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Fujii, Yoshie

Iritani, Jun

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Jun Iritani and Yoshie Fujii

*Graduate School of Economics, Kobe University*

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## **Abstract**

The objective of this note is to show that the models of tax competition are not necessarily consistent with the general equilibrium. In this field, there is an important but implicit assumption, i.e., the continuity of the response functions. The continuity does not necessarily hold because the set of variables satisfying the constraints in the maximization problem is not convex. The present paper offers a counter example against the model of Zodrow-Mieszkowski [7] where no equilibria exist.

Key words: tax competition, general equilibrium, local public good

*JEL classification:* H71,H77

**correspondence address:** Jun Iritani, Graduate School of Economics, Kobe University, 2-1, Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan. iritani@econ.kobe-u.ac.jp

Yoshie Fujii, 4-8-30-101 Yahata-cho Nada-ku, Kobe 657-0051, Japan. 976d216e@yku.kobe-u.ac.jp

# A Note on Non-Existence of Equilibria in Tax Competition Models

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## 1 Introduction

The model on tax competition now abounds in literature (e.g., see Wilson [5,6], Zodrow-Mieszkowski [7], Wildasin[3,4], and others). The theory of tax competition has become one of central themes in tax theories. The behaviors of local governments are described by use of optimal tax theory. An agent, here a local government, determines its local tax rate so that it can attain the maximum inhabitants' welfare with given tax rates of other regions. The equilibrium under tax competition is attained when every region determines exactly the same tax rate that is assumed by the other regions and when the rental rate of capital gives the equality between the total demand and the total supply of capital.

In their exploring work, Zodrow and Mieszkowski presented a model in the simplest general equilibrium setting. By their contribution we can easily capture the concept of the tax competition to obtain the proposition of underprovision of local public good. A characteristic feature in their model is that the regions are assumed to be identical. By this we can observe the mutual interdependency among the regions by studying only one region's behavior. And thus the possible equilibrium under Zodrow-Mieszkowski type tax competition is described by a pair of one local government's maximization and the market equilibrium condition for capital.

The objective of this note is to show that there exists an unsolved problem in the model of Zodrow-Mieszkowski [7]. They assumed implicitly the continuity of the response functions, that is, the continuity of optimal local tax rate with respect to parameters. Unfortunately, however, the discussion like the optimal tax can not elude a possible discontinuity because the set of variables satisfying the constraints in the maximization is not convex. In the present paper, we give a counter example against Zodrow-Mieszkowski model where no equilibria exist. Furthermore, we shall point out that almost all the models on tax competition will face with the same kind of difficulty (e.g., see models of Bucovetsky [1], DePater and Myers [2], Wildasin [3, 4], and Wilson [5, 6]).

The difficulty of the existence of equilibria under tax competition lies in the fact that the budget set of a local government is not necessarily convex. The behavior of a local government itself is well defined. It is not, however, consistent with market equilibrium. Hence, it will be a challenging open question for economists to construct a well defined tax competition model in a general equilibrium setting.

## 2 Zodrow-Mieszkowski Model

### 2.1 The Model

The basic model in the present paper follows that of Zodrow-Mieszkowski [7]. A nation consists of  $N$  identical jurisdictions. The nation as a whole has a given endowment of capital  $\bar{K}$ .  $K$  is the amount of capital employed in a jurisdiction. The capital is perfectly mobile across jurisdictions so that the capital in each jurisdiction earns the same net return  $r$ .

One kind of output is produced in each jurisdiction by perfectly competitive firm of which production technology is represented by a twice differentiable decreasing return to scale production function:

$$F(K), F' > 0, F'' < 0. \quad (1)$$

One unit of output can be transformed into one unit of either private good  $C$  or local public services  $P$ . Each jurisdiction has a fixed population, which is immobile. The residents in a region own an equal share of national capital stock. The residents have the identical utility function  $u(C, P)$  which is strictly quasi-concave and twice differentiable.

$P$  is financed either by a specific property tax  $T$  on capital or by a head tax  $H$ . The jurisdiction's budget constraint is

$$P \leq TK + H. \quad (2)$$

The amount of private good consumed by the residents in the jurisdiction can not exceeds the sum of local quasi-rents  $F(K) - (r + T)K$  and capital income  $r\bar{K}/N$  they receive, that is,

$$C \leq F(K) - (r + T)K + r\frac{\bar{K}}{N} - H, \quad (3)$$

where the price of output is unity. This is the budget constraint of the residents. We assume that  $H$  is fixed in the jurisdiction.

The profit-maximization by private firms requires that

$$F'(K) = r + T. \quad (4)$$

We can rewrite the equation by using the function  $\phi \stackrel{\text{def}}{=} F'^{-1}$  as  $K = \phi(r + T)$ . To clear the capital market,  $r$  must satisfy the following equilibrium condition:

$$\frac{\bar{K}}{N} = \phi(r + T). \quad (5)$$

From (4), it holds that  $r + T = F'(\phi(r + T))$ . And thus, we obtain  $1 = F''(\phi(r + T))d\phi/dT$

Each jurisdiction maximizes the welfare of the residents by the choice of the tax rate while considering the rental ratio as a constant, that is,

$$\max_{T, C, P} u(C, P) \text{ subject to } \begin{cases} C \leq F(\phi(r + T)) - (r + T)\phi(r + T) + r\frac{\bar{K}}{N} - H, \\ P \leq T\phi(r + T) + H. \end{cases} \quad (6)$$

The first constraint in (6) is the feasibility condition and the second is the jurisdiction's budget. Since the solution to the problem (6) depends on  $r$ , we can express this dependence as  $T(r)$ . Finally, we find the equilibrium under tax competition if we solve (5) with respect to  $r$ , which completes the model of Zodrow-Mieszkowski. To put it analytically,

The equilibrium under tax competition is represented by a solution to the simultaneous equations, (5) and a necessary condition of (6)

We must emphasize that the following implicit assumptions are made.

(A)  $F$  satisfies (1) and is well behaved and  $u$  is sufficiently smooth.

## 2.2 Non-Convexity

A problem may occur when we are to study the tax competition of Zodrow-Mieszkowski. The jurisdiction's problem (6) is not a usual convex programming problem. That is, the constraints in (6) do not necessarily give a convex set of variables. The first constraint in (6) has no problems. The second constraint, however, does not necessarily ensure us of convexity of the constraint set. And thus we are not sure whether  $T(r)$  is a continuous function with respect to  $r$  or not. This causes a difficulty in assuring the existence of the solutions to (5) and (6). We shall show in this subsection that a possible non-convexity exists.

From now on, put  $H = 0$  and assume (2) and (3) to hold in equalities. And thus, we express the constraints of (6) as,

$$C = \theta(r, r + T), \quad P = T\phi(r + T).$$

From well-known duality discussions, it holds that

$$\frac{d\theta}{dT} = -\phi(r + T) < 0. \tag{7}$$

For simplicity, we write  $\phi(r + T)$  as  $\phi(T)$ , since each jurisdiction determines  $T$  with paying no attentions to the repercussion of  $T$  to  $r$ .

From (7),  $\theta$  is a one-to-one and onto function between  $C$  and  $T$ . This fact enables us to treat the system as if the choice variables in the problem (6) were  $C$  and  $P$ . Motivated thus, defining  $\zeta(T) \stackrel{\text{def}}{=} \theta(\phi(T))$ , we obtain the inverse function  $\psi$  of  $\zeta$ :

$$\psi : C \mapsto T, \quad \psi = \zeta^{-1}.$$

Since  $C = \theta(\psi(C))$  is an identity, it is clear that

$$1 = \frac{d\theta}{dT} \frac{d\psi}{dC}.$$

Therefore it holds that

$$\frac{d\psi}{dC} = -\frac{1}{\phi} < 0.$$

On the other hand, since the level of public services is  $P = \psi(C) \times \phi(\psi(C))$ , we have:

$$\begin{aligned}\frac{dP}{dC} &= -1 - \frac{\psi(C)}{\phi(\psi(C))} \frac{1}{F''}, \\ \frac{d^2P}{dC^2} &= \frac{1}{(\phi(\psi(C)))^2 F''} \left( 1 - \frac{\psi(C)}{F'' \phi(\psi(C))} - \frac{\psi(C) F'''}{(F'')^2} \right).\end{aligned}\quad (8)$$

If  $P$  is a concave function with respect to  $C$ , the solution  $T^*$  of the problem is expressed as a continuous function with respect to the parameter  $r$ . Unfortunately, we can not determine the sign of  $d^2P/dC^2$  when the sign of  $F'''$  is positive. The set of variables satisfying constraints in (6) is not necessarily convex. This causes a possibility that there are no equilibria in the model of Zodrow-Mieszkowski. Needless to say, this fact suggests very important problem remains unsolved in the study of tax competition.

### 2.3 An example

As explained in the previous subsection, there may remain a serious problem in the Zodrow-Mieszkowski model. Here, we construct an example in which there are no equilibria by specifying production function and utility function. In this event, it is noteworthy that our example satisfies the implicit assumption **(A)**.

We specify the production function as:

$$F(K) = \begin{cases} aK^\gamma + bK & \text{if } K \in [0, \frac{1}{4}) \\ -d \exp(-\delta(K-1)) + fK^2 + \frac{33}{32}d \exp(\delta) & \text{if } K \in [\frac{1}{4}, 1) \\ g(K - \frac{5}{6})^\varepsilon + h & \text{if } K \in [1, \infty), \end{cases}\quad (9)$$

$$F'(K) = \begin{cases} a\gamma K^{(\gamma-1)} + b & \text{if } K \in (0, \frac{1}{4}) \\ d\delta \exp(-\delta(K-1)) + 2fK & \text{if } K \in (\frac{1}{4}, 1) \\ g\varepsilon (K - \frac{5}{6})^{\varepsilon-1} & \text{if } K \in (1, \infty), \end{cases}\quad (10)$$

and

$$F''(K) = \begin{cases} a\gamma(\gamma-1)K^{(\gamma-2)} & \text{if } K \in (0, \frac{1}{4}) \\ -d\delta^2 \exp(-\delta(K-1)) + 2f & \text{if } K \in (\frac{1}{4}, 1) \\ g\varepsilon(\varepsilon-1) (K - \frac{5}{6})^{\varepsilon-2} & \text{if } K \in (1, \infty). \end{cases}\quad (11)$$

We determine  $d$  and  $f$  as follows:

$$d = \frac{4}{\delta(\delta+1)}, \quad f = \frac{1}{2} - \frac{2}{\delta+1},$$

where  $\delta$  is a positive number satisfying

$$-4 - 2\delta + \frac{33}{8} \exp(\delta) = \frac{5}{2} \delta(\delta+1).\quad (12)$$

Let us show the existence of  $\delta$  satisfying (12). Put  $\delta = 0.4$  and we observe that the value of the left hand side in (12) (=1.3537) is strictly less than the right hand side(=1.4). Put  $\delta = 2.0$  and we have the reverse inequality, i.e., (left hand side)= 22.4798 > 15 =(right hand side). Therefore there exists  $\delta$  satisfying (12). Such a  $\delta$  which satisfies (12) is 0.82664 approximately.

The values of the parameters  $a, \gamma, b, h, \varepsilon, g$  are determined so that  $F(K)$  may be twice continuously differentiable,  $F'(K) > 0$ ,  $F''(K) < 0$ ,  $\lim_{K \rightarrow 0} F'(K) = \infty$  and  $\lim_{K \rightarrow \infty} F'(K) = 0$ . It is clear that  $F(K)$  is twice continuously differentiable in the open intervals of  $(0, 1/4)$ ,  $(1/4, 1)$ , and  $(1, \infty)$ . Therefore it suffices for us to study two cases  $K = 1$  and  $K = 1/4$ .

Letting  $K$  tend to unity from below in the second equations in (9), (10), and (11), we obtain

$$\begin{aligned} F(1-) &= -d + f + d \frac{33}{32} \exp(\delta) = -\frac{4}{\delta(\delta+1)} + \frac{1}{2} - \frac{2}{\delta+1} + \frac{33}{8\delta(\delta+1)} \exp(\delta) \\ &= \frac{1}{\delta(1+\delta)} \times \left\{ -4 - 2\delta + \frac{33}{8} \exp(\delta) \right\} + \frac{1}{2} = 3, \\ F'(1-) &= d\delta + 2f = \frac{4}{\delta+1} + 1 - \frac{4}{\delta+1} = 1, \\ F''(1-) &= -d\delta^2 + 2f = -\frac{4}{\delta(\delta+1)}\delta^2 + 1 - \frac{4}{\delta+1} \\ &= \frac{-4\delta^2 - 4\delta}{\delta(1+\delta)} + 1 = -3. \end{aligned}$$

For  $F(K)$  to be twice continuously differentiable at  $K = 1$ , the parameters  $\varepsilon, g, h$  should satisfy:

$$\begin{aligned} F(1+) &= g \left( \frac{1}{6} \right)^\varepsilon + h = 3, \\ F'(1+) &= g\varepsilon \left( \frac{1}{6} \right)^{\varepsilon-1} = 1, \\ F''(1+) &= g\varepsilon(\varepsilon-1) \left( \frac{1}{6} \right)^{\varepsilon-2} = -3. \end{aligned}$$

The equalities are satisfied when  $\varepsilon = 0.5$ ,  $g = 2/\sqrt{6}$ , and  $h = 8/3$ .

When  $K = 1/4$ , we obtain the following equalities:

$$\begin{aligned} F(1/4-) &= -d \exp\left(\frac{3}{4}\delta\right) + f \frac{1}{16} + d \frac{33}{32} \exp(\delta) \\ &= -\frac{4}{\delta(\delta+1)} \exp\left(\frac{3}{4}\delta\right) + \frac{1}{32} - \frac{1}{8(\delta+1)} + \frac{33}{8\delta(\delta+1)} \exp(\delta) = 1.282475, \\ F'(1/4-) &= d\delta \exp\left(\frac{3}{4}\delta\right) + \frac{1}{2}f = \frac{4 \exp\left(\frac{3}{4}\delta\right)}{\delta+1} + \frac{1}{4} - \frac{1}{\delta+1} = 3.773178, \\ F''(1/4-) &= -d\delta^2 \exp\left(\frac{3}{4}\delta\right) + 2f = -\frac{4}{\delta+1} \delta \exp\left(\frac{3}{4}\delta\right) + 1 - \frac{4}{\delta+1} = -4.554781, \end{aligned}$$

where the values in the right hand sides are approximate figures. For  $F(K)$  to be twice continuously differentiable at  $K = 1/4$ , the parameters  $a$ ,  $\gamma$ ,  $b$  must satisfy:

$$\begin{aligned} F(1/4+) &= a \left(\frac{1}{4}\right)^\gamma + \frac{1}{4}b = 1.282475, \\ F'(1/4+) &= \gamma a \left(\frac{1}{4}\right)^{\gamma-1} + b = 3.773178, \\ F''(1/4+) &= a\gamma(\gamma-1) \left(\frac{1}{4}\right)^{\gamma-2} = -4.554781. \end{aligned}$$

The equalities hold when  $a = 6.75645$ ,  $\gamma = 0.83929$ , and  $b = -3.31259$ .

We can show that the production function defined above has usual properties. From the conditions above, we obtain  $a > 0$ ,  $d > 0$ ,  $g > 0$ ,  $\delta > 0$ ,  $0 < \gamma < 1$ , and  $0 < \varepsilon < 1$ . It is clear that  $F'''(K) > 0$  in the intervals  $(0, 1/4)$ ,  $(1/4, 1)$ , and  $(1, \infty)$ . And thus  $F''(K)$  is globally increasing. It must hold that  $F''(K) < 0$  for all  $K \in (0, \infty)$ , since  $F''(1) < 0$ . Similarly, we obtain  $F'(K) > 0$  for all  $K \in (0, \infty)$ . Finally it is clear that  $\lim_{K \rightarrow 0} F'(K) = \infty$  and  $\lim_{K \rightarrow \infty} F'(K) = 0$ . This shows that the production function has the desirable properties.

The utility function is specified in Cobb-Douglas form, i.e.,

$$u(C, P) = C^\alpha P^\beta, \quad \alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 1.$$

It is clear that the marginal rate of substitution  $S(C, T)$  of  $C$  to  $P$  is

$$S(C, T) = \frac{\alpha P}{\beta C}.$$

The exogenous parameter  $\bar{K}/N$  is assumed to satisfy  $\bar{K}/N = 1$ .

Suppose that the equilibrium under tax competition had existed in this setting. Then it must hold that

$$\begin{aligned} K = \phi(\psi(C)) &= \frac{\bar{K}}{N} = 1 \quad \text{and} \quad F'(\phi(\psi(C))) = r + T, \\ P &= T\phi(\psi(C)) = T, \\ C &= F(\phi(\psi(C))) - (r + T)\phi(\psi(C)) + r\frac{\bar{K}}{N} = 3 - T. \end{aligned} \tag{13}$$

From the equation (13) together with the fact that  $F'(1) = 1$ , we obtain  $r + T = 1$  so that

$$0 \leq T \leq 1. \tag{14}$$

The slope  $-dP/dC$  of the constraint  $P = \psi(C) \times (\phi(\psi(C)))$  at the allocation  $(C, P) = (3 - T, T)$  must equal to the marginal rate substitution at the allocation if the optimal exists, that is, from (8) and (13)

$$-\frac{dP}{dC} \Big|_{(C,P)=(3-T,T)} = 1 - \frac{T}{3} = S(3 - T, T) = \frac{\alpha}{\beta} \frac{T}{3 - T}.$$



Hereafter, we confine the utility to the one satisfying  $\alpha/\beta < 4/3$ .

(i) Suppose that  $3 - T > 0$  is the case. Then we have

$$1 - \frac{T}{3} = \frac{\alpha}{\beta} \frac{T}{3 - T} < \frac{4}{3} \left( \frac{T}{3 - T} \right) \quad \text{i.e.} \quad T^2 - 10T + 9 < 0.$$

This implies that the solution satisfies  $1 < T < 3$ .

(ii) Next consider the case  $3 - T < 0$  and we have

$$1 - \frac{T}{3} = \frac{\alpha}{\beta} \frac{T}{3 - T} > \frac{4}{3} \left( \frac{T}{3 - T} \right) \quad \text{i.e.} \quad T^2 - 10T + 9 < 0.$$

The solution must satisfy  $3 < T < 9$ . Both solutions of (i) and (ii) contradict (14).

This is an example where we have no equilibria in the Zodrow-Mieszkowski tax competition model.

### 3 Discussion

We have shown that there is an existential difficulty in the Zodrow-Mieszkowski model. They implicitly assumed that the optimal tax rate of each region should be continuous with respect to parameters. Many papers on tax competition have assumed the same kind of continuity of response function as in Zodrow-Mieszkowski. We shall discuss, here, that many models on tax competition will face with the same kind of difficulty as in the Zodrow-Mieszkowski model once we comprehend models under general equilibrium setting.

#### 3.1 Wilson model

The public service production function  $S(K_P, L_P)$  is introduced in Wilson [5]. In addition to this, two kinds of commodities, the local and the national goods are considered. We simplify, here, the Wilson model to the one with single consumption good since the basic structure of the model remains intact. Let the consumption good production function be  $F(K_C, L_C)$ . The functions  $S(\cdot, \cdot)$  and  $F(\cdot, \cdot)$  are homogenous of degree one and well behaved. The jurisdiction under consideration contains  $\bar{L}$  number of residents who hold the capital  $\bar{K}$ . Let the residents utility be  $u(C, P)$ . And thus the model is described by the following relations:

$$\begin{aligned} \text{residents' budget} & \quad C \leq r\bar{K} + w\bar{L}, \\ \text{local government budget} & \quad rK_P + wL_P \leq TK_C, \\ \text{profit maximization} & \quad F_K(K_C, L_C) = r + T, \quad F_L(K_C, L_C) = w, \\ \text{labor market} & \quad L_C + L_P = \bar{L}. \end{aligned} \tag{15}$$

The price of consumption good is assumed to be unity. The local government levies a tax on capital by the rate  $T$ .

Note that the equilibrium condition for capital market is not considered but that the equilibrium condition for labor market,  $\bar{L} = L_C + L_P$  is taken into consideration. The rental ratio  $r$  is assumed to be given. And thus we can say that Wilson [5] is a partial equilibrium model.

From (15), we can obtain the well known functional relation  $k_C = \phi(r + T)$  and  $k_C \stackrel{\text{def}}{=} K_C/L_C$ . The wage  $w$  is given by the function  $w(r + T)$ . The demand for the consumption good is determined by consumers' budget, then the consumers' behavior implies

$$C(r, r + T) = r\bar{K} + w(r + T)\bar{L}.$$

And thus the demand for capital  $K_C(r, r + T)$  and labor  $L_C(r, r + T)$  to produce consumption good are

$$\begin{aligned} L_C(r, r + T) &= C(r, r + T) \times \frac{1}{f(\phi(r + T))}, \\ K_C(r, r + T) &= L_C(r, r + T) \times \phi(r + T), \end{aligned}$$

where  $f(k) \stackrel{\text{def}}{=} F(k, 1)$ . The maximization problem of the jurisdiction is:

$$\max_{T, K_P, L_P, C, P} u(C, P) \text{ subject to } \begin{cases} C \leq r\bar{K} + w(r + T)\bar{L} \\ rK_P + w(r + T)L_P \leq TK_C(r, r + T) \\ P \leq S(L_P, K_P) \\ L_P + L_C(r, r + T) \leq \bar{L} \end{cases} \quad (16)$$

The government budget is expressed in the more simple form when the labor market is balanced, i.e.,

$$\begin{aligned} rK_P &\leq TK_C(r, r + T) - w(r + T)L_P \\ &= TL_C(r, r + T)\phi(r + T) - w(r + T)(\bar{L} - L_C(r, r + T)) \\ &= L_C(r, r + T)(T\phi(r + T) + w(r + T)) - w(r + T)\bar{L} \\ &= L_C(r, r + T)\{f(\phi(r + T)) - r\phi(r + T)\} - w(r + T)\bar{L} \\ &= r\bar{K} + w(r + T)\bar{L} - r\phi(r + T)L_C(r, r + T) - w(r + T)\bar{L} \\ &= r(\bar{K} - K_C(r, r + T)). \end{aligned}$$

And thus (16) is reduced to:

$$\max_{T, K_P} u(r\bar{K} + w(r + T)\bar{L}, S(\bar{L} - K_P/\phi(r + T), K_P)) \text{ subject to } K_C(r, r + T) + K_P \leq \bar{K}. \quad (17)$$

There are no existential difficulties due to non-convexity as far as the problem (17) is concerned. The fact that  $r$  is constant leads us to this conclusion. It does not, however, imply that the Wilson model is consistent with the general equilibrium. We face with the problem caused by non-convexity once we are to determine the rental ratio  $r$ . Suppose that the rental ratio is determined in the global capital market. For example, let  $\bar{K}^i, K_C^i(r, r + T_i), K_P^i$  be

the initial holding of capital, private demand for capital and public demand for capital in the  $i$ -th region respectively. Then the rental ratio must satisfy:

$$\sum_i \{K_C^i(r, r + T_i) + K_P^i\} = \sum_i \bar{K}^i.$$

And thus, Wilson's model may have problem as in Zodrow-Mieszkowski model if we deal with it in the general equilibrium setting.

### 3.2 Wildasin model

This subsection discusses the model of Wildasin [4] which is different from Zodrow-Mieszkowski [7] and Wildasin [3] <sup>1</sup>. He developed the models containing non-price-taking jurisdiction in the sense that the local governments know the way how their choices of tax rates have effects on the global rental ratio  $r$ .

The functions  $u_i(C_i, P_i)$  and  $F_i(K_i)$  are the utility and the production functions of the  $i$ -th jurisdiction,  $i = 1, \dots, n$  respectively. The individual in locality  $i$  owns the share  $\theta_i \geq 0$  of the capital stock which is given by  $\bar{K}$ . We simplify the model of Wildasin [4] to:

$$\text{residents' budget} \quad C_i \leq F_i(K_i) - (r + T_i)K_i + \theta_i r \bar{K} \quad i = 1, \dots, n \quad (18)$$

$$\text{local government budget} \quad T_i K_i \leq P_i \quad i = 1, \dots, n \quad (19)$$

$$\text{profit maximization} \quad F_i'(K_i) = r + T_i, \quad i = 1, \dots, n \quad (20)$$

$$\text{equilibrium condition} \quad \sum K_i = \bar{K}. \quad (21)$$

(18) is the budget constraint of an individual since  $\theta_i r \bar{K}$  is the rental the residents receive. The system (18) – (21) implies that the economy is considered in the general equilibrium setting.

From (20), we have

$$K_i = K_i(T_i + r). \quad (22)$$

From the relation (21) together with (22), the rental ratio  $r$  is represented by a function of tax rates:

$$r = r(\tau), \quad \tau = (T_1, \dots, T_n).$$

The assumption that the governments are not price takers implies that governments know the function  $r(\tau)$  and the revenue  $P_i(\tau) = T_i K_i(T_i + r(\tau))$ . He gave two kinds of equilibrium concepts. One is T-equilibrium. Given  $\hat{\tau} = (\hat{T}_1, \dots, \hat{T}_n)$ , the jurisdiction  $i$  solves the problem:

$$\begin{aligned} & \max_{T_i, C_i} u_i(C_i, P_i(\tau)) \\ & \text{subject to} \quad \begin{cases} C_i \leq F_i(K_i(T_i + r(\tau))) - (r(\tau) + T_i)K_i(T_i + r(\tau)) + \theta_i r(\tau) \bar{K} \\ \tau = (\hat{T}_1, \dots, \hat{T}_{i-1}, T_i, \hat{T}_{i+1}, \dots, \hat{T}_n) \end{cases} \quad (23) \end{aligned}$$

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<sup>1</sup>The models of Zodrow and Mieszkowski [7] and Wildasin [3] have the same structure.

$i = 1, 2, \dots, n$ . The solution  $\tilde{T}_i$  to (23) gives us a function  $f_i: (\hat{T}_1, \dots, \hat{T}_n) \mapsto \tilde{T}_i$ . T-equilibrium is a Nash equilibrium satisfying

$$f_i(T_1^*, \dots, T_n^*) = T_i^*, \quad i = 1, 2, \dots, n.$$

In this problem, we can not expect the functions  $P_i(\tau)$ 's are concave functions and that  $f_i(\tau)$  is continuous.

Another equilibrium is called P-equilibrium. Solving the equations  $P_i = P_i(\tau)$ ,  $i = 1, \dots, n$  with respect to  $\tau$  and we obtain a function  $\tau(\zeta)$ ,  $\zeta = (P_1, \dots, P_n)$ . Assume that  $\tau(\zeta)$  exists and is sufficiently smooth. P-equilibrium is a Nash equilibrium to the problems:

$$\begin{aligned} & \max_{P_i, C_i} u_i(C_i, P_i) \\ & \text{subject to } \begin{cases} C_i \leq F_i(K_i(\tau_i(\zeta) + \tau(\zeta))) \\ \quad - (r(\tau(\zeta)) + T_i)K_i(\tau_i(\zeta) + \tau(\zeta)) + \theta_i r(\tau(\zeta))\bar{K} \\ P_j = P_j^*, \quad i \neq j, \end{cases} \end{aligned}$$

$i = 1, 2, \dots, n$ . Unfortunately, the function in the right hand side of the first constraint is not necessarily convex. Therefore Wildasin [4] faces with the same kind of difficulty as in the model of Zodrow-Mieszkowski [7].

### 3.3 Bucovetsky, Wilson, and De Pater-Myers models

The models of Bucovetsky [1], Wilson [6] and De Pater-Myers [2] extended the model of tax competition to the case with two asymmetric jurisdictions. The model of De Pater-Myers [2] is reduced to the models of Bucovetsky [1] and Wilson [6] when the regions' production technologies are identical. A nation consists of two regions 1 and 2. Each region  $i$  is assumed to contain a fixed number of individuals  $L_i$ .  $k_i$ ,  $c_i$ , and  $q_i$  are the capital-labor ratio, the amount of private good consumption per worker, and the public good consumption per worker in region  $i$ , respectively. Their models are described by the following system:

$$\begin{aligned} \text{production function} & \quad f(k_i), \quad i = 1, 2 \\ \text{utility function} & \quad u(c_i, q_i), \quad i = 1, 2 \\ \text{residents' budget} & \quad c_i \leq f(k_i) - (r + T_i)k_i + r\bar{k} \quad i = 1, 2 \end{aligned} \tag{24}$$

$$\text{local government budget} \quad q_i \leq T_i k_i \quad i = 1, 2 \tag{25}$$

$$\text{profit maximization} \quad r = f'(k_1) - T_1 = f'(k_2) - T_2, \tag{26}$$

$$\text{equilibrium condition} \quad L_1 k_1 + L_2 k_2 \leq L_1 \bar{k} + L_2 \bar{k}, \quad \text{with equality for } r > 0. \tag{27}$$

The system (24)–(27) is a general equilibrium model.

From (26), we obtain

$$k_i = \phi(T_i + r). \tag{28}$$

From (28), (27) is rewritten as follows:

$$L_1 k_1(r + T_1) + L_2 k_2(r + T_2) \leq L_1 \bar{k} + L_2 \bar{k}$$

This condition with equality determines the rental rate  $r$  depending on the pair  $(T_i, T_j)$ ,  $i \neq j$ . This relation is given by a function  $r = r(T_i, T_j)$  for region  $i$ .

Each jurisdiction chooses its optimal tax rate  $T_i$ , given the tax rate  $T_j^*$  of the other. Therefore, the jurisdiction  $i$ 's maximization problem is as follows:

$$\begin{aligned} & \max_{T_i, p_i, c_i} u_i(c_i, q_i) \\ & \text{subject to } \begin{cases} c_i \leq f(k_i(r(T_i, T_j^*) + T_i)) - (r(T_i, T_j^*) + T_i)k_i(r(T_i, T_j^*) + T_i) \\ \quad + r(T_i, T_j^*)k \\ q_i \leq T_i k_i(r(T_i, T_j^*) + T_i) \end{cases} \end{aligned} \quad (29)$$

The set of variables satisfying constraints in (29) is not necessarily convex. The constraints in (29) does not necessarily exhibit the convexity. The same discussion can be applied to the models of Bucovetsky [1], Wilson [6] and De Pater-Myers [2].

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