0 - r}{\varepsilon} + O(\varepsilon^2) \\
&\quad \text{for all } (t, r) \in (0, +\infty) \times (0, +\infty).
\end{aligned}$$

Moreover, we can prove that for any $k \in \mathbb{N}$,

$$\begin{aligned}
(3.53) \quad U(t, r) &= U_0(z) + O(\varepsilon^k), \quad z = \frac{R_0 - r}{\varepsilon} + O(\varepsilon^{k-1}) \\
&\quad \text{for all } (t, r) \in (0, +\infty) \times (0, +\infty).
\end{aligned}$$

4 Some remarks

This section is devoted to the comparison between the formal asymptotic expansion in [18] (cf. [5]) with ours obtained in the previous section. Furthermore we also compare the motion of $\{\Gamma^\varepsilon(t)\}_{t \geq 0}$ with that in [16].

4.1 Comparison with the expansion in [18]

In [18] Loreti and March give asymptotic expansions of a pair $(u^\varepsilon, \mu^\varepsilon)$ of solutions of (1.4), which is equivalent to (1.1). In particular the inner expansions are as follows:

$$(4.1) \quad u^\varepsilon(t, x) = U_0(z) + \varepsilon^2 \left(|A|^2 - \frac{1}{2} H^2 \right) P_2(z) + O(\varepsilon^3),$$

$$(4.2) \quad \mu^\varepsilon(t, x) = -\varepsilon H U_{0,z}(z) + \varepsilon^2 H^2 Q_2(z) + O(\varepsilon^3),$$

where $z = d^\varepsilon(t, x)/\varepsilon$, $d^\varepsilon(t, x)$ is the signed distance function to the set $\{u^\varepsilon(t, \cdot) = 0\}$, U_0 is the function obtained in step 1 of subsection 2.2 and U_2 and V_2 are the solutions of

$$\begin{aligned} P_{2,zz} - W''(U_0)P_2 &= zU_{0,z}, \quad \lim_{z \rightarrow \pm\infty} P_2(z) = 0, \quad P_2(0) = 0 \\ Q_{2,zz} - W''(U_0)Q_2 &= U_{0,zz}, \quad \lim_{z \rightarrow \pm\infty} Q_2(z) = 0, \quad Q_2(0) = 0, \end{aligned}$$

respectively. Noting that $P_2 = 2\tilde{U}_2$ which is given in step 3 of subsection 2.2, we see that (4.1) coincides with (3.2) up to the order ε^2 .

We derive (4.2) from (3.2). For $|Dd^\varepsilon| = 1$, we calculate that

$$\begin{aligned} -\varepsilon^2 \Delta u &= -U_{0,zz} - \varepsilon U_{0,z} \Delta(d_0 + \varepsilon^2 d_2) \\ &\quad - \varepsilon^2 (2|A|^2 - H^2) \{ \tilde{U}_{2,zz} - \varepsilon \tilde{U}_{2,z} \Delta(d_0 + \varepsilon^2 d_2) \} + O(\varepsilon^3), \\ &= -U_{0,zz} - \varepsilon U_{0,z} (-H - \varepsilon z |A|^2 - \varepsilon^2 z^2 |B|^3) \\ &\quad - \varepsilon^2 (2|A|^2 - H^2) \{ \tilde{U}_{2,zz} + \varepsilon \tilde{U}_{2,z} (-H - \varepsilon z |A|^2 - \varepsilon^2 z^2 |B|^3) \} + O(\varepsilon^3), \\ W'(u) &= W'(U_0 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \varepsilon^4 U_4 + O(\varepsilon^5)) \\ &= W'(U_0) + \varepsilon^2 (2|A|^2 - H^2) W''(U_0) \tilde{U}_2 + O(\varepsilon^3). \end{aligned}$$

Thus we have by the equation of (3.25)

$$\begin{aligned} \mu^\varepsilon &= -\varepsilon^2 \Delta u + W'(u) \\ &= \varepsilon H U_{0,z} + \varepsilon^2 |A|^2 z U_{0,z} - \frac{\varepsilon^2}{2} (2|A|^2 - H^2) z U_{0,z} + O(\varepsilon^3) \\ &= \varepsilon H U_{0,z} + \frac{\varepsilon^2}{2} H^2 z U_{0,z} + O(\varepsilon^3). \end{aligned}$$

Since the function $zU_{0,z}/2$ is the unique solution of $\mathcal{A}_1 u = U_{0,zz}$ satisfying (3.4) - (3.6), we obtain $Q_2 = zU_{0,z}/2$. Notice that $\Delta d_0 = H$ in [18] and [5]. Thus the expansion of μ^ε coincides with that in [18] and [5].

The author considers that we need such a higher order formal expansion as we have obtained in the previous section when estimating the (optimal) rate of convergence of $\{\Gamma_\varepsilon(t)\}_{t \geq 0}$ to $\{\Gamma(t)\}_{t \geq 0}$. In particular the expansion $d_\varepsilon = d_0 + \varepsilon^2 d_2 + O(\varepsilon^3)$ suggests that the rate of convergence of $\{\Gamma_\varepsilon(t)\}_{t \geq 0}$ to $\{\Gamma(t)\}_{t \geq 0}$ is of order ε^2 .

Using (3.2), we are able to have the following formal asymptotic expansion for μ^ε , which is more

precise than (4.2):

$$(4.3) \quad \begin{aligned} \mu^\varepsilon(t, x) = & -\varepsilon H U_{0,z}(z) + \frac{\varepsilon^2}{2} H^2 z U_{0,z}(z) \\ & + \varepsilon^3 \left[\left\{ -4|B|^3 c_1 + H^3 \left(\frac{z^2}{4} + c_1 \right) - \Delta d_2 \right\} U_{0,z} + (2|A|^2 - H^2) H \tilde{U}_{2,z} \right] \\ & + O(\varepsilon^4), \end{aligned}$$

Remark 4.1. (1) Assume that there exists a Willmore flow $\{\Gamma(t)\}_{t \geq 0}$ and that it is sufficiently smooth. Then we are able to define (3.2), (3.3) and (4.3) near $\bigcup_{t \geq 0} \{t\} \times \Gamma(t)$ and to expect that (3.2) gives an approximation to the solution of (1.1) by using Fei–Liu [12, Theorem 1.1]. However we cannot apply [12, Theorem 1.1] because it requires a higher order approximation than (3.2).

(2) In the case of $N = 3$ and radial symmetry, (3.53) suggests that $\Gamma^\varepsilon(t) := \{x \in \mathbb{R}^3 \mid u^\varepsilon(t, |x|) = 0\}$ moves slowly. The case $N = 1$ seems to be similar although it is not treated in this article. We refer Bronsard–Kohn [7] and Bronsard–Hilhorst [6], in which the authors consider the slow dynamics for the Allen–Chan equation and the Cahn–Hilliard equation in one space dimension.

4.2 Comparison the motion of the set $\{u(t, \cdot) = 0\}$ with that by the threshold-type algorithm with using (1.5)

In section 2 we formally observed that $\{\Gamma(t)\}_{t \geq 0}$ moves by (1.7) in any space dimension. In other words $\{\Gamma_\varepsilon(t)\}$ approximately moves by (1.7) in any space dimension. Then let us compare the motion generated by (1.1) with that by (1.5).

Let $u = u(t, x)$ be a solution of (1.5) and define a function $U(t, x)$ and a set $C(t)$ by

$$\begin{aligned} U(t, x) &:= u((3a)^4 t, x) - 3u((2a)^4 t, x) + 3u(a^4 t, x) - \frac{1}{2}, \\ C(t) &= \{U(t, \cdot) \geq 0\}. \end{aligned}$$

Then by [16] we see that $\{\partial C(t)\}_{0 \leq t \leq 1}$ approximately moves by

$$V = \begin{cases} -\Delta_{\partial C(t)} H - H|A|^2 + \frac{1}{2} H^3 & (N = 2, 3), \\ -\Delta_{\partial C(t)} H - H|A|^2 + \frac{1}{2} H^3 - 2 \sum_{\substack{i,j,k=1,\dots,N-1 \\ i < j < k}} \kappa_i \kappa_j \kappa_k & (N \geq 4) \end{cases}$$

Comparing the equation (1.7) with this, we see that the threshold-type algorithm by (1.5) is somehow similar to (1.1) for $N = 2, 3$ and that it is different for $N \geq 4$.

As for the approximation to the motion by mean curvature, Bence, Merriman and Osher proposed in [4] a threshold-type algorithm by using the initial value problem for the second order heat equation:

$$\begin{cases} u_t = \Delta u & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \chi_{C_0}(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

where C_0 is a given compact set in $\mathbb{R}^N$ and χ_{C_0} is the characteristic function for C_0 . Vivier pointed out in [20] that this algorithm is similar to the Allen–Cahn equation (1.6). Actually Leoni [17] and Goto–Ishii–Ogawa [14] proved the convergence of threshold-type algorithms by applying the method of the matched asymptotic expansion of solutions of (1.6). Based on these results, the author had a question whether the algorithm by [16] is similar to the approximation by (1.1). However, these seem to be different from each other.

However, the (1.1) suggests that we need to modify (1.5) to construct a threshold-type algorithm for the motion by (1.7) in any space dimension.

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